

Wasserstein distributionally robust co-design of a fractional-order dual-resonator elastic metamaterial for broadband vibration attenuation

Asokan Vasudevan¹, Yogeesh Nijalingappa^{2,3,4,*}, Rashed Abu Hammour^{5,6}, Poornachandran William^{7,8}, Suleiman Ibrahim Mohammad¹, Torki M. Al-Fawwaz⁹

¹ Faculty of Business and Communications, INTI International University, Nilai 71800, Malaysia; asokan.vasudevan@newinti.edu.my (AV); dr_sliman@yahoo.com (SIM)

² Mathematics & Natural Sciences, Gulf University for Science & Technology, Safat 13060, Kuwait

³ Department of Mathematics, Government First Grade College, Tumkur 572102, India

⁴ Research Fellow, INTI International University, Nilai 71800, Malaysia

⁵ Faculty of Technical Education, Al-Ahliyya Amman University, Amman 19328, Jordan; r.abuhammour@ammanu.edu.jo

⁶ Hourani Center for Applied Scientific Research (HCASR), Al-Ahliyya Amman University, Amman 19328, Jordan

⁷ School of Computer Science and Technology, Karunya Institute of Technology and Sciences, Coimbatore 641114, India; william@karunya.edu

⁸ Centre for Research and Consultancy, UNITAR International University, Petaling Jaya 47301, Malaysia

⁹ College of Commerce and Business, Lusail University, Lusail P.O. Box 9717, Qatar; talfawwaz@lu.edu.qa

* Corresponding author: Yogeesh Nijalingappa, yogeesh.n@ka.gov.in

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Abstract: Locally resonant elastic metamaterials provide subwavelength attenuation but remain sensitive to uncertainty in stiffness, damping, and fractional order. This study develops a Wasserstein distributionally robust co-design framework that jointly selects the mass split, resonance tuning, loss scales, and fractional orders of a dual-resonator periodic cell. Bloch dispersion is condensed into a complex effective-mass model, and a feasibility-aware loss balances integrated attenuation, threshold coverage, internal-motion amplification, and resonance separation. Five bounded uncertainties in host stiffness, resonator stiffnesses, common loss scale, and fractional-order drift are represented in a finite 1-Wasserstein ambiguity set trained on 64 scrambled Sobol scenarios with 256 adversarial candidates. The proposed design is independently compared with equal-mass, nominal, and conditional-value-at-risk alternatives over 4,096 unseen scenarios. Across this validation set, the Wasserstein design reduced worst loss by 73.68% and increased fifth-percentile integrated attenuation by 32.50% relative to the equal-mass baseline while satisfying the internal-motion constraint. It allocated 56.03% of the resonator mass to the lower branch and selected tuning frequencies of 27.20 and 40.31 Hz. Exact antiresonance verification, uncertainty envelopes, finite-chain attenuation-equivalent insertion loss, and local sensitivity analysis support the numerical results. The framework provides a reproducible basis for robust broadband attenuation and subsequent specimen-level calibration and experimental validation.

Keywords: Bloch–Floquet analysis; optimal transport; attenuation band; Kelvin–Voigt damping; uncertain dynamics; risk-sensitive design; quasi-Monte Carlo validation

1. Introduction

Media with periodicity attenuate vibration via Bragg scattering or local resonance [1]. Bragg gaps need wavelengths comparable to cell spacing; local resonance produces subwavelength dynamic inertia that is useful for low frequencies.

The pioneering locally resonant concept [2] has since developed into a broad class of phononic, acoustic, and mechanical metamaterials, as summarized in recent field perspectives [3,4].

Bloch–Floquet reduction projects an infinite lattice onto one cell, while dynamic homogenization represents hidden coordinates through effective inertia [5]. Recent studies extend this foundation to fractional homogenization [6], multi-resonant metamaterial beams [7], and broadband meta-shells [8]. Together, these studies provide the modeling basis for overlapping attenuation lobes, but they do not determine how two fractional resonances should be co-designed under uncertain properties.

Dissipation makes band-gap analysis complex-valued, so attenuation varies continuously rather than as a conservative pass/stop classification [9]. Recent viscoelastic, tailored-resonator, and composite meta-slab studies show that controlled damping and resonance placement can broaden useful attenuation without relying on a single negative-mass peak [10–12].

A single integer-order dashpot does not adequately represent the power-law memory that characterizes many viscoelastic attachments. Fractional constitutive models compactly interpolate elastic and viscous response [13, 14]. Studies of fractionally damped phononic crystals [15], power-law response [16], and fractional-order resonators [17] show how frequency-dependent storage and loss shape attenuation, thereby motivating fractional order as a design variable rather than a fixed constitutive parameter.

Host stiffness, resonator stiffness, damping, and fitted order vary with manufacturing and material conditions, so nominal designs may perform poorly outside the training distribution. With limited scenarios, moment-based robustness can be sensitive to the assumed ambiguity description. Wasserstein DRO instead limits the optimal-transport distance from the empirical law and supports tractable finite-sample dual formulations [18–20]. However, existing DRO formulations do not resolve the metamaterial-specific coupling among complex branch selection, band coverage, and local-motion feasibility.

CVaR controls the upper empirical loss tail [21], whereas Wasserstein DRO permits probability shifts toward nearby adverse tolerances subject to a transport budget [19,22]. The numerical workflow combines differential evolution [23] with disjoint scrambled Sobol samples [24,25], followed by multiobjective and sensitivity diagnostics.

Experimental and numerical studies have broadened attenuation through additive manufacturing [26], multi-resonator beams [7], multiscale absorbers [27], multi-resonant shells [8], tailored finite rods [11], and metabarriers [28]. These contributions primarily address geometry or damping, whereas fractional-resonator studies [6, 15, 17] and robust-optimization theory [18–20, 22, 29–31] remain largely separate. Consequently, the unresolved gap is a unified formulation that jointly selects mass split, two resonances, two fractional loss scales, and two orders under distributional ambiguity and validates the resulting design independently. Meeting this gap requires consistent effective-mass condensation, continuous complex-wavenumber branch selection, fractional calibration, amplitude control, and finite-chain interpretation.

Novel contributions

This study contributes: complex effective mass and Bloch attenuation for a dual fractional Kelvin–Voigt cell; a loss balancing attenuation, coverage, motion, and resonance separation; a five-dimensional 1-Wasserstein dual trained on 64 Sobol scenarios with 256 adversarial candidates; and independent 4,096-scenario validation, antiresonance verification, critical-scenario insertion loss, sensitivity, and a robustness frontier. It provides a numerical basis for later fabrication and testing.

The remainder of the manuscript is organized as follows. Section 2 formulates the periodic cell, fractional constitutive model, deterministic loss, uncertainty set, optimization procedure, and validation metrics. Section 3 presents analytical verification, optimized designs, uncertainty envelopes, stress tests, finite-chain interpretation, and sensitivity results. Section 4 discusses the physical and computational implications and limitations, and Section 5 summarizes the conclusions and future work.

2. Materials and methods

The methodology contributes three linked elements: a dual-resonator fractional effective-mass formulation with continuity-preserving complex-wavenumber retrieval; a feasibility-aware attenuation-and-coverage loss; and a finite 1-Wasserstein co-design workflow with independent validation.

