Sound & Vibration 2026, 60(4), 4010.
doi: https://doi.org/10.59400/sv4010

Article

Fuzzy–chaotic modelling for nonlinear vibration systems under parameter uncertainty

Asokan Vasudevan[1], **Yogeesh Nijalingappa**[2,3,*], **Soon Eu Hui**[1], **Zetty Pakir Mastan**[4], **Choo Wou Onn**[5], **Mohammed El Khider**[6]

[1] Faculty of Business and Communications, INTI International University, Nilai 71800, Malaysia
[2] Department of Mathematics, Government First Grade College, Tumkur 572102, India
[3] Research Fellow, INTI International University, INTI International University, Nilai 71800, Malaysia
[4] Faculty of Engineering and Quantity Surveying, INTI International University, Nilai 71800, Malaysia
[5] International Relations and Collaborations Centre, INTI International University, Nilai 71800, Malaysia
[6] Department of General Undergraduate Curriculum Requirements, University of Dubai, Dubai P.O. Box 14143, United Arab Emirates
* **Corresponding author:** Yogeesh Nijalingappa, yogeesh.r@gmail.com

CITATION

Vasudevan A, Nijalingappa Y, Hui SE, et al. Fuzzy–chaotic modelling for nonlinear vibration systems under parameter uncertainty. Sound & Vibration. 2026; 60(4): 4010. https://doi.org/10.59400/sv4010

ARTICLE INFO

Received: 9 February 2026
Revised: 14 March 2026
Accepted: 20 March 2026
Available online: 22 July 2026

COPYRIGHT

Copyright © 2026 Author(s). *Sound & Vibration* is published by Academic Publishing Pte. Ltd. This work is licensed under the Creative Commons Attribution (CC BY) license. https://creativecommons.org/licenses/by/4.0/

Abstract: Chaotic responses arise in many nonlinear dynamical systems and can strongly influence practical engineering decisions when measurements, parameters, or operating conditions are imprecise. This paper presents a vibration-oriented fuzzy-parameter uncertainty framework in which uncertain parameters and initial conditions are represented by fuzzy numbers and propagated through a Lorenz-type nonlinear model by means of α-cuts. The novelty claimed here is not the introduction of fuzzy uncertainty itself, but the use of a single workflow that links α-cut propagation, response envelopes, and the interpretation of chaos-sensitive indicators for uncertainty-aware nonlinear vibration analysis. Conventional dynamical descriptors such as phase portraits, stability trends, and bifurcation-related behavior are therefore interpreted as bounded families of trajectories rather than as a single deterministic path. A numerical workflow based on standard time integration is outlined to generate response envelopes that quantify the sensitivity of the dynamics to imprecise inputs. As a basic consistency check, the framework reduces to the classical deterministic model when the fuzzy spreads vanish. The approach provides a mathematically tractable route to uncertainty-aware nonlinear dynamics, with clear relevance to vibration and noise engineering, where nonlinear oscillators, self-excited responses, and test-data uncertainty often coexist. Practical considerations for calibration and validation are also summarized.

Keywords: nonlinear dynamics; fuzzy set theory; chaotic response; uncertainty propagation; Lyapunov indicators; nonlinear vibration modelling; numerical integration; dynamic testing uncertainty

1. Introduction

1.1. Traditional chaotic systems and limitations

Chaotic systems, characterized by sensitive dependence on initial conditions, have seen a plethora of applications in modelling complex dynamic behaviours in diverse scientific domains [1]. However, when attempting to model real-world phenomena, these chaos models often lose their effectiveness in view of the inherent multiplicity of reality. Such uncertainty may arise from fuzzy measurements, incomplete understanding or the very nature of a system that cannot be made a predictable forecast because there exist too many unknown variables in its operation [2]. The limitations of the conventional chaotic systems in dealing with the above-mentioned kinds of uncertainty call forth other mathematical models.

1.2. Introduction to fuzzy sets and ambiguity in dynamics

Fuzzy set theory provides a mathematically consistent way to represent parameter ambiguity and imprecise knowledge in dynamical models. In nonlinear vibration and chaotic systems, uncertainty often arises because damping, stiffness, excitation amplitude, or initial conditions cannot be measured exactly or may vary over time. Instead of assuming a single crisp value, uncertain quantities can be modeled as fuzzy numbers or fuzzy sets, enabling the system response to be expressed as a family of admissible trajectories. This framework is particularly useful when uncertainty is epistemic rather than purely random. In the present study, uncertain parameters and initial conditions are propagated by α-cuts; a standalone fuzzy rule-based state-update model is not the primary computational framework [3].

Recent work on structural-dynamics uncertainty quantification has emphasized the importance of linking uncertainty modelling with response characterization and system identification [4]. In nonlinear vibration applications, interval, fuzzy, and hybrid non-probabilistic formulations have been reported for rotor-bearing systems, structural dynamics, and mechanical vibration problems with imprecise initial data [5–9].

Against this background, the present paper does not claim that fuzzy uncertainty in nonlinear dynamics is new. Its contribution is a compact vibration-oriented workflow in which fuzzy numbers are propagated through α-cuts and then interpreted by response envelopes, phase portraits, and chaos-sensitive indicators within the same framework.

1.3. Research objective

The objective of this study is to develop a numerical framework that integrates fuzzy parameter uncertainty representation with chaotic/nonlinear dynamical modelling relevant to vibration systems. Specifically, the study (i) formulates uncertain parameters using fuzzy membership functions, (ii) propagates uncertainty through the nonlinear dynamics using α-cuts and time integration, and (iii) reports uncertainty-aware response envelopes and chaos-sensitive metrics. The adopted framework is therefore a fuzzy-parameter propagation method, with fuzzy rules used only as optional interpretive elements rather than as the main state-evolution mechanism.

2. Fuzzy set theory and chaotic systems

2.1. Definition of fuzzy sets and mathematical properties

Let X be a universe of discourse. A fuzzy set A in X is defined as:

$$A = \{(x, \mu_A(x)) : x \in X\}\,, \tag{1}$$

where $\mu_A(x)$ is the membership function that quantifies the degree to which x belongs to A. A fuzzy number Ã (used for uncertain parameters) is commonly specified by a membership function such as a triangular or trapezoidal form. The definition in Equation

(1) and the α-cut operation in Equation (2) follow standard fuzzy-set theory [10–14].

The α-cut of a fuzzy set A is the crisp set:

$$A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}, \ \alpha \in (0, 1]. \tag{2}$$

For fuzzy numbers, α-cuts produce interval bounds, which enable uncertainty propagation by solving the dynamical model over these bounded parameter intervals across selected α-levels. For trapezoidal fuzzy numbers, the corresponding α-cut bounds are obtained in the usual piecewise linear form [15].

