Sound & Vibration 2026, 60(2), 3993.
doi: https://doi.org/10.59400/sv3993

Article

Free and forced vibrations of structurally inhomogeneous rod mechanical systems

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CITATION

Safarov I, Teshaev M, Ishmamatov M, et al. Free and forced vibrations of structurally inhomogeneous rod mechanical systems. Sound & Vibration. 2026; 60 (2): 3993. https://doi.org/10.59400/sv3993

ARTICLE INFO

Received: 24 February 2026
Revised: 20 March 2026
Accepted: 25 March 2026
Available online: 20 April 2026

Abstract: The paper investigates free and forced vibrations of structurally inhomogeneous viscoelastic rod mechanical systems. A mechanical system is considered in which the rheological properties of the deformable elements differ significantly: some elements are elastic, while others are viscoelastic with different hereditary functions. Massive deformable elements have finite volumes, whereas massless elements have finite or negligibly small volumes. The deformable elements of the system are made of viscoelastic materials, such as polymers and polymer-based composites, whose physical properties are described by linear Boltzmann–Volterra hereditary constitutive relations with integral difference kernels. The main objective of the work is to study the dissipative properties of such structurally inhomogeneous rod mechanical systems. Moreover, in free oscillations of the system, the manifestation of dissipation reduces to the attenuation of oscillations, the attenuation rate quantitatively assesses the dissipative properties of the system; in steady-state forced oscillations, the dissipative properties are most pronounced in resonant modes and lead to finite values of resonant amplitudes. The natural frequencies and forms of oscillations are determined from the condition that the determinant of the system, calculated by the Müller-Gauss method, is equal to zero. For the considered mechanical system, the fundamental possibility of significantly intensifying dissipative processes in dynamical systems and reducing the resonant amplitudes of principal oscillations due to the convergence of corresponding natural frequencies is shown.

Keywords: natural frequency; steady-state oscillations; oscillation form; dissipative properties; forced oscillations; viscoelastic material

1. Introduction

Modern mechanical engineering, aviation, construction, and other structures typically consist of various mechanical elements in the form of packages, slabs, plates, shells, and attached masses with a large number of elastic and viscoelastic connections made from materials with different rheological properties. Different rheological properties of the materials in a structure significantly affect the dynamic state of machines and their elements during oscillatory processes. Therefore, the study of vibrations in mechanical systems with elements possessing different rheological properties is an important and relevant problem in mechanics.

The problem of the linear analysis of natural and forced vibrations of elastic oscillatory systems has now been solved with considerable completeness. However, structurally heterogeneous mechanical systems (mechanical systems with different rheological parameters) have been insufficiently studied to date. In works by Mardonov et al. [1] and Mirzaev and Shomurodov [2], the natural and forced vibrations of a structurally heterogeneous mechanical system with a finite number of degrees of freedom (six and twelve) were investigated. For the first time, the concept of a global damping coefficient was introduced, which determines the maximum energy dissipation in the mechanical system as a whole. In works by Mirsaidov et al. [3] and Lu et al. [4], the damping of natural vibrations of a structurally heterogeneous mechanical system with a finite number of degrees of freedom was studied depending on the stiffness parameters of massless elements, and the results of the calculations presented in Mardonov et al. [1] and Mirzaev and Shomurodov [2] were confirmed. Literature [5, 6] are devoted to the dynamic analysis of plates and cylindrical shells supported at internal points and carrying several attached concentrated masses. Some of the deformable elements are viscoelastic. The mechanical system under consideration is treated as structurally heterogeneous. Based on the obtained numerical results, the maximum value of the global damping coefficient was determined as a function of the geometric and physico-mechanical parameters of the mechanical system. In works by Hou and Jiang [7] and Karimov et al. [8], the attenuation of waves in two-layer and three-layer extended structurally heterogeneous plates is considered. The dependence of the global damping coefficient on the wave number was investigated. It was found that the global damping coefficient for a structurally heterogeneous mechanical system is a non-monotonic function of the wave number. Works by Diala and Ezeh [9] and Valeev and Zotov [10] are devoted to the practical application of the theory of viscoelastic mechanical systems. In particular, it is applied to the protection of radio-electronic equipment from vibration, and the conditions for designing damping and vibration-isolation systems are presented. In the case of a linear elastic system with a finite number of degrees of freedom, the problem of determining natural frequencies and mode shapes can be completely solved using standard subroutines [11, 12] and methods of passive vibration-isolation systems. In recent years, rapid progress has been observed in the development of technologies for active vibration suppression, and it has been reported that vibrations can be suppressed with greater efficiency [13, 14]. However, these studies have mainly been carried out for low-order systems, and only a few investigations have been conducted using control models that take into account the complex vibration characteristics of multi-degree-of-freedom systems. For active control of vibration isolators with several degrees of freedom, the traditional absolute-velocity feedback strategy can suppress vibrations but leads to deviations in platform positioning. The problems of external friction are most simply and clearly addressed by introducing the Rayleigh dissipation function. As a result, first-order time derivatives appear in the linear equations of the second kind of Lagrange [15, 16]. The main difficulty here is related to determining the Rayleigh coefficient while taking into account the specific configuration of the system and the real properties of the resisting medium. Strong vibration and noise caused by the operation of high-speed trains not only cause discomfort to passengers in railway cars but also seriously disturb the living conditions and environment of residents along the railway line. Therefore, it is very important to reduce the vibration and noise of the structure and improve its mechanical environment. At present, most researchers tend to conclude that the most adequate description of the real dissipative behavior of materials is provided by the hereditary theory [17]. This and other methods have been applied to solving dynamic problems of viscoelasticity, including problems of vibration protection. In works by Huang et al. [18], the forced and natural vibrations of a structurally heterogeneous mechanical system consisting of a rigid body and attached masses are investigated. In these studies, the natural and forced (steady-state) vibrations of a mechanical system with a finite number of degrees of freedom were analyzed. The influence of structural heterogeneity on the dissipative properties of the mechanical system was demonstrated.