2.1. Periodic cell and dimensional scaling

Figure 1 defines the infinite one-dimensional lattice. Each host mass M is connected to its nearest neighbors by stiffness K . Two internal masses m_1 and m_2 are connected to the host coordinate $x_n(t)$ by fractional Kelvin–Voigt elements. Their absolute coordinates are $y_{1n}(t)$ and $y_{2n}(t)$, and a denotes the cell length. The displacement vector of cell n is

$$\mathbf{u}_n(t) = \begin{bmatrix} x_n(t) & y_{1n}(t) & y_{2n}(t) \end{bmatrix}^T. \quad (1)$$

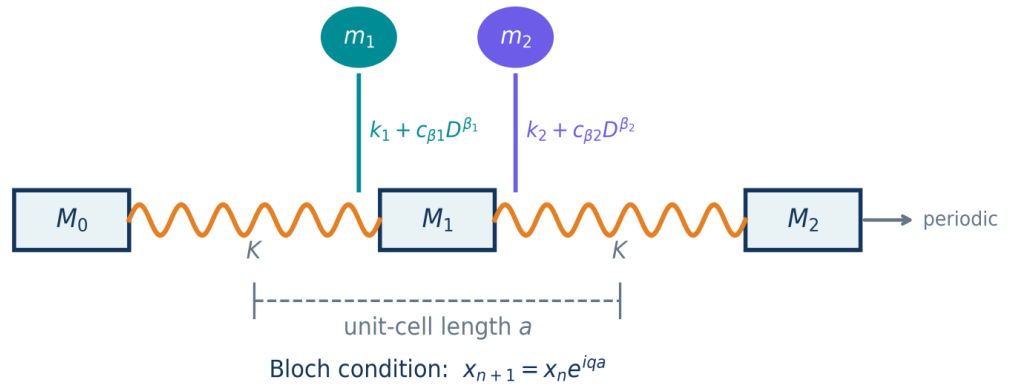


Figure 1. Periodic dual-resonator fractional elastic metamaterial.

Each cell contains one host mass and two internally attached resonators; the constitutive operators are $k_i + c_{\beta i} D^{\beta_i}$ and the cell boundary satisfies the Bloch phase relation.

The reference angular frequency and the corresponding ordinary frequency are defined by

$$\omega_0 = \sqrt{\frac{K}{M}}, \quad f_0 = \frac{\omega_0}{2\pi}. \quad (2)$$

The normalized excitation frequency is

$$\Omega = \frac{\omega}{\omega_0} \quad (3)$$

The host parameters were $M = 1.80$ kg, $K = 58,000$ N m⁻¹, and $a = 0.085$ m, giving $\omega_0 = 179.506$ rad s⁻¹ and $f_0 = 28.569$ Hz. The total resonator-to-host mass ratio was fixed at $\mu_t = 0.18$, hence

$$m_t = m_1 + m_2 = \mu_t M = 0.18(1.80) = 0.324 \text{ kg} \quad (4)$$

The optimization band $\Omega \in [0.45, 1.85]$ therefore corresponds to

$$f \in [0.45f_0, 1.85f_0] = [12.856, 52.853] \text{ Hz}. \quad (5)$$

The fixed model parameters and performance settings are summarized in **Table 1**.

Table 1. Fixed model parameters and performance settings.

Quantity	Symbol	Value	Unit or interpretation
Host mass	M	1.80	kg
Host coupling stiffness	K	58,000	N m ⁻¹
Unit-cell length	a	0.085	m
Total resonator mass ratio	μ_t	0.18	m_t/M
Total resonator mass	m_t	0.324	kg
Reference frequency	f_0	28.569	Hz
Design-frequency interval	$[\Omega_L, \Omega_U]$	0.45–1.85	12.856–52.853 Hz
Attenuation threshold	α_0	0.12	neper cell ⁻¹
Local-amplification limit	$H_{\max}^{\text{lim}}$	8.0	dimensionless
Finite-chain length	N_c	12	cells

2.2. Fractional Kelvin–Voigt resonators

For $0 < \beta_i \leq 1$, the Caputo derivative of a sufficiently regular relative displacement $z_i(t) = y_i(t) - x(t)$ is [13,14,32–35]:

$${}^C D_t^{\beta_i} z_i(t) = \frac{1}{\Gamma(1 - \beta_i)} \int_0^t \frac{\dot{z}_i(\tau)}{(t - \tau)^{\beta_i}} d\tau. \quad (6)$$

The resonator force is

$$F_i(t) = k_i z_i(t) + c_{\beta_i} {}^C D_t^{\beta_i} z_i(t) \quad (7)$$

Under the harmonic convention $z_i(t) = \Re\{Z_i e^{i\omega t}\}$, the fractional derivative becomes $(i\omega)^{\beta_i} Z_i$, using the principal branch [15–17].

$$(i\omega)^{\beta_i} = \omega^{\beta_i} \left[\cos\left(\frac{\pi\beta_i}{2}\right) + i \sin\left(\frac{\pi\beta_i}{2}\right) \right] \quad (8)$$

The complex attachment stiffness is consequently

$$S_i(\omega) = k_i + c_{\beta i} (i\omega)^{\beta_i} \quad (9)$$

Its storage and loss parts are

$$S_i'(\omega) = k_i + c_{\beta i} \omega^{\beta_i} \cos\left(\frac{\pi\beta_i}{2}\right) \quad (10)$$

$$S_i''(\omega) = c_{\beta i} \omega^{\beta_i} \sin\left(\frac{\pi\beta_i}{2}\right). \quad (11)$$

Let $r_i = \omega_i/\omega_0$ be the tuning ratio, with $\omega_i = \sqrt{k_i/m_i}$. Then

$$k_i = m_i (r_i \omega_0)^2 \quad (12)$$

The fractional coefficient is calibrated by a dimensionless loss scale ζ_i at ω_i :

$$c_{\beta i} = \frac{2\zeta_i k_i}{\omega_i^{\beta_i}} \quad (13)$$

For $\beta_i = 1$, Equation (13) reduces to $c_i = 2\zeta_i k_i/\omega_i = 2\zeta_i m_i \omega_i$, the conventional viscous result. For fractional order, $c_{\beta i}$ has units $\text{Ns}^{\beta_i} \text{m}^{-1}$. Positivity of k_i , $c_{\beta i}$ and $0 < \beta_i \leq 1$ makes the loss term in Equation (11) non-negative for $\omega > 0$.

2.3. Complex effective mass and Bloch attenuation

The resonator equilibrium in the frequency domain is

$$-\omega^2 m_i Y_{in} + S_i(\omega) (Y_{in} - X_n) = 0. \quad (14)$$

Solving for the internal-to-host displacement ratio gives

$$R_i(\omega) = \frac{Y_{in}}{X_n} = \frac{S_i(\omega)}{S_i(\omega) - m_i \omega^2} \quad (15)$$

The force transmitted by resonator i to the host can therefore be written as an inertial contribution:

$$S_i (Y_{in} - X_n) = m_i \omega^2 R_i X_n \quad (16)$$

The host equilibrium is

$$-M\omega^2 X_n + K (2X_n - X_{n-1} - X_{n+1}) - \sum_{i=1}^2 S_i (Y_{in} - X_n) = 0 \quad (17)$$

Substitution of Equation (16) produces the complex effective mass [1,36]:

$$M_{eff}(\omega) = M + \sum_{i=1}^2 m_i \frac{S_i(\omega)}{S_i(\omega) - m_i \omega^2} \quad (18)$$

The Bloch ansatz is [5,9,37]:

$$X_n = X e^{i(nqa - \omega t)}, \quad Y_{in} = Y_i e^{i(nqa - \omega t)} \quad (19)$$

Thus $X_{n+1} + X_{n-1} = 2X_n \cos(qa)$, and the condensed dispersion relation is

$$4K \sin^2\left(\frac{qa}{2}\right) = \omega^2 M_{\text{eff}}(\omega) \quad (20)$$