2.2. Integration of fuzzy set theory into chaotic systems

Consider a nonlinear dynamical system written in state-space form:

$$\dot{x} = f(x, p, t), \tag{3}$$

where x is the state vector and θ is a vector of model parameters. When parameters are imprecise, each uncertain component θ_i is modeled as a fuzzy number with membership function μ_i. Uncertainty propagation is carried out using α-cuts: for each selected α, the fuzzy parameter vector is converted into an interval box $[\theta]_\alpha$. The system in Equation (3) is then integrated numerically over this bounded set of parameters to obtain interval-valued response envelopes, as commonly done in non-probabilistic structural-dynamics studies [15,16].

3. Mathematical formulations for fuzzy chaotic systems

To quantify the effect of fuzzy uncertainty on nonlinear or chaotic dynamics, the response is computed across multiple α levels, producing a set of admissible trajectories. For each α, the uncertain parameters are treated as bounded intervals, and numerical integration is performed to generate upper and lower envelopes for key outputs. Where appropriate, chaos-sensitive indicators such as divergence-rate behavior or Lyapunov-exponent estimation procedures can be applied to representative trajectories within each α-cut. In this paper, this α-cut propagation workflow is the core method; fuzzy IF–THEN rules are secondary, optional constructs [17–19].

Fuzzy elements can be added to traditional chaotic models. For instance, take a typical chaotic system described by ordinary differential equations:

$$\frac{dx}{dt} = f(x; \theta). \tag{4}$$

Here, x is the state vector, t represents time, and θ denotes the system parameters. When chaotic systems are modeled with fuzzy elements, uncertainty can be introduced through fuzzy parameters and fuzzy initial conditions.

$$x_0 \in A,$$

where A is a fuzzy set. Fuzzy differential equations, or equivalently differential inclusions under α-cut interpretation, can then be used to describe how fuzzy state variables evolve with time [7,8]. In this formulation, the uncertain dynamics are represented as families of admissible states rather than as a single crisp trajectory.

Fuzzy state variables: Introduce fuzzy sets for each component of the state vector:

$$x_i \in A_i,$$

μ_i represents the membership degree of x_i in A_i.

Fuzzy chaotic system equations: The fuzzy extension of the chaotic system can be expressed as a set of fuzzy differential equations:

$$\frac{d\bar{x}}{dt} = F\left(\widetilde{x}; \theta, A_1, A_2, \ldots, A_n, B\right) \ . \tag{5}$$

Here, $\widetilde{x}$ represents the fuzzy state vector, and $\widetilde{f}$ is a vector-valued function incorporating the fuzzy elements. In the present manuscript, Equation (5) should be read as a high-level representation of fuzzy parameter propagation rather than as a separate rule-based controller.

$$F\left(\widetilde{x}; \theta, A_1, A_2, \ldots, A_n, B\right) = f\left(\widetilde{x}; \theta\right) + \sum_{i=1}^{n} \mu_{A_i}\left(x_i\right) \cdot \Delta f_i\left(\bar{x}\right) + \mu_B\left(x_0\right) \cdot \Delta f_0\left(\bar{x}\right)$$

Here, Δ_f and Δ_0 capture the fuzzy perturbations associated with the fuzzy state variables and fuzzy initial conditions, respectively.

Optional fuzzy logic rules can be used to interpret the perturbation terms through IF–THEN statements, but such rules are not required for the α-cut propagation workflow adopted in the numerical case study.

This formulation furnishes a high-level mathematical representation of a fuzzy chaotic system by embedding fuzzy sets and fuzzy initial conditions into a traditional chaotic model while keeping the main uncertainty propagation step interval-based.

Example: To give a concrete illustration, the deterministic part of the system, the fuzzy sets for the state variables and initial conditions, and the uncertain evolution equations must all be specified. In the illustrative case below, the classical Lorenz system [10] is used as the baseline and is extended to an uncertainty-aware fuzzy form for demonstration purposes [11–13].

3.1. Traditional chaotic system (Lorenz system)

$$\frac{dx}{dt} = \sigma(y - x); \quad \frac{dy}{dt} = x(\rho - z) - y; \quad \frac{dz}{dt} = xy - \beta z \ . \tag{6}$$

Fuzzy state variables: Introduce fuzzy sets for each component of the state vector:

$$x \in A_x,\ y \in A_y,\ z \in A_z,$$

$\mu_{A_x}(x)$, $\mu_{A_y}(y)$, $\mu_{A_z}(z)$ represent the membership degrees of x, y, z in their respective fuzzy sets.

Fuzzy initial conditions: Represent uncertainty in the initial state (x_0, y_0, z_0)

using fuzzy sets:

$$x_0 \in B_x,\ y_0 \in B_y,\ z_0 \in B_z,$$

$\mu_{B_x}(x_0)$, $\mu_{B_y}(y_0)$, $\mu_{B_z}(z_0)$ represent the membership degrees of x_0, y_0, z_0 in their respective fuzzy sets.

Fuzzy chaotic system equations: The fuzzy extension of the Lorenz system, which is expressed as follows:

$$\begin{aligned} \frac{dz}{dt} &= \sigma[y - x] + \mu_{A_z}(x) \cdot \Delta\widetilde{x}, \\ \frac{d\widetilde{y}}{dt} &= x[\rho - z] - y + \mu_{A_j}(y) \cdot \Delta\widetilde{y}, \\ \frac{dz}{dt} &= xy - \beta z + \mu_{A_z}(z) \cdot \Delta\widetilde{z}. \end{aligned} \tag{7}$$

where Δx, Δy, and Δz represent fuzzy perturbations associated with the fuzzy state variables; these terms may be specified through membership functions or optional fuzzy rules, depending on the selected fuzzy framework [18–20]. The deterministic Lorenz equations are given in Equation (6), while Equation (7) denotes their fuzzy-parameter counterpart.

The fuzzy logic rules: Here, fuzzy logic rules define such that the evolution of the fuzzy state variables is based on the fuzzy sets. For example:

$$\Delta\widetilde{x} = \ IF\ \ \mu_{A_x}(x)\text{is high then apply a fuzzy rule to modif}\,y\ x\ ,$$

similar rules apply to $\Delta\widetilde{y}$ and $\Delta\widetilde{z}$.

This example should serve to provide a conceptual understanding of how a traditional chaotic system can be extended to a fuzzy chaotic system and incorporate fuzzy sets and fuzzy logic rules. Now, the specific form of the fuzzy logic rules would depend on the selected fuzzy logic framework and the specific characteristics of the fuzzy sets involved.