Unlike previous studies, the present work is devoted to the investigation of vibrations in structurally heterogeneous rod mechanical systems. The influence of structural heterogeneity on the dissipative properties of the mechanical system as a whole is examined. The deformable elements of the system are made of viscoelastic materials, for example, polymer materials and their composites, whose physical properties are described by linear relations [19].

$$\sigma_{ij} = \widetilde{\lambda}_n \varepsilon_{kk} \delta_{ij} + 2\widetilde{\mu}_n \varepsilon_{ij}, \tag{1}$$

where $\widetilde{\lambda}$ and $\widetilde{\mu}$ are the operator moduli of elasticity:

$$\widetilde{\lambda}_n \phi(t) = \lambda_{0n} \left[\phi(t) - \int_0^t R_{\lambda n}(t-\tau)\phi(\tau)d\tau \right], \widetilde{\mu}_n \phi(t) = \mu_{0n} \left[\phi(t) - \int_0^t R_{\mu n}(t-\tau)\phi(\tau)d\tau \right], \tag{2}$$

$R_{\lambda n}(t-\tau)$ and $R_{\mu n}(t-\tau)$ are the relaxation kernels; λ_{0n} and μ_{0n} are the instantaneous elastic moduli; $\phi(t)$ is an arbitrary function of time; $\sigma_{ij}, \varepsilon_{ij}$ denote the components of the stress and strain tensors, respectively.

The usual requirements of integrability, continuity (except at $t = \tau$), sign definiteness, and monotonicity are imposed on the influence functions $R_{\lambda n}(t-\tau)$ and $R_{\mu n}(t-\tau)$:

$$R > 0, \ \frac{dR(t)}{dt} \leq 0, \ \ 0 < \int_0^\infty R(t)\, dt < 1.$$

If some of the deformable elements are elastic, then the hereditary kernels are equal to zero. Poisson's ratio v_n in the proposed formulation of the problem is assumed to be constant. This means that for a structurally homogeneous viscoelastic system, the mode shapes of natural vibrations coincide with the eigenvectors of the corresponding elastic problem [20]. The subject of investigation is the dissipative properties of such structurally heterogeneous mechanical systems. In the case of free vibrations of the system, the manifestation of dissipation is reduced to the damping of oscillations, and the damping rate quantitatively characterizes the dissipative properties of the system.

For steady-state forced vibrations, the dissipative properties are most clearly manifested in resonance regimes and lead to finite values of resonance amplitudes.