Equivalently,

$$\cos(qa) = 1 - \frac{\omega^2 M_{\text{eff}}(\omega)}{2K} \quad (21)$$

The complex wavenumber is retrieved as

$$q(\omega) = \frac{1}{a} \cos^{-1} \left[1 - \frac{\omega^2 M_{\text{eff}}(\omega)}{2K} \right] \quad (22)$$

with the branch chosen so that $\text{Im}(qa) \geq 0$ and continuity is preserved between adjacent frequency samples. At the first grid point, admissible inverse-cosine candidates satisfying $\text{Im}(qa) \geq 0$ and $\text{Re}(qa) \in [0, \pi]$ are considered; at each subsequent point, the candidate producing the smallest change in qa from the preceding frequency sample is retained. The attenuation per cell is

$$\alpha(\omega) = \text{Im}(qa) \quad \text{nepers cell}^{-1} \quad (23)$$

The real phase index is

$$\kappa(\omega) = \frac{\text{Re}(qa)}{\pi} \quad (24)$$

For lossless dispersion plots, the three degrees of freedom give the generalized eigenproblem [7,38]:

$$[\mathbf{K}(q) - \omega^2 \mathbf{M}_c] \phi = 0. \quad (25)$$

where

$$\mathbf{M}_c = \text{diag}(M, m_1, m_2) \quad (26)$$

$$\mathbf{K}(q) = \begin{bmatrix} 4K \sin^2(qa/2) + k_1 + k_2 & -k_1 & -k_2 \\ -k_1 & k_1 & 0 \\ -k_2 & 0 & k_2 \end{bmatrix} \quad (27)$$

The ordered positive roots $\omega_j(q)$, $j = 1, 2, 3$, define the acoustic and two optical branches. Their normalized group velocities are evaluated by centered differences,

$$v_{g,j}(q) = \frac{d\omega_j}{dq} \approx \frac{\omega_j(q + \Delta q) - \omega_j(q - \Delta q)}{2\Delta q} \quad (28)$$

2.4. Design variables and deterministic performance loss

The mass-allocation variable η enforces the total-mass constraint exactly:

$$m_1 = \eta m_t, \quad m_2 = (1 - \eta) m_t \quad (29)$$

The complete design vector is

$$\mathbf{d} = \begin{bmatrix} \eta & r_1 & r_2 & \zeta_1 & \zeta_2 & \beta_1 & \beta_2 \end{bmatrix}^T \quad (30)$$

The bounds were $0.25 \leq \eta \leq 0.75$, $0.70 \leq r_1 \leq 1.10$, $1.10 \leq r_2 \leq 1.60$, $0.02 \leq \zeta_i \leq 0.15$, and $0.55 \leq \beta_i \leq 1.00$. A minimum tuning separation avoids redundant resonators:

$$r_2 - r_1 \geq \Delta r_{\min} = 0.18. \quad (31)$$

On a uniform grid of $N_\Omega = 501$ frequencies, with $\Omega_j = 0.45 + (j - 1)(1.40/500)$ for $j = 1, \dots, 501$, normalized integrated attenuation is

$$A(\mathbf{d}, \boldsymbol{\xi}) = \frac{1}{\Omega_U - \Omega_L} \int_{\Omega_L}^{\Omega_U} \alpha(\Omega; \mathbf{d}, \boldsymbol{\xi}) d\Omega \quad (32)$$

Trapezoidal evaluation used in the computation is

$$A \approx \frac{1}{\Omega_U - \Omega_L} \sum_{j=1}^{N_\Omega-1} \frac{\alpha_j + \alpha_{j+1}}{2} (\Omega_{j+1} - \Omega_j) \quad (33)$$

Threshold coverage measures the fraction of the design interval exceeding $\alpha_0 = 0.12$:

$$C(\mathbf{d}, \boldsymbol{\xi}) = \frac{1}{N_\Omega} \sum_{j=1}^{N_\Omega} \mathbb{I}[\alpha_j \geq \alpha_0] \quad (34)$$

The maximum internal-motion amplification is

$$H_{\max}(\mathbf{d}, \boldsymbol{\xi}) = \max_{i \in \{1,2\}} \max_{1 \leq j \leq N_\Omega} |R_i(\Omega_j)| \quad (35)$$

The squared amplification penalty is

$$P_H = \left[\max \left(0, \frac{H_{\max}}{H_{\max}^{\lim}} - 1 \right) \right]^2 \quad (36)$$

The separation penalty is

$$P_r = \left[\max \left(0, \frac{\Delta r_{\min} - (r_2 - r_1)}{\Delta r_{\min}} \right) \right]^2 \quad (37)$$

The scenario loss combines attenuation magnitude, coverage, and feasibility:

$$\ell(\mathbf{d}, \boldsymbol{\xi}) = 0.65(1 - A) + 0.35(1 - C) + 8P_H + 4P_r \quad (38)$$

The attenuation and coverage weights, 0.65 and 0.35, sum to one and prioritize band-averaged attenuation while retaining substantial emphasis on the fraction of the band above the prescribed threshold. The amplification and separation coefficients, 8 and 4, make normalized feasibility violations dominate typical variations in the bounded performance terms; amplification receives twice the penalty because excessive internal motion directly limits physical realizability. These coefficients

were fixed before comparing all four designs so that performance differences arise from the design objectives rather than retuning the loss. The design variables and their admissible bounds are listed in **Table 2**.

Table 2. Design variables and admissible bounds.

Variable	Meaning	Lower bound	Upper bound	Proposed value
η	Lower-resonator mass fraction	0.25	0.75	0.5603
r_1	Lower tuning ratio	0.70	1.10	0.9521
r_2	Upper tuning ratio	1.10	1.60	1.4108
ζ_1	Lower loss scale	0.02	0.15	0.1335
ζ_2	Upper loss scale	0.02	0.15	0.1491
β_1	Lower fractional order	0.55	1.00	0.7431
β_2	Upper fractional order	0.55	1.00	0.7292

2.5. Uncertainty model and Wasserstein ambiguity set

Five standardized uncertain variables are collected in

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_K & \xi_{k1} & \xi_{k2} & \xi_c & \xi_\beta \end{bmatrix}^\top \in [-1, 1]^5 \quad (39)$$

Their physical mapping is

$$K(\boldsymbol{\xi}) = K_0 (1 + 0.08\xi_K) \quad (40)$$

$$k_i(\boldsymbol{\xi}) = k_{i0} (1 + 0.06\xi_{ki}), \quad i = 1, 2 \quad (41)$$

$$c_{\beta i}(\boldsymbol{\xi}) = c_{\beta i,0} (1 + 0.20\xi_c) \quad (42)$$

$$\beta_i(\boldsymbol{\xi}) = \text{clip}(\beta_{i0} + 0.05\xi_\beta, 0.50, 1.00) \quad (43)$$

The bounded uncertainty definitions used throughout the numerical workflow are summarized in **Table 3**.

Table 3. Bounded uncertainty definition used for training, adversarial search, and independent validation.