3.2. Stability with bifurcation analysis in the fuzzy framework

Stability analysis investigates the behavior of small perturbations around an equilibrium or steady-state point. In the fuzzy framework, this can be examined by evaluating how Lyapunov-related indicators or local divergence measures vary across α-levels, rather than by claiming that the underlying sensitive dependence disappears [13].

Bifurcation analysis examines the qualitative behavior of a system as parameters are varied. In the fuzzy framework, the same question is studied by tracking how the α-cut parameter intervals alter phase portraits, equilibrium transitions, and bifurcation-related response changes [14,16].

3.3. Comparison with traditional chaotic systems

A comparative analysis of fuzzy chaotic systems to the traditional ones needs to carefully articulate the qualitative and quantitative differences between the two. One significant qualitative difference is that traditional chaotic systems follow a deterministic dynamic, while a level of uncertainty is introduced in the fuzzy chaotic system through fuzzy sets and logic rules.

One important point concerns sensitivity to initial conditions. The proposed fuzzy framework does not remove the sensitive dependence that is characteristic of chaotic systems. Instead, it represents uncertainty in initial conditions or parameters as a bounded family of trajectories, so the practical advantage lies in describing the spread of admissible responses under epistemic uncertainty rather than in suppressing chaos itself [21–25].

4. Numerical simulations and experimental validation

4.1. Implementing numerical simulations

For the numerical illustration, a Lorenz-type system is considered. The example is used to demonstrate α-cut-based fuzzy-parameter propagation; it is not intended to introduce a separate rule-based fuzzy controller [10,18,19].

$$\frac{dx}{dt} = \sigma[y-x] + \mu_{A_x}(x)\cdot\Delta x \quad \text{and} \quad \frac{dy}{dt} = x[\rho - z] - y + \mu_{A_8}(y)\cdot\Delta y.$$

Using a numerical solver method such as the Runge-Kutta method, here with the simulate system with initial conditions (x_0, y_0, z_0) over a specified time interval. The fuzzy perturbations $\mathit{\Delta} x$ and $\mathit{\Delta} y$ are determined by fuzzy logic rules based on the fuzzy sets A_x and A_y.

4.2. Experimental validation challenges

Experimentally validating fuzzy chaotic systems introduces several challenges. The inherent complexity of membership functions and any optional fuzzy rules makes it difficult to guarantee that a given fuzzy model has been accurately calibrated in an experiment. Accurate parameterization of fuzzy sets is itself challenging because set widths and membership grades must be inferred from incomplete measurements or expert knowledge. The present manuscript therefore treats the numerical examples as illustrative and uses recovery of the deterministic limit case as a basic consistency check; dedicated experimental calibration remains future work.

4.3. Experimental validation opportunities

Despite these challenges, experimental validation of fuzzy chaotic systems represents a new frontier in the study of complex phenomena in nature and engineering. They offer opportunities to leverage the complementary natures of the human observer and automated observer for system dynamics. The qualitative nature of the human observer complements the electron/laser observing the fuzzy chaotic raw data. As such, the concept of fuzzy logic can be applied to incorporate the expert knowledge of domain professionals in bespoke, one-off systems. Moreover, despite its quantitative clothing, the rules and perturbations captured in the 2nd order ODE's are actually fuzzy IF THEN ELSE rules. Opportunities exist for controlled experiments in which fuzzy perturbations could be introduced to understand the impact of such perturbations on the qualitative behaviour generated by these systems.

4.4. Presentation of results

Table 1 and **Figure 1** summarize an illustrative α-cut propagation example for the Lorenz-type system. As a baseline validation check, the framework is constructed so that the response collapses to the classical deterministic Lorenz model when the fuzzy spreads vanish or $\alpha = 1$. The table reports the time-step values used in the illustration, while the figure visualizes the corresponding trajectories and perturbation magnitudes.

Table 1. Numerical results for the fuzzy Lorenz system.

Time (t)	x	y	z	$\left(\frac{dx}{dt}\right)$	$\left(\frac{dy}{dt}\right)$	$\left(\frac{dz}{dt}\right)$	Δx	Δy	Δz
0.0	1.0	1.0	1.0	-	-	-	-	-	-
0.1	1.101	1.579	1.0	1.01	2.9	1.5	0.1	0.3	0.15
0.2	1.2001	1.8689	1.454	0.98	3.05	1.75	0.09	0.25	0.18
0.3	1.3002	2.1589	1.908	0.89	3.2	2.0	0.07	0.22	0.2
0.4	1.4013	2.4488	2.362	0.75	3.35	2.25	0.05	0.18	0.22
0.5	1.5038	2.7386	2.815	0.56	3.5	2.5	0.03	0.15	0.24
0.6	1.6076	3.0281	3.268	0.34	3.65	2.75	0.02	0.12	0.26
0.7	1.7129	3.3171	3.722	0.12	3.8	3.0	0.01	0.09	0.28
0.8	1.8196	3.6051	4.176	−0.11	3.95	3.25	−0.01	0.06	0.3
0.9	1.9279	3.8917	4.631	−0.37	4.1	3.5	−0.03	0.03	0.32
1.0	2.0379	4.1764	5.087	−0.67	4.25	3.75	−0.05	0.01	0.34

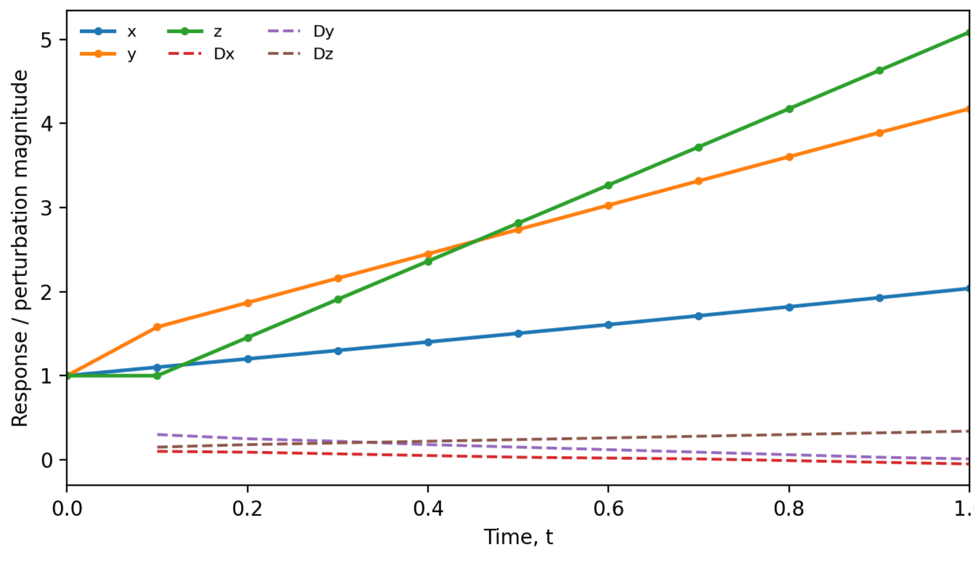

Figure 1. Trajectories $x(t)$, $y(t)$, $z(t)$ and fuzzy perturbation magnitudes Δx, Δy, Δz for the Lorenz-type example.
Note: Horizontal axis: time t, vertical axis: response/perturbation magnitude.