The integral terms in the hereditary relations describing the rheological properties of deformable elements are generally small compared to the instantaneous elastic terms. This, together with the assumption of an oscillatory character of motion, makes it possible to apply the freezing procedure. In this case, Equation (2) takes the following form:

$$\bar{\lambda}_n = \lambda_{0n}\left[1 - \mathit{\Gamma}^c_{n\lambda}(\omega_R) - i\ \mathit{\Gamma}^s_{n\lambda}(\omega_R)\right]\ , \bar{\mu}_n = \mu_{0n}\left[1 - \mathit{\Gamma}^c_{n\mu}(\omega_R) - i\ \mathit{\Gamma}^s_{n\mu}(\omega_R)\right], \quad (3)$$

where $\mathit{\Gamma}^c_{n\lambda,n\mu}(\omega_R) = \int_0^\infty R_{n\lambda,n\mu}(\tau)\cdot cos\omega_R\tau d\tau$, $\mathit{\Gamma}^s_{n\lambda,n\mu}\ (\omega_R) = \int_0^\infty R_{n\lambda,n\mu}(\tau)$ $sin\omega_R\tau d\tau$, $\mathit{\Gamma}^c_{n\lambda,n\mu}(\omega_R)$, $\mathit{\Gamma}^s_{n\lambda,n\mu}\ (\omega_R)$—the sine and cosine Fourier transforms of the kernels of the *n*-th distributed element; ω_R is the real part of the complex vibration frequency of the system ($\omega = \omega_R + i\omega_I$).

Physical relations for deformable elements of zero volume:

$$F_e = -\bar{c}_e\Delta e = -c_{0e}\left[1 - \mathit{\Gamma}^c_e(\omega_R) - i\mathit{\Gamma}^s_e(\omega_R)\right]\Delta e, \quad (4)$$

where c_{0e} are the instantaneous stiffnesses of the deformable elements of zero volume, and Δe are the increments of the deformable elements.

Together with the assumption of an oscillatory character of motion, this allows the application of the freezing procedure [6], which leads to the following complex physical relations for deformable elements of zero volume:

$$F_e = -c_e\Delta e = -c_e\left[1 - \mathit{\Gamma}^c_e(\omega_R) - i\mathit{\Gamma}^s_e(\omega_R)\right]\Delta e. \quad (5)$$

Two operating modes of the system are considered—free and forced vibrations. Free vibrations refer to motions in which all points of the system oscillate with the same frequencies and damping coefficients (but with different complex amplitudes). It is assumed that external forces are absent during natural vibrations.

Forced vibrations occur under stationary (periodic) or non-stationary external actions. The vibration regime (steady-state or transient) depends on the nature of the external excitations. In this context, the frequency is complex for natural vibrations and real for forced vibrations. In the first case, the complex natural frequency corresponds to the frequency of decaying oscillations, with its imaginary part representing the damping coefficient of the system's natural vibrations. In the second case, ω_R coincides with the frequency of the forced vibrations. For natural vibrations, relations (5) are approximate, whereas for forced vibrations they are exact.

Some of the deformable elements may be elastic; in this case, the hereditary kernels describing the rheological properties of the elements are identically zero. A system in which the rheological properties of the deformable elements are identical (i.e., the hereditary kernels of the elements are equal) will be referred to as dissipatively homogeneous, whereas a system with differing rheological characteristics is dissipatively heterogeneous. In the particular case where external forces are absent, the free damped vibrations of the system are considered; when external forces are present, forced vibrations are analyzed.

The main problem is to investigate the dissipative (damping) properties of the system as a whole, as well as to study its stress-strain state. In free vibrations, dissipation manifests as the decay of natural oscillations. The rate of decay quantitatively characterizes the system's dissipative properties: the higher the decay rate, the greater the dissipation. In steady-state forced vibrations, the system's dissipative properties are most prominently manifested in resonance regimes, leading to finite values of resonance amplitudes. In the case of non-stationary vibrations, dissipative properties become apparent when determining the system's stress-strain state. For steady-state forced vibrations, the resonance amplitudes serve as a quantitative measure of the system's dissipative properties, with higher dissipation corresponding to lower resonance amplitudes of forced vibrations.

To quantitatively characterize the system's overall dissipative properties, two measures are proposed: the minimum decay rate of natural vibrations and the maximum resonance amplitude.

Introduce the concepts of a global damping coefficient and a global resonance amplitude. The dissipative properties of a system are primarily determined by its damping characteristics, which are generally inapplicable to dissipatively heterogeneous systems. The investigation of how the level of a system's dissipative properties depends on its parameters constitutes the main focus of this study.

It has been established that the global damping characteristics of a dissipatively heterogeneous system are determined not only (and not primarily) by the viscoelastic properties of the system's elements, but also by the interaction of oscillations of different natural modes. These interactions are significantly influenced by the system's structure, design, geometry, size, presence of elastic connections, and the mutual arrangement of the elements within the system as a whole.

2. Problem statement and solution methods

We analyze the dynamic effects in a structurally inhomogeneous mechanical structure, shown in **Figure 