Component	Standardized variable	Physical transformation	Full range
Host stiffness	ξ_K	$K_0 (1 + 0.08\xi_K)$	$\pm 8\%$
Resonator 1 stiffness	ξ_{k1}	$k_{10} (1 + 0.06\xi_{k1})$	$\pm 6\%$
Resonator 2 stiffness	ξ_{k2}	$k_{20} (1 + 0.06\xi_{k2})$	$\pm 6\%$
Common fractional loss scale	ξ_c	$c_{\beta i,0} (1 + 0.20\xi_c)$	$\pm 20\%$
Common order drift	ξ_β	$\beta_{i0} + 0.05\xi_\beta$	± 0.05

The $N = 64$ training scenarios $\hat{\boldsymbol{\xi}}_n$ were generated by a five-dimensional scrambled Sobol sequence [24, 25] and transformed to $[-1, 1]^5$. The empirical distribution is

$$\hat{\mathbb{P}}_N = \frac{1}{N} \sum_{n=1}^N \delta_{\hat{\boldsymbol{\xi}}_n} \quad (44)$$

For probability measures $\mathbb{Q}$ and $\hat{\mathbb{P}}_N$ on $[-1, 1]^5$, the 1-Wasserstein distance under the Euclidean ground metric is [18, 20, 29, 30, 39]:

$$W_1(\mathbb{Q}, \widehat{\mathbb{P}}_N) = \inf_{\pi \in \Pi(\mathbb{Q}, \widehat{\mathbb{P}}_N)} \int \|\boldsymbol{\xi} - \boldsymbol{\xi}'\|_2 d\pi(\boldsymbol{\xi}, \boldsymbol{\xi}') \quad (45)$$

The ambiguity set of radius $\varepsilon = 0.075$ is

$$\mathcal{P}_\varepsilon = \left\{ \mathbb{Q} : W_1(\mathbb{Q}, \widehat{\mathbb{P}}_N) \leq \varepsilon \right\} \quad (46)$$

The radius $\varepsilon = 0.075$ was selected a priori as a moderate transport budget in the standardized uncertainty space: it permits local probability redistribution beyond the 64 empirical points while remaining small relative to the support diameter $2\sqrt{5}$. It was held fixed for every design comparison and is treated as a modeling hyperparameter pending calibration from repeated specimens.

The Wasserstein-robust design solves

$$\min_{\mathbf{d} \in \mathcal{D}} \sup_{\mathbb{Q} \in \mathcal{P}_\varepsilon} \mathbb{E}_{\mathbb{Q}}[\ell(\mathbf{d}, \boldsymbol{\xi})] \quad (47)$$

Kantorovich duality produces the following reformulation [18,29,30]:

$$\sup_{\mathbb{Q} \in \mathcal{P}_\varepsilon} \mathbb{E}_{\mathbb{Q}}[\ell] = \inf_{\lambda \geq 0} \left\{ \lambda \varepsilon + \frac{1}{N} \sum_{n=1}^N \sup_{\boldsymbol{\xi} \in [-1,1]^5} \left[\ell(\mathbf{d}, \boldsymbol{\xi}) - \lambda \|\boldsymbol{\xi} - \widehat{\boldsymbol{\xi}}_n\|_2 \right] \right\} \quad (48)$$

The inner supremum was approximated on the value $N_a = 256$ independent scrambled Sobol adversarial candidates $\boldsymbol{\zeta}_m$:

$$\widehat{g}_n(\lambda, \mathbf{d}) = \max_{1 \leq m \leq N_a} \left[\ell(\mathbf{d}, \boldsymbol{\zeta}_m) - \lambda \|\boldsymbol{\zeta}_m - \widehat{\boldsymbol{\xi}}_n\|_2 \right] \quad (49)$$

The numerical dual objective is

$$\widehat{J}_W(\mathbf{d}) = \min_{\lambda \geq 0} \left[\lambda \varepsilon + \frac{1}{N} \sum_{n=1}^N \widehat{g}_n(\lambda, \mathbf{d}) \right] \quad (50)$$

The optimal multiplier λ^* is the marginal cost of shifting probability mass in standardized uncertainty space. A near-zero value allows for almost unobstructed relocation; a high value strongly prices transport. The computed value was $\lambda^* = 0.06611$.

2.6. Comparator objectives, optimization, and validation

The nominal comparator minimizes the zero-uncertainty loss,

$$\widehat{J}_N(\mathbf{d}) = \ell(\mathbf{d}, 0) \quad (51)$$

For losses $L_n = \ell(\mathbf{d}, \widehat{\boldsymbol{\xi}}_n)$, the empirical CVaR comparator at confidence $\gamma = 0.95$ minimizes [21,40]:

$$\widehat{J}_C(\mathbf{d}) = \min_{t \in \mathbb{R}} \left[t + \frac{1}{(1-\gamma)N} \sum_{n=1}^N (L_n - t)_+ \right] \quad (52)$$

where $(x)_+ = \max(x, 0)$. The fixed baseline used equal mass allocation, $r_1 = 0.88$, $r_2 = 1.35$, $\zeta_1 = \zeta_2 = 0.055$, and $\beta_1 = \beta_2 = 1$. Each optimization used differential evolution [23] with a population of 18 candidates per dimension, mutation factor in $[0.45, 1.00]$, crossover probability 0.80, and a fixed seed. Convergence required a relative population-objective spread below 10^{-6} or exhaustion of 140 generations. The nominal, CVaR, and Wasserstein searches required 1,598, 1,446, and 1,718 objective evaluations, respectively.

Independent validation used $N_v = 4,096$ new Sobol scenarios excluded from training and adversarial search. For any metric Z_s , the sample mean was

$$\bar{Z} = \frac{1}{N_v} \sum_{s=1}^{N_v} Z_s \quad (53)$$

and the empirical p -quantile was

$$Q_p(Z) = \inf \left\{ z : \frac{1}{N_v} \sum_{s=1}^{N_v} \mathbb{I}[Z_s \leq z] \geq p \right\} \quad (54)$$

For a finite chain of $N_c = 12$ cells, an attenuation-equivalent insertion loss was computed as

$$\text{IL}_{12}(\Omega) = 20 \log_{10}(e) N_c \alpha(\Omega) = 8.686(12) \alpha(\Omega) \text{ dB} \quad (55)$$

Thus the design threshold $\alpha_0 = 0.12$ corresponds to

$$\text{IL}_{12,0} = 8.686(12)(0.12) = 12.508 \text{ dB} \quad (56)$$

This insertion-loss estimate isolates the exponential Bloch decay and does not include end reflections or impedance mismatch; it is therefore a finite-length interpretation of the infinite-lattice result, not a full transfer-matrix experiment. The numerical workflow and sample separation are summarized in **Table 4**.

Table 4. Numerical workflow and sample separation.

Stage	Method	Scenarios or settings	Purpose
Frequency integration	Uniform trapezoidal grid	501 frequencies	Evaluate attenuation and coverage
Nominal optimization	Differential evolution	Zero-uncertainty model	Deterministic comparator
CVaR optimization	Differential evolution	64 training scenarios, $\gamma = 0.95$	Tail-risk comparator
Wasserstein DRO	Differential evolution + scalar dual minimization	64 training + 256 adversarial scenarios	Proposed distributionally robust design
Stress test	Scrambled Sobol sequence	4,096 independent scenarios	Out-of-sample comparison
Local frontier	Bounded perturbation cloud	640 feasible candidates	Attenuation–coverage trade-off

2.7. Analytical verification and sensitivity

An exact check is available for a lossless cell with one resonator of mass ratio $\mu = m/M$ and tuning ratio r . Setting the real effective mass to zero gives

$$\frac{M_{\text{eff}}}{M} = 1 + \mu \frac{r^2}{r^2 - \Omega^2} = 0 \quad (57)$$

The antiresonance zero is

$$\Omega_z = r\sqrt{1 + \mu} \quad (58)$$

while the pole is

$$\Omega_p = r \quad (59)$$

Numerical roots were compared with Equations (58) and (59) at $r = 0.9$ for $\mu \in \{0.05, 0.10, 0.18\}$. Relative error was

$$e_{\text{rel}} = 100 \frac{|\Omega_z^{\text{num}} - \Omega_z^{\text{exact}}|}{\Omega_z^{\text{exact}}} \% \quad (60)$$

For local design interpretation, the normalized centered sensitivity of the Wasserstein objective to component d_j was [41–43]:

$$S_j = \frac{d_j}{\hat{J}_W} \frac{\partial \hat{J}_W}{\partial d_j} \quad (61)$$

with

$$\frac{\partial \hat{J}_W}{\partial d_j} \approx \frac{\hat{J}_W(\mathbf{d} + h_j \mathbf{e}_j) - \hat{J}_W(\mathbf{d} - h_j \mathbf{e}_j)}{2h_j}, \quad h_j = 0.005(U_j - L_j) \quad (62)$$

The local robust frontier used two maximization objectives: the 10th-percentile integrated attenuation and 10th-percentile threshold coverage. Candidate a dominates candidate b if [44]:

$$A_{0.10}^{(a)} \geq A_{0.10}^{(b)}, \quad C_{0.10}^{(a)} \geq C_{0.10}^{(b)} \quad (63)$$

with at least one strict inequality.