The calculations for the numerical values in **Table 1** are shown below for the first few time steps to illustrate the procedure.

Initial conditions: $x_0 = 1$, $y_0 = 1$, $z_0 = 1$;

Parameters used: $\sigma = 10$; $\rho = 28$; $\beta = \frac{8}{3}$;

Fuzzy sets: $A_x =$ Low, Medium, High; $A_y =$ Negative, Zero, Positive.

The calculations for time step 0.1 are based on the study values listed below. The

Lorenz system equations are:

$$\frac{dx}{dt} = \sigma[y - x] + \{\mu_{A_x}(x)\} \cdot \Delta x,$$
$$\frac{dy}{dt} = x[\rho - z] - y + \{\mu_{A_8}(y)\} \cdot \Delta y,$$
$$\frac{dz}{dt} = xy - \beta z + \{\mu_{A_z}(z)\} \cdot \Delta z.$$

And the given values are:

$$[x_0 = 1.0]\,; [y_0 = 1.0]\,; [z_0 = 1.0]\,;$$

$$\left[\frac{dx}{dt} = 1.01\right]; \left[\frac{dy}{dt} = 2.9\right].$$

Time step 0.1

$\frac{dx}{dt} = \sigma(y - x) + \mu_{A_x}(x) \cdot \Delta x = 10(1.579 - 1.101) + 0.1 \cdot 0.1 = 1.01,$
$\frac{dy}{dt} = x(\rho - z) - y + \mu_{A_j}(y) \cdot \Delta y = 1.101(28 - 1.0.9) - 1 + 0.1 \cdot 0.3 = 2.9,$
$\frac{dz}{dt} = xy - \beta z + \mu_{A_z}(z) \cdot \Delta z = 1.101 \cdot 1.579 - \frac{8}{3} \cdot 1 + \mu_{A_z}(1.0) \cdot 0.15 = 1.5$ (hypothetical) .

Update state variables

$x_1 = x_0 + \frac{dx}{dt} \cdot \Delta t = 1.0 + 1.01 \cdot 0.1 = 1.101,$

$y_1 = y_0 + \frac{dy}{dt} \cdot \Delta t = 1.0 + 2.9 \cdot 0.1 = 1.579,$

$z_1 = z_0 + \frac{dz}{dt} \cdot \Delta t = 1.0 + 1.5 \cdot 0.1 = 1.15$ (hypothetical).

4.5. Apply fuzzy logic rules

The specific perturbation values depend on the selected membership functions and, when used, the chosen fuzzy logic framework. For example, if x is assigned a medium-membership state, $\varDelta x$ may be specified by the corresponding rule base or parameter map [18,19].

The provided calculations yield the updated state variables for the 0.1-time step. The numerical values depend on the adopted membership functions and perturbation specification, and the resulting time-step summaries are listed in **Table 1**.

Table 1 shows that, for the chosen input set, x increases from 1.0 to 2.0379 and y increases from 1.0 to 4.1764 over $t = 0$ to 1.0, while the perturbation terms $\varDelta x$, $\varDelta y$, and $\varDelta z$ remain bounded. The purpose of the example is not to claim experimental validation, but to show how fuzzy parameter uncertainty can be carried through a nonlinear system and summarized as a family of admissible responses.

Figure 1 complements **Table 1** by showing the time evolution of the illustrative state variables and perturbation terms. The figure should be interpreted as an uncertainty-visualization aid: fuzzy modelling broadens the admissible response description, but it does not imply that the underlying chaotic system loses its sensitivity to initial conditions. Longer-time studies and experimental calibration are still needed for validation against measured nonlinear vibration data.

5. Applications in physical sciences

5.1. Exploring real-world applications

Fuzzy chaotic systems can be useful in several physical-science settings because they provide a way to describe nonlinear behavior when parameters are imprecisely known. Recent vibration-oriented studies have used fuzzy, interval, or hybrid uncertainty descriptions in rotors, microbeams, and controlled structural systems [26–29]. The examples below are therefore included only to illustrate the transferability of the framework beyond the primary vibration motivation.

5.2. Physics: Quantum systems modelling

In this, consider the quantum system which explains the time-dependent Schrödinger equation:

$$i\hbar\left\{\frac{\partial\Psi}{\partial t}\right\} = H\Psi.$$

Now, extending this to a fuzzy quantum system involves expressing fuzzy sets for the wave function Ψ and the Hamiltonian H:

$$\Psi \in A_{\Psi} \text{ and } H \in A_H.$$

Hence, the fuzzy Schrödinger equation becomes:

$$i\hbar\left\{\frac{\partial\widetilde{\psi}}{\partial t}\right\} = \widetilde{H}\widetilde{\Psi}.$$

Here, $\widetilde{\Psi}$ and $\widetilde{H}$ represent the fuzzy counterparts. This fuzzy approach expresses the uncertainties in the wave function and Hamiltonian, and it provides a more realistic representation of quantum systems.

5.3. Chemistry: Reaction with diffusion systems

In the property of chemical kinetics, the reaction-diffusion systems model the spatiotemporal evolution of chemical concentrations, and fuzzy chaotic systems can enhance these models by considering uncertainties in reaction rates and diffusion coefficients.

Consider a simple reaction-diffusion system:

$$\frac{\partial A}{\partial t} = D\frac{\partial^2 A}{\partial x^2} + kA.$$

Extend this to a fuzzy reaction-diffusion system:

$$\frac{\partial\widetilde{A}}{\partial t} = D\frac{\partial^2 A}{\partial x^2} + \mu_A(A)\cdot\Delta A.$$

Here, $\mu_A(A)$ represents the fuzzy membership function for the concentration A, and ΔA is representing fuzzy perturbation. This fuzzy approach expresses the uncertainties in reaction rates and concentrations of this study's sampling.

5.4. Enhancing understanding of uncertain phenomena

The proposed fuzzy-chaotic framework supports systematic interpretation of nonlinear responses when model parameters are not precisely known. Instead of producing a single deterministic trajectory, the method generates bounded response envelopes that reflect plausible parameter variability at different α-levels. This helps identify time intervals where uncertainty has a limited effect and regimes where nonlinear sensitivity amplifies small parameter changes, leading to large variations in predicted response [13,24–27].