3. Results

3.1. Exact verification and optimized physical designs

Table 5 confirms the analytical antiresonance calculation. The largest relative error across the three mass ratios was $1.46 \times 10^{-8}\%$, and the pole remained numerically at 0.900000. Because Equation (58) follows directly from the condensed effective mass and is independent of frequency discretization, it verifies the sign convention, mass normalization, root finder, and pole placement used in the dual-resonator code.

Table 5. Exact lossless single-resonator verification at $r = 0.9$.

Mass ratio μ	Exact zero Ω_z	Numerical zero	Relative error (%)	Exact/numerical pole
0.05	0.922225569	0.922225569	1.23×10^{-8}	0.900000/0.900000
0.10	0.943927963	0.943927963	1.46×10^{-8}	0.900000/0.900000
0.18	0.977650244	0.977650244	1.53×10^{-9}	0.900000/0.900000

Table 6 lists the optimized physical designs. The proposed design assigns 0.18154 kg to the lower resonator and 0.14246 kg to the upper resonator. In mass-ratio form,

$$\mu_1 = \frac{m_1}{M} = 0.10085, \quad \mu_2 = \frac{m_2}{M} = 0.07915 \quad (64)$$

and $\mu_1 + \mu_2 = 0.18$ exactly. The corresponding tuning frequencies were

$$f_1 = r_1 f_0 = 0.95211(28.569) = 27.201 \text{ Hz} \quad (65)$$

$$f_2 = r_2 f_0 = 1.41080(28.569) = 40.305 \text{ Hz} \quad (66)$$

Table 6. Baseline and optimized dual-resonator designs.

Design	m_1/m_t	r_1	r_2	ζ_1	ζ_2	β_1	β_2
Equal-mass baseline	0.5000	0.8800	1.3500	0.0550	0.0550	1.0000	1.0000
Nominal optimum	0.5068	0.8831	1.3867	0.1313	0.1482	0.6960	0.7620
CVaR robust	0.5373	0.9385	1.3897	0.1413	0.1477	0.7349	0.6859
Proposed Wasserstein DRO	0.5603	0.9521	1.4108	0.1335	0.1491	0.7431	0.7292

The optimized fractional orders (0.7431 and 0.7292) are significantly different from the integer-order baseline. Instead of a purely viscous term concentrating all loss, Equation (8) spreads dissipation over frequency. By using Equation (12), the stiffnesses were found to be 5,302.73 and 9,136.62 N m⁻¹. By applying Equation (13), the fractional coefficients were calculated to be 31.039 N s^{0.7431} m⁻¹ and 48.144 N s^{0.7292} m⁻¹.

3.2. Nominal effective mass, attenuation, and dispersion

Figure 2 presents the complex effective-mass response and Bloch attenuation. The baseline exhibits two closely spaced attenuation peaks and a central dip below the 0.12-neper threshold. The proposed Wasserstein design raises both lobes and fills the intervening valley. Consistent with an objective based on integrated attenuation and coverage rather than peak attenuation alone, its maximum is slightly lower than that of the CVaR design, but its high-frequency shoulder is considerably flatter.

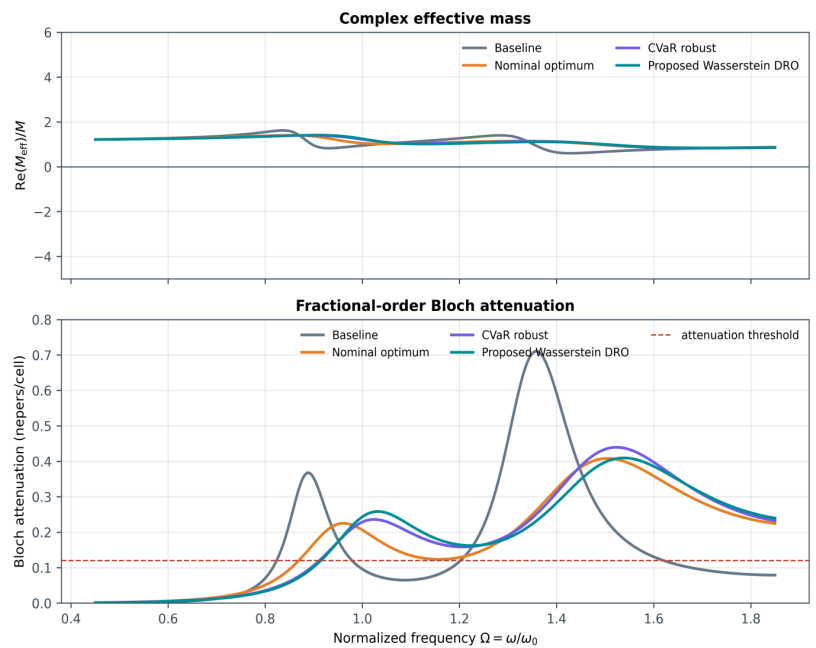


Figure 2. Nominal dynamic response. Top: real part of normalized complex effective mass. Bottom: attenuation per cell over the design-frequency interval; the dashed line is the 0.12-neper threshold.

These changes are quantified with nominal metrics in **Table 7**. This design has increased the integrated attenuation from 0.39457 to 0.51409 (a gain of 30.29% over baseline). We also see a rise in threshold coverage, from 40.92% to 66.87%. The maximum attenuation is intentionally reduced from 0.71047 to 0.40953 nepers $cell^{-1}$ while the mean attenuation increases from 0.16422 to 0.18527. More importantly, the maximum local amplification decreases from 9.159 to 4.471, so the baseline violation of the limit $H_{\max}^{\lim} = 8$ translates into a sizeable margin.

Table 7. Nominal performance over $0.45 \leq \Omega \leq 1.85$.

Design	Loss	Integrated attenuation A	Coverage C (%)	Mean α	Peak α	$H_{\max}$
Equal-mass baseline	0.72485	0.39457	40.92	0.16422	0.71047	9.159
Nominal optimum	0.41080	0.49806	70.06	0.17937	0.40811	4.749
CVaR robust	0.41671	0.51182	67.07	0.18817	0.43953	4.333
Proposed Wasserstein DRO	0.41635	0.51409	66.87	0.18527	0.40953	4.471

The undamped branches in **Figure 3** elucidate the hybridization of the two resonators with the acoustic host. The first branch approaches the lower internal frequency marginally, the second bridges the interval between two resonance scales and the third stays optical throughout the reduced Brillouin zone. Avoided crossings occur around both tuning ratios. Since damping turns these conservative gaps into smooth complex-wavenumber attenuation regions, we interpret **Figure 3** in the context of a structural explanation and not as an attenuation metric itself.