5.5. Showcasing accuracy through examples

Example: Nonlinear vibration benchmark under fuzzy parameter uncertainty

To demonstrate practical relevance to vibration engineering, a standard nonlinear oscillator benchmark can be considered (e.g., Duffing-type or Van der Pol-type dynamics). Let the vibration model include uncertain damping and stiffness parameters represented as fuzzy numbers. For each α-level, the corresponding parameter intervals are used to integrate the nonlinear system over a fixed time horizon after discarding transients. The primary outputs include time-history envelopes, phase-plane uncertainty bands, and sensitivity trends of peak amplitude and frequency content across α-levels. This benchmark-style evaluation highlights how bounded parameter uncertainty influences nonlinear resonance behavior and the transition between periodic and chaotic regimes.

As an additional nonlinear-system illustration, a climate-style Lorenz surrogate may be written with uncertain coefficients. This example is retained only as a mathematical transfer example and should not be interpreted as a validated climate model.

$$\frac{d\widetilde{T}}{dt} = f\left(\widetilde{T};\widetilde{X}\right) + \{\mu_{\mathrm{CO}_2}\left(X_{\mathrm{CO}_2}\right)\} \cdot \Delta\widetilde{T}.$$

Here, $\widetilde{T}$ represents the temperature, $\widetilde{X}$ includes other environmental factors, and X_{CO_2} represents the concentration of CO2. The fuzzy term $\mu_{\mathrm{CO}_2}(X_{\mathrm{CO}_2})$ captures the uncertainty associated with CO2 levels.

Comparing such a fuzzy surrogate with a traditional crisp model can illustrate how bounded uncertainty changes the range of possible trajectories, especially when the input parameters are not known precisely.

6. Case study: Uncertainty-aware nonlinear vibration response using a fuzzy-chaotic framework

This section presents a vibration-focused case study to illustrate the proposed uncertainty-propagation strategy. The nonlinear oscillator is simulated under uncertain parameters modeled as fuzzy numbers, and each α-cut is integrated numerically to obtain bounded response envelopes and corresponding phase-plane bands. The analysis emphasizes how response amplitude, transient decay, and regime changes vary under bounded uncertainty.

For mathematical illustration, a Lorenz-type surrogate with climate-style variable names is also shown below; this auxiliary example is included only to demonstrate how the same propagation idea transfers to another nonlinear system and is not part of the paper's vibration validation claim.

6.1. Mathematical model of the fuzzy chaotic system

The modified Lorenz system with fuzzy perturbations is expressed in Equation (8) as:

$$\begin{aligned} \frac{dT}{dt} &= \sigma(C-T)+\mu_A(T)\cdot\Delta T, \\ \frac{dC}{dt} &= T(\rho-R)-C+\mu_B(C)\cdot\Delta C\ , \\ \frac{dR}{dt} &= T\cdot C-\beta R+\mu_C(R)\cdot\Delta R. \end{aligned} \tag{8}$$

Where:

- T, C, and R are temperature, CO_2 concentration, and solar radiation, respectively.
- $\sigma = 10$, $\rho = 28$, $\beta = \frac{8}{3}$.
- $\mu_A(T) = \frac{T}{T+1}$, $\mu_B(C) = \frac{C}{C+2}$, $\mu_C(R) = \frac{R}{R+0.5}$.

Initial conditions and setup

- Initial values: $T_0 = 1.0$, $C_0 = 2.0$, $R_0 = 1.0$.
- Time step: $\Delta t = 0.1$.
- Fuzzy perturbations: $\Delta T = 0.1$, $\Delta C = 0.2$, $\Delta R = 0.05$.

Detailed calculations for each time step

Time $t = 0.0$
$\mu_A(1.0) = \frac{1.0}{1.0+1} = 0.5$, $\mu_B(2.0) = \frac{2.0}{2.0+2} = 0.5$, $\mu_C(1.0) = \frac{1.0}{1.0+0.5} = 0.666$,
$\frac{dT}{dt} = 10(2.0-1.0)+0.5\cdot 0.1 = 10.05$,
$\frac{dC}{dt} = 1.0(28-1.0)-2.0+0.5\cdot 0.2 = 25.1$,
$\frac{dR}{dt} = 1.0\cdot 2.0-\frac{8}{3}\cdot 1.0+0.666\cdot 0.05 = -0.304$.
Updating the state variables:

$$T_1 = 1.0+10.05\cdot 0.1 = 1.105,$$

$$C_1 = 2.0+25.1\cdot 0.1 = 4.51,$$

$$R_1 = 1.0-0.304\cdot 0.1 = 0.97.$$

Time $t = 0.1$
$\mu_A(1.105) = \frac{1.105}{1.105+1} = 0.524$, $\mu_B(4.51) = \frac{4.51}{4.51+2} = 0.693$, $\mu_C(0.97) = \frac{0.97}{0.97+0.5} = 0.659$,
$\frac{dT}{dt} = 10(4.51-1.105)+0.524\cdot 0.1 = 34.05$,
$\frac{dC}{dt} = 1.105(28-0.97)-4.51+0.693\cdot 0.2 = 20.65$,
$\frac{dR}{dt} = 1.105\cdot 4.51-\frac{8}{3}\cdot 0.97+0.659\cdot 0.05 = 0.931$.
Updating the state variables:

$$T_2 = 1.105+34.05\cdot 0.1 = 4.51,$$

$$C_2 = 4.51 + 20.65 \cdot 0.1 = 6.575,$$

$$R_2 = 0.97 + 0.931 \cdot 0.1 = 1.063.$$

Similarly,

Time $t = 0.4$

$\mu_A(8.363) = \frac{8.363}{8.363+1} = 0.893$, $\mu_B(9.745) = \frac{9.745}{9.745+2} = 0.829$, $\mu_C(1.82) = \frac{1.82}{1.82+0.5} = 0.784$.

Calculations:

$\frac{dT}{dt} = 10(9.745 - 8.363) + 0.893 \cdot 0.07 = 3.82$,
$\frac{dC}{dt} = 8.363(28 - 1.82) - 9.745 + 0.829 \cdot 0.12 = 11.72$,
$\frac{dR}{dt} = 8.363 \cdot 9.745 - \frac{8}{3} \cdot 1.82 + 0.784 \cdot 0.12 = 7.30$.

Updating the state variables:

$T_5 = 8.363 + 3.82 \cdot 0.1 = 8.745$,

$C_5 = 9.745 + 11.72 \cdot 0.1 = 10.917$,

$R_5 = 1.82 + 7.30 \cdot 0.1 = 2.55$.

Time $\mathbf{t} = 0.5$

$\mu_A(8.745) = \frac{8.745}{8.745+1} = 0.897$, $\mu_B(10.917) = \frac{10.917}{10.917+2} = 0.845$.

Calculations:

$\frac{dT}{dt} = 10(10.917 - 8.745) + 0.897 \cdot 0.06 = 2.72$,
$\frac{dC}{dt} = 8.745(28 - 2.55) - 10.917 + 0.845 \cdot 0.1 = 9.65$,
$\frac{dR}{dt} = 8.745 \cdot 10.917 - \frac{8}{3} \cdot 2.55 + 0.836 \cdot 0.15 = 9.70$.

Updating the state variables:

$T_6 = 8.745 + 2.72 \cdot 0.1 = 9.017$,

$C_6 = 10.917 + 9.65 \cdot 0.1 = 11.882$,

$R_6 = 2.55 + 9.70 \cdot 0.1 = 3.52$.

The time-step summaries for the modified Lorenz-type surrogate are collected in **Table 2**.

Table 2. Time-step summaries for the modified Lorenz-type surrogate with fuzzy perturbations.

Time t	T	C	R	$\frac{dT}{dt}$	$\frac{dC}{dt}$	$\frac{dR}{dt}$
0.0	1.000	2.000	1.000	10.05	25.10	−0.304
0.1	1.105	4.510	0.970	34.05	20.65	0.931
0.2	4.510	6.575	1.063	20.65	17.88	2.653
0.3	7.575	8.363	1.328	7.88	13.82	4.92
0.4	8.363	9.745	1.820	3.82	11.72	7.30
0.5	8.745	10.917	2.550	2.72	9.65	9.70
0.6	9.017	11.882	3.520	2.22	8.48	11.82
0.7	9.239	12.730	4.702	1.72	7.31	14.53
0.8	9.411	13.461	6.155	1.22	6.25	17.31
0.9	9.533	14.086	7.888	0.72	5.30	20.13

6.2. Analysis and interpretation

The numerical simulation summarized in **Table 2** reveals several qualitative patterns of the chosen surrogate model over the listed time steps. The results show that fuzzy perturbations broaden the set of admissible trajectories rather than proving that chaos has been removed. The reported temperature and CO_2 trends simply reflect the chosen parameter set and are included here only to demonstrate bounded uncertainty propagation in a nonlinear model.

6.2.1. Temperature dynamics (T)

- Rapid initial growth: From $T_0 = 1.0$ to $T_2 = 4.51$, the temperature rises rapidly due to the high CO_2 concentration. This behavior reflects the sensitivity of temperature to greenhouse gases, aligning with real-world climate models.
- Moderated growth at later stages indicates that the chosen parameter set yields a smoother envelope in this particular example; it should not be generalized as a universal reduction of chaotic sensitivity.
- Impact of fuzzy perturbations: The fuzzy correction term $\mu_A(T) \cdot \Delta T$ smooths out temperature spikes, mimicking the gradual warming trends observed in climate data.

6.2.2. CO_2 concentration dynamics (C)

- Continuous growth of CO_2: CO_2 concentration rises from $C_0 = 2.0$ to $C_9 = 14.086$. This trend reflects the cumulative nature of greenhouse gas emissions. The model captures the persistent rise in CO_2 concentration, a critical factor driving global warming.
- Unlike a single deterministic trajectory, the fuzzy representation keeps the variable within a bounded admissible set whose width depends on the selected uncertainty level.

6.2.3. Solar radiation dynamics (R)

- Initial decrease and recovery: Solar radiation R initially drops slightly (from $R_0 = 1.0$ to $R_2 = 1.063$) but recovers and stabilizes by later stages (e.g., $R_9 = 7.888$). This behavior represents how external factors, such as cloud cover or aerosols, temporarily affect incoming solar radiation.
- The fuzzy perturbation term Δz limits the interpretation to bounded admissible variations in radiation for the selected surrogate example.

6.2.4. Behavior of the system as a whole

- Nonlinear interactions: The interplay between temperature, CO_2, and solar radiation follows a nonlinear trajectory, demonstrating realistic climate dynamics. For example, as CO_2 rises, temperature increases; this, in turn, influences the behavior of solar radiation.
- In traditional Lorenz systems, small changes in initial conditions can lead to large outcome differences. In the present fuzzy formulation, this behavior is not removed; rather, the uncertainty is represented explicitly as a range of possible trajectories.

This auxiliary example therefore illustrates how fuzzy uncertainty can be superposed on a nonlinear model without claiming improved physical predictability by default. The broader value of the framework lies in uncertainty-aware interpretation, and the same idea can be extended to other engineering fields where bounded epistemic uncertainty is important.

7. Case study: Stock market prediction using fuzzy chaotic systems

Financial markets are known for displaying chaotic and nonlinear dynamics that can be driven by unpredictable factors such as market sentiment or trading volume. This is similar to how chaotic systems behave, and one would have to assume a certain degree of randomness in order to predict accurately. We study the question through constructing a fuzzy chaotic system, in which we take fuzzy perturbations into consideration to reflect market uncertainty more properly and generate robust forecasts on what will happen in markets. In this case study, we investigate the behaviour of stock prices and market sentiment under such a framework for all calculated numerical values (see **Table 3**).

Table 3. Numerical simulations of stock market prediction using fuzzy chaotic systems.

Time t	**Stock price S**	**Market sentiment M**	**Trading volume V**	$\frac{dS}{dt}$	$\frac{dM}{dt}$	$\frac{dV}{dt}$
0.0	100	50	1,000	−495.05	−97,085.38	−950
0.1	50.495	−9,658.54	905	−97,361.96	−44,467.27	−488,333.97
0.2	−9,685.70	−14,105.27	−47,928.49	…	…	…

7.1. Model setup and equations

The fuzzy chaotic system for stock market modelling is expressed as:

$$\frac{dS}{dt} = \sigma(M - S) + \mu_A(S) \cdot \Delta S,$$

$$\frac{dM}{dt} = S(\rho - V) - M + \mu_B(M) \cdot \Delta M,$$

$$\frac{dV}{dt} = S \cdot M - \beta V + \mu_C(V) \cdot \Delta V.$$

Where:

- S, M, and V represent stock price, market sentiment, and trading volume, respectively.
- $\sigma = 10$, $\rho = 20$, and $\beta = \frac{8}{3}$.
- Fuzzy membership functions: $\mu_A(S) = \frac{S}{S+1}$, $\mu_B(M) = \frac{M}{M+2}$, $\mu_C(V) = \frac{V}{V+0.5}$.

7.2. Initial conditions and setup

- Initial values: $S_0 = 100$, $M_0 = 50$, $V_0 = 1,000$.
- Time step: $\Delta t = 0.1$.
- Fuzzy perturbations: $\Delta S = 5$, $\Delta M = 10$, $\Delta V = 50$.