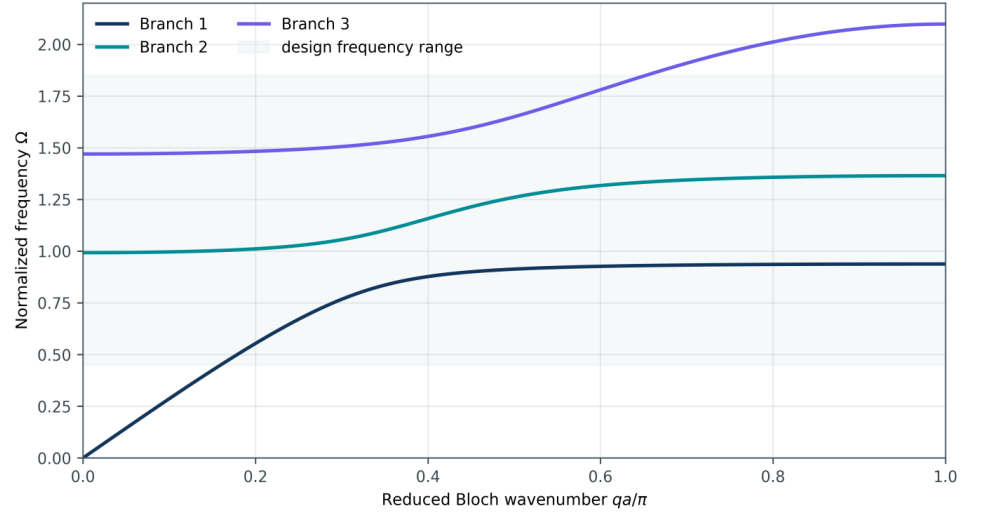


Figure 3. Lossless generalized-eigenvalue branches for the proposed cell. The shaded region indicates the frequency interval optimized using the dissipative complex-wavenumber model.

3.3. Attenuation envelopes under uncertain properties

Figure 4 compares the validation envelopes. A deeper valley occurs near $\Omega \approx 1.18$ together with a wider 5th–95th percentile spread in the first attenuation lobe. The Wasserstein design broadens the first lobe, raises the inter-lobe median, and maintains a stable lower envelope across a wider frequency range. Through Equation (48), probability transport emphasizes adverse combinations of high host stiffness, asymmetric detuning, and low damping that may not appear in the sparse training set.

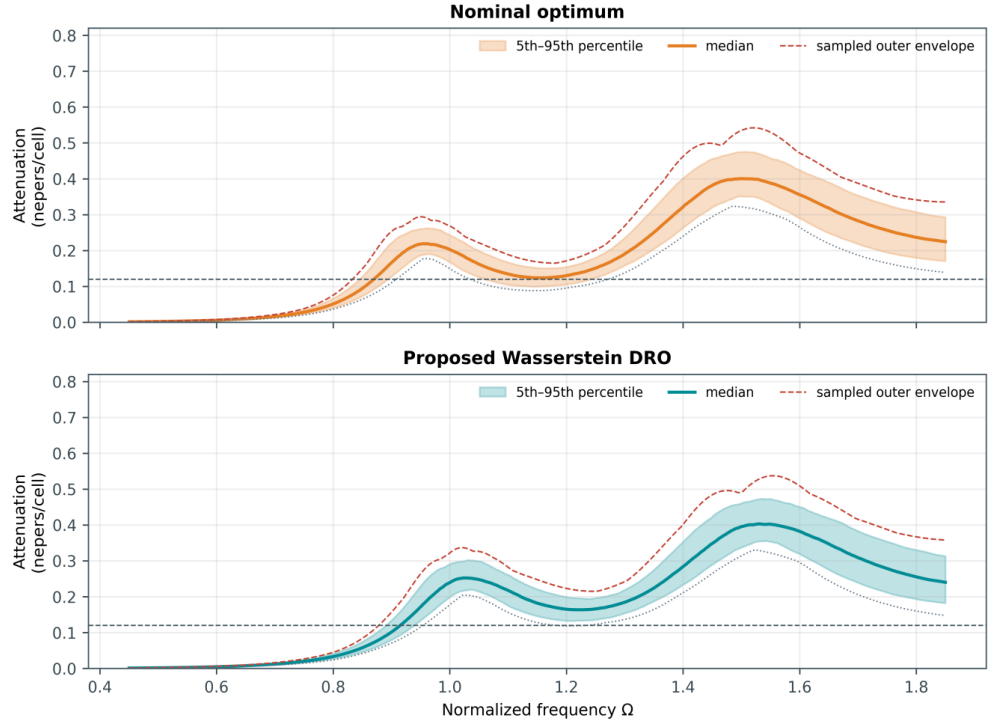


Figure 4. Independent 4,096-scenario attenuation envelopes.

Note: Solid lines are medians, colored bands are 5th–95th percentiles, dashed red lines are sampled maxima, dotted lines are sampled minima, and the horizontal dashed line is the 0.12-neper threshold.

Although nominal coverage is 70.06%, its fifth-percentile validation coverage is 59.23%. The proposed design trades 3.19 nominal percentage points for a 5.84-point fifth-percentile gain, strengthening the lower tail.

3.4. Independent 4,096-scenario stress test

Figure 5 and **Table 8** summarize the independent stress test. The baseline distributions are broad and weak; its mean loss is 0.86339, its 95th-percentile loss is 1.56195, and its worst sampled loss is 1.76785. The proposed design reduces those quantities to 0.41808, 0.43437, and 0.46532. The reductions are

$$100 \left(1 - \frac{0.41808}{0.86339} \right) = 51.58\% \quad (67)$$

$$100 \left(1 - \frac{0.43437}{1.56195} \right) = 72.19\% \quad (68)$$

$$100 \left(1 - \frac{0.46532}{1.76785} \right) = 73.68\% \quad (69)$$

Table 8. Independent 4,096-scenario validation metrics.

Design	Mean loss	95th-percentile loss	Worst loss	Mean A	5th-percentile A	Mean C (%)	5th-percentile C (%)	Worst $H_{\max}$
Equal-mass baseline	0.86339	1.56195	1.76785	0.39382	0.36440	41.25	35.93	11.476
Nominal optimum	0.42732	0.47144	0.49878	0.49590	0.46702	66.65	59.23	6.099
CVaR robust	0.41834	0.43380	0.46879	0.50970	0.48229	66.96	65.07	5.556
Proposed Wasserstein DRO	0.41808	0.43437	0.46532	0.51187	0.48283	66.75	65.07	5.706

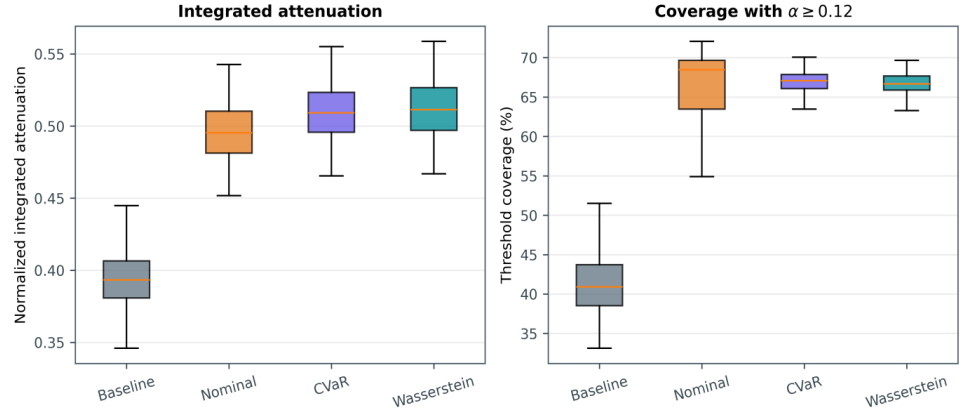


Figure 5. Box plots from 4,096 validation scenarios. Left: normalized integrated attenuation. Right: percentage of the target band with attenuation at least 0.12 nepers cell⁻¹.