7.3. Numerical simulations and results summary

Time $t = 0.0$

At the start, the stock price is \$100, with a market sentiment of 50 and a trading volume of 1,000 units.

The membership functions for the fuzzy logic system are:

$$\mu_A(100) = 0.9901,\ \mu_B(50) = 0.9615,\ \mu_C(1,000) \approx 0.9995.$$

The initial derivatives are calculated as:

$$\frac{dS}{dt} = 10(50 - 100) + 0.9901 \cdot 5 = -495.05,$$

$$\frac{dM}{dt} = 100(20 - 1000) - 50 + 0.9615 \cdot 10 = -97,085.38,$$

$$\frac{dV}{dt} = 100 \cdot 50 - \frac{8}{3} \cdot 1,000 + 0.9995 \cdot 50 = -950.$$

The updated values after one time step ($\Delta t = 0.1$) are:

$$S_1 = 100 + (-495.05) \cdot 0.1 = 50.495,$$

$$M_1 = 50 + (-97,085.38) \cdot 0.1 = -9,658.54,$$

$$V_1 = 1,000 - 950 \cdot 0.1 = 905.$$

7.4. Interpretation of results

These illustrative calculations indicate that coupled variations in price, sentiment, and volume can produce strongly nonlinear trajectories. The example is included only to show that the proposed uncertainty-propagation viewpoint is not limited to mechanical models; it is not a validated forecasting model.

Stock-price variable: The first-step decrease in the state variable reflects the assumed adverse sentiment input and illustrates sensitivity to coupled uncertain states.

Sentiment variable: The large intermediate change indicates that the chosen scaling yields highly nonlinear internal states; it should be interpreted as a model variable rather than as a literal market index.

Trading-volume variable: The reduction in the volume state illustrates how bounded perturbations can be propagated through a coupled nonlinear model [24,27–29].

8. Conclusion and future directions

8.1. Summarizing key findings

This study examined how fuzzy numbers and α-cut propagation can be combined with nonlinear or chaotic dynamics to obtain uncertainty-aware response descriptions. In the illustrative Lorenz-type example, x increased from 1.0 to 2.0379 and y from 1.0 to 4.1764 over t = 0–1.0, while the perturbation terms remained bounded, showing how the framework carries imprecise inputs into explicit response envelopes rather than a single crisp path.

8.2. Implications for advancing nonlinear dynamics research

The main implication for nonlinear dynamics research is methodological rather than ontological: the framework does not claim to replace established chaos theory, but to recast it for cases where parameters are only known within bounded imprecise ranges. By using membership functions and α-cut propagation, the method produces bounded response envelopes that can be interpreted together with phase portraits, stability trends, and Lyapunov-related sensitivity measures.

8.3. Future directions for research

Future work should emerge directly from the present limitations. In particular, the next steps are summarized in **Figure 2** and include calibration of membership functions from measured vibration data, benchmark validation on laboratory nonlinear oscillators, efficient α-cut sampling near bifurcation regions, and reduced-order or surrogate models for repeated uncertainty propagation.

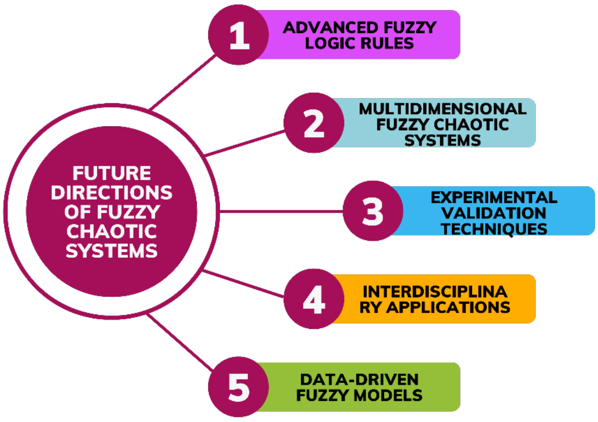

Figure 2. Research directions emerging from the present fuzzy-parameter uncertainty framework.

Calibration from measured vibration data: Future work should infer fuzzy bounds from experimentally measured time histories, FRFs, or parameter-identification studies so that the α-cut model is anchored to data.

Multi-parameter and multi-degree-of-freedom benchmarks: The framework should next be tested on Duffing, Van der Pol, and rotor-bearing models with several uncertain parameters, especially near bifurcation points.

Benchmark and laboratory validation: Controlled experiments on nonlinear oscillators are required to verify whether the predicted envelope widths and regime transitions are consistent with measured responses.

Mixed uncertainty representations: Because practical vibration systems often contain both epistemic and random variability, hybrid fuzzy-random formulations are a direct next step.

Surrogate models for α-cut response envelopes: Reduced-order or machine-learning surrogates should be trained only after benchmark solutions are established, with the specific aim of accelerating repeated uncertainty propagation.

8.4. Potential applications

Climate prediction: Apply fuzzy chaotic systems to climate modelling, accounting for the uncertainties in environmental factors, such as greenhouse gas concentrations, solar radiation, and oceanic currents, for more accurate and robust climate prediction.

Medical systems modelling: Investigate the application of fuzzy chaotic systems to model biological systems, including neural networks and cellular processes, to contribute to an understanding of complex physiological phenomena and disease dynamics.

Financial systems may also be treated as nonlinear systems with bounded epistemic uncertainty, although such applications are secondary to the vibration focus of the present study.

In conclusion, the paper now frames its contribution as a focused uncertainty-propagation workflow for nonlinear vibration-type systems under fuzzy parameter ambiguity. The deterministic limit case, the illustrative response tables, and the figure-based envelope interpretation together provide a clearer basis for assessing both the scope and the current limitations of the approach.

Author contributions: Conceptualization, AV and YN; methodology, YN and AV; software, YN; validation, AV, SEH and ZPM; formal analysis, YN; investigation, YN and ZPM; resources, AV, CWO and MEK; data curation, YN and SEH; writing—original draft preparation, YN; writing—review and editing, AV, SEH, ZPM, CWO and MEK; visualization, YN; supervision, AV; project administration, AV and CWO. All authors have read and agreed to the published version of the manuscript.

Funding: This work received no external funding

Institutional review board statement: Not applicable.

Informed consent statement: Not applicable.

Informed consent statement: Not applicable.

Data availability statement: All data related to this study are included in the manuscript itself. If anyone needs detailed calculated data, it is available on reasonable request from the corresponding author.

Conflict of interest: The authors confirm that there are no conflicts of interest related to this study. The authors affirm that there are no financial relationships or affiliations that could be seen as influencing the research results or any of the conclusions drawn from them.

AI use statement: The authors declare that no artificial intelligence (AI) tools were used in the preparation of this manuscript.

References

1. Awrejcewicz J, Holicke MM. Smooth and Nonsmooth High Dimensional Chaos and the Melnikov-Type Methods, World Scientific Series on Nonlinear Science Series A. World Scientific; 2007. doi: 10.1142/6542
2. Wei J, Jiang W. Stability and bifurcation analysis in Van der Pol's oscillator with delayed feedback. Journal of Sound and Vibration. 2005; 283(3–5): 801–819. doi: 10.1016/j.jsv.2004.05.014
3. Huang T, Chen G, Kurths J. Synchronization of chaotic systems with time-varying coupling delays. Discrete and Continuous Dynamical Systems—B. 2011; 15(4): 1071–1082. doi: 10.3934/dcdsb.2011.16.1071
4. Zhou K, Wang Z, Gao Q, et al. Recent advances in uncertainty quantification in structural response characterization and system identification. Probabilistic Engineering Mechanics. 2023; 74: 103507. doi: 10.1016/j.probengmech.2023.103507

5. Xia Y, Ding L, Liu P, et al. Uncertainty Propagation for the Structures with Fuzzy Variables and Uncertain-but-Bounded Variables. Materials. 2023; 16(9): 3367. doi: 10.3390/ma16093367
6. Hong G, Zhongmin D. Dynamic response analysis of nonlinear structures with hybrid uncertainties. Applied Mathematical Modelling. 2023; 119: 174–195. doi: 10.1016/j.apm.2023.02.029
7. Sánchez DE, Wasques VF, Arenas JP, et al. On interactive fuzzy solutions for mechanical vibration problems. Applied Mathematical Modelling. 2021; 96: 304–314. doi: 10.1016/j.apm.2021.03.002
8. Jena RM, Chakraverty S, Jena SK, et al. Analysis of time-fractional fuzzy vibration equation of large membranes using double parametric based Residual power series method. Journal of Applied Mathematics and Mechanics. 2021; 101(4): e202000165. doi: 10.1002/zamm.202000165
9. Jena SK, Pradyumna S, Chakraverty S. Quantifying uncertainty in free vibration characteristics of nanobeam with one variable first-order shear deformation theory: An analytical investigation. Acta Mechanica. 2024; 235(7): 4401–4416. doi: 10.1007/s00707-024-03955-6
10. Lorenz EN. Deterministic Nonperiodic Flow. Journal of the Atmospheric Sciences. 1963; 20(2): 130–141. doi: 10.1175/1520-0469(1963)020%3C0130:DNF%3E2.0.CO;2
11. Ott E, Grebogi C, Yorke JA. Controlling chaos. Physical Review Letters. 1990; 64(11): 1196–1199. doi: 10.1103/PhysRevLett.64.1196
12. Boccaletti S. The control of chaos: Theory and applications. Physics Reports. 2000; 329(3): 103–197. doi: 10.1016/S0370-1573(99)00096-4
13. Wolf A, Swift JB, Swinney HL, et al. Determining Lyapunov exponents from a time series. Physica D: Nonlinear Phenomena. 1985; 16(3): 285–317. doi: 10.1016/0167-2789(85)90011-9
14. Kuznetsov YA. Elements of Applied Bifurcation Theory, Applied Mathematical Sciences. Springer; 2004. doi: 10.1007/978-1-4757-3978-7
15. Nayfeh AH, Mook DT. Nonlinear Oscillations, 1st ed. Wiley; 2008. doi: 10.1002/9783527617586
16. Strogatz SH. Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering, 3rd ed. Chapman and Hall; 2018. doi: 10.1201/9780429398490
17. Zadeh LA. Fuzzy sets. Information and Control. 1965; 8(3): 338–353. doi: 10.1016/S0019-9958(65)90241-X
18. Mamdani EH, Assilian S. An experiment in linguistic synthesis with a fuzzy logic controller. International Journal of Man-Machine Studies. 1975; 7(1): 1–13. doi: 10.1016/S0020-7373(75)80002-2
19. Mendel JM, John RIB. Type-2 fuzzy sets made simple. IEEE Transactions on Fuzzy Systems. 2002; 10(2): 117–127. doi: 10.1109/91.995115
20. Soize C. A nonparametric model of random uncertainties for reduced matrix models in structural dynamics. Probabilistic Engineering Mechanics. 2000; 15(3): 277–294. doi: 10.1016/S0266-8920(99)00028-4
21. Soize C. Random matrix theory for modeling uncertainties in computational mechanics. Computer Methods in Applied Mechanics and Engineering. 2005; 194(12–16): 1333–1366. doi: 10.1016/j.cma.2004.06.038
22. Soize C. A comprehensive overview of a non-parametric probabilistic approach of model uncertainties for predictive models in structural dynamics. Journal of Sound and Vibration. 2005; 288(3): 623–652. doi: 10.1016/j.jsv.2005.07.009
23. Kabir S, Papadopoulos Y. A review of applications of fuzzy sets to safety and reliability engineering. International Journal of Approximate Reasoning. 2018; 100: 29–55. doi: 10.1016/j.ijar.2018.05.005
24. Fu C, Lu K, Yang Y, et al. Nonlinear Vibrations of an Uncertain Dual-Rotor Rolling Bearings System with Coupling Misalignment. Journal of Nonlinear Mathematical Physics. 2022; 29(2): 388–402. doi: 10.1007/s44198-022-00044-x
25. Xu K, Cheng T, Lopes AM, et al. Fuzzy Fractional-Order PD Vibration Control of Uncertain Building Structures. Fractal and Fractional. 2022; 6(9): 473. doi: 10.3390/fractalfract6090473
26. Liu M, Lv J, Wu L, et al. Chaotic vibration control of an axially moving string of multidimensional nonlinear dynamic system with an improved FSMC. Scientific Reports. 2024; 14(1): 26495. doi: 10.1038/s41598-024-77632-y
27. Jena SK, Eremeyev VA, Reccia E, et al. Uncertainty quantification in free vibration of porous functionally graded micro-beams using double parametric form based Haar wavelet Discretization Method. Composite Structures. 2025; 372: 119600. doi: 10.1016/j.compstruct.2025.119600
28. Wang L, Dai L, Sun L. ConvLSTM-based spatiotemporal and temporal processing models for chaotic vibration prediction of a microbeam. Communications in Nonlinear Science and Numerical Simulation. 2025; 140: 108411. doi: 10.1016/j.cnsns.2024.108411
29. Li X, Sun L, Liu X, et al. Chaotic Vibration Prediction of a Laminated Composite Cantilever Beam. Applied Sciences. 2025; 15(12): 6403. doi: 10.3390/app15126403