Against the nominal optimum, Wasserstein DRO reduces mean loss by 2.16%, 95th-percentile loss by 7.86%, worst loss by 6.71%, and worst local amplification by 6.45%. It also raises mean integrated attenuation by 3.22%, fifth-percentile attenuation by 3.39%, and fifth-percentile coverage by 9.86% in relative terms. The CVaR and Wasserstein designs are closer: Wasserstein DRO has 0.74% lower worst loss and 0.43% higher mean integrated attenuation, whereas CVaR has 0.13% lower 95th-percentile loss and 2.71% lower worst amplification. These crossovers are physically plausible because CVaR concentrates on the empirical upper loss tail, whereas the Wasserstein objective prices nearby distribution shifts over the entire training support.

The fifth-percentile attenuation improvement over the baseline is

$$100 \left(\frac{0.48283}{0.36440} - 1 \right) = 32.50\% \quad (70)$$

and the worst amplification is reduced by

$$100 \left(1 - \frac{5.706}{11.476} \right) = 50.28\% \quad (71)$$

These results show that the proposed loss did not obtain robustness by allowing unbounded internal motion. All optimized designs remain below the limit in the independent sample, but the Wasserstein design gives a stronger worst-loss margin than the nominal solution.

3.5. Critical scenario and finite-chain engineering interpretation

The critical sampled scenario for the proposed design was validation index 509, with

$$\xi_{\text{crit}} = [0.98757 \quad -0.71445 \quad 0.94825 \quad -0.98133 \quad 0.01602]^T \quad (72)$$

Its physical transformations are

$$\frac{K}{K_0} = 1.07901, \quad \frac{k_1}{k_{10}} = 0.95713, \quad \frac{k_2}{k_{20}} = 1.05689, \quad \frac{c_{\beta i}}{c_{\beta i,0}} = 0.80373 \quad (73)$$

with a common order shift $\Delta\beta = 0.000801$. This combination raises the host scale, detunes the two resonators in opposite directions, and nearly attains the minimum damping multiplier. It is a band-separation and dissipation problem that can be mechanically interpreted as opposed to an arbitrary corner.

In this case, the complex Bloch attenuation is mapped to the 12-cell estimate in Equation (55) (**Figure 6**). The baseline insertion loss has narrow high peaks but falls below the 12.508-dB threshold over broad intervals between them. The optimized designs reduce the isolated peaks but maintain useful attenuation over a wider band. The Wasserstein response is particularly smooth along its high-frequency shoulder and remains comparable to the CVaR design.

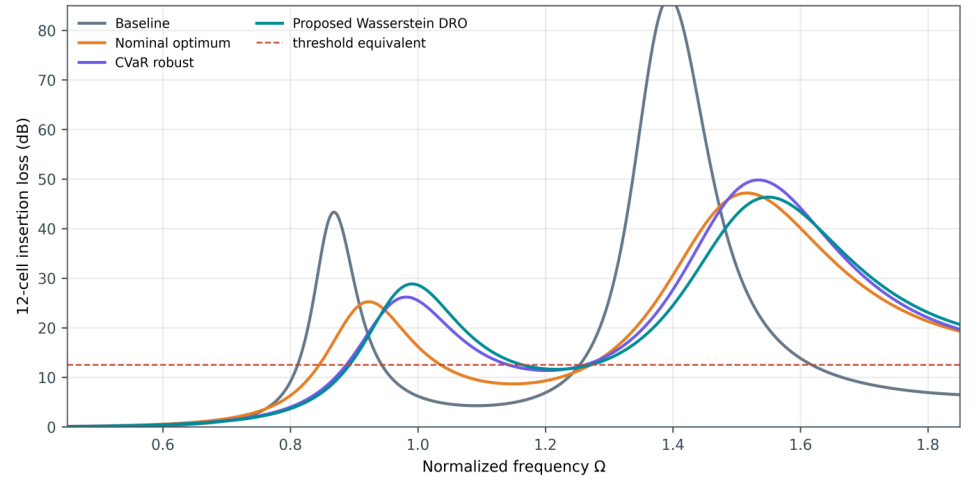


Figure 6. Twelve-cell attenuation-equivalent insertion loss in the critical proposed-design scenario. The dashed line is the 12.508-dB value corresponding to 0.12 nepers cell⁻¹.

The practical implication is that **Figure 6** represents attenuation-equivalent decay rather than exact chain transmission under arbitrary boundary conditions. For a 12-cell chain, each 0.10-neper increase in α changes the exponential amplitude factor by

$$\Delta IL_{12} = 8.686(12)(0.10) = 10.423 \text{ dB} \quad (74)$$

Thus, a modest change in the complex Bloch exponent can accumulate appreciably over a practical number of cells. An experimental realization should include termination impedances, interface losses, host damping, and force-transmission measurements.

3.6. Local sensitivity and robust performance frontier

Figure 7 shows that tuning ratios dominate local sensitivity: r_2 has the largest positive coefficient ($S = 0.1121$) and r_1 the largest negative ($S = -0.09946$). Increasing r_2 raises loss, while increasing r_1 lowers it locally. Loss-scale sensitivities (-0.03650 and -0.02613) favor added dissipation; fractional-order sensitivities are 0.03030 and 0.01206 , and mass allocation has $S = 0.01591$.

Tuning tolerances therefore merit tighter control than small-order errors. Because r_1 and r_2 interact through band overlap, these local signs may reverse after large changes and do not replace global reoptimization.

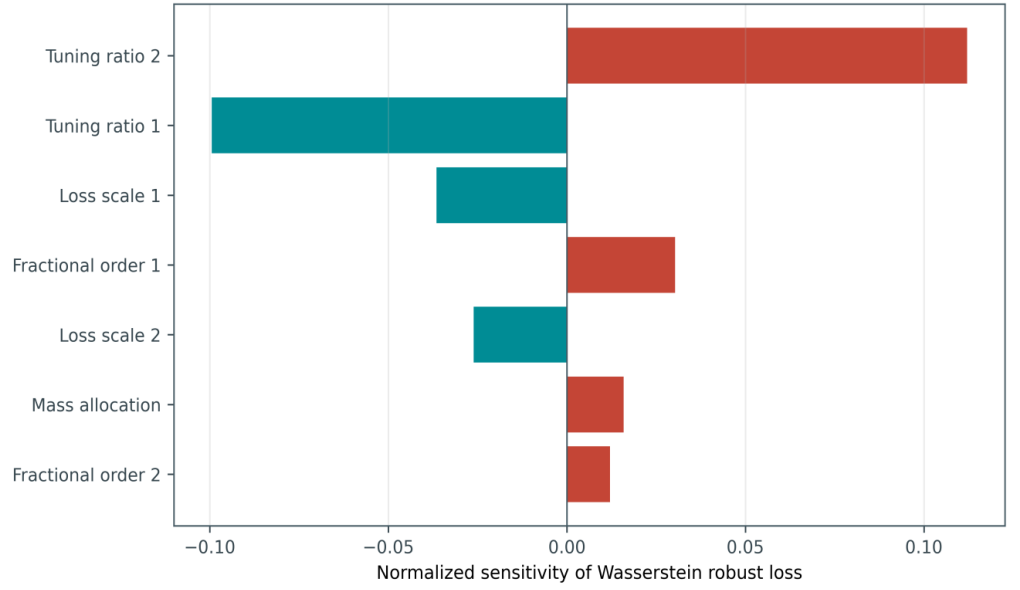


Figure 7. Centered normalized sensitivities of the proposed design. Positive bars indicate that a local parameter increase raises robust loss; negative bars indicate a local decrease in robust loss.

Figure 8a gives an active transport optimum at $\lambda^* = 0.06611$. Among 640 feasible local perturbations in **Figure 8b**, 24 are nondominated; the marked compromise attains 10th-percentile attenuation of 0.48649 and coverage of 65.145%.

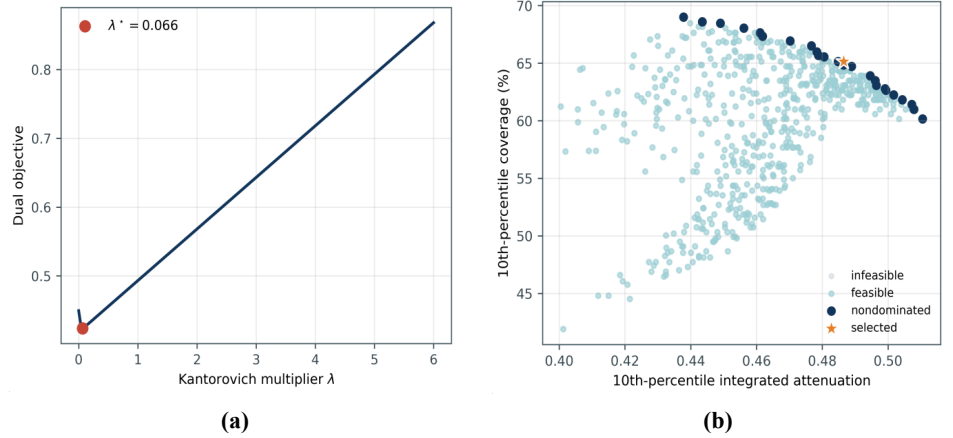


Figure 8. Robust-design diagnostics. **(a)** Kantorovich dual objective versus transport multiplier. **(b)** Feasible local designs and the nondominated frontier for 10th-percentile attenuation and threshold coverage; the star marks the selected compromise.

The frontier trades attenuation magnitude against coverage of the valley near $\Omega \approx 1.2$. With ambiguous laws, the chosen interior point preserves a balanced, constrained profile without maximizing one criterion.

4. Discussion

Fractional order actively shaped the attenuation band. In Equation (8), β_i controls the frequency exponent and the storage-to-loss phase; reducing it below unity redistributes dissipation over frequency. The optimized values $\beta_1 = 0.7431$ and $\beta_2 = 0.7292$ show that two similar non-integer orders produced a flatter attenuation

profile than the integer-order baseline. This agrees with recent fractional-resonator and homogenization studies [6,15–17], while extending them by coupling the constitutive orders directly to a Wasserstein-robust Bloch design. The improvement is therefore attributed to frequency-distributed dissipation rather than simply to increased damping magnitude.

Distributional ambiguity altered both mass allocation and tuning. The nominal design assigned 50.68% of the resonator mass to the lower branch at $r_1 = 0.8831$, whereas the Wasserstein design increased these values to 56.03% and 0.9521, respectively; the upper tuning ratio increased from 1.3867 to 1.4108. This coordinated shift strengthened the inter-lobe attenuation valley while sacrificing only a small amount of nominal coverage. The result is consistent with recent multi-resonant beam and shell studies [7,8,11,12,45–47], which show that broadband attenuation depends on controlled modal overlap. In the present model, that overlap is represented by the trajectory of the complex $M_{eff}(\omega)$ rather than by isolated negative-effective-mass poles alone [1, 9, 36]. The 3.19 percentage-point reduction in nominal coverage was offset by a 5.84-point increase in fifth-percentile coverage, indicating that the redistribution primarily improved lower-tail performance.

CVaR and Wasserstein DRO impose different risk geometries. Equation (52) averages the upper empirical loss tail, whereas the Wasserstein model reallocates probability within the bounded tolerance domain and charges a transport cost. Consequently, a nearby adverse state can influence the design even when it is not among the largest training losses. The positive optimal transport multiplier and the identified critical scenario confirm that the ambiguity constraint was active. This mechanism explains why the Wasserstein design achieved lower mean and worst loss than the nominal design, while remaining close to the CVaR solution in the 95th-percentile metrics.

Computational efficiency was enabled by analytical condensation: Equations (18) and (22) reduced each frequency evaluation to complex scalar operations. For more detailed geometries, the loss in Equation (48) could be supplied by a finite-element Bloch surrogate without changing the Wasserstein dual architecture. The one-dimensional cell therefore serves as a transparent benchmark for separating constitutive, resonance, and distributional effects. However, computational cost would then scale with the surrogate construction and the number of adversarial candidates.

Sampling bias was reduced by keeping the training, adversarial, and validation samples disjoint and by using a validation set 64 times larger than the training set. This separation prevents the reported lower-tail improvements from being evaluated on the same scenarios that shaped the design. The exact antiresonance roots verified the condensation, while common bounds, a common loss function, and reported algorithm settings made the four designs directly comparable and reproducible.

Limitations remain. The linear, one-dimensional, infinite-lattice model neglects host damping, nonlinear contact, geometric modes, interface compliance, and spatially correlated variability. The uncertainty bounds and Wasserstein radius were modeling choices, and the 256-candidate adversarial set approximated, rather than exactly solved, the inner supremum in Equation (50). Differential evolution also provides

no global optimality certificate. Consequently, specimen-level identification and transfer-function measurements are required before the reported attenuation gains can be interpreted as experimental performance.

Future work should identify k_i , $c_{\beta i}$ and β_i across temperature and manufacturing batches, calibrate the transport radius from repeated specimens, and compare measured complex wavenumbers with Equation (22). The common uncertainty shifts should then be replaced by correlated cell-level random fields, and finite-element Bloch surrogates should be used for geometrically detailed cells. Additive manufacturing provides a practical route to fabricate the proposed allocation [26], while the same ambiguity-aware formulation can be extended to multi-resonator barriers and metabarriers [28]. These steps would test whether the numerical lower-tail gains persist under boundary, interface, and cell-to-cell variability.

5. Conclusion

This study developed and independently validated a Wasserstein distributionally robust co-design framework for a dual-resonator elastic metamaterial with fractional Kelvin–Voigt attachments. The framework condensed Bloch dispersion into a complex effective-mass model and jointly optimized mass allocation, resonance tuning, damping scales, and fractional orders under five coupled property uncertainties. For the 1.80-kg host cell with an 18% resonator mass budget, the optimized design allocated 56.03% of resonator mass to the lower branch, selected resonance frequencies of 27.20 and 40.31 Hz, and reduced worst sampled loss by 73.68% relative to the equal-mass integer-order baseline.

Future work should first identify the fractional stiffness, damping coefficients, and orders from repeated physical specimens over the intended temperature range. It should then calibrate the Wasserstein radius from measured between-specimen variability and validate the predicted complex wavenumber and 12-cell insertion loss using controlled boundary impedances. Finally, the model should be extended to include host damping, interface compliance, nonlinear effects, and spatially correlated cell-to-cell variability before application to larger multi-resonator barriers.

These steps would establish whether the lower-tail gains observed in the 4,096-scenario numerical test translate to fabricated structures and would provide an evidence-based route from measured variability to broadband vibration-control design.

Author contributions:

Conceptualization, AV and YN; methodology, YN and RAH; software, YN and PW; validation, RAH, PW and SIM; formal analysis, YN and RAH; investigation, YN, PW and SIM; resources, AV and TMAF; data curation, YN and PW; writing—original draft preparation, YN; writing—review and editing, AV, YN, RAH, PW, SIM and TMAF; visualization, YN and PW; supervision, YN, AV and TMAF; project administration, AV and YN. All authors have read and agreed to the published version of the manuscript.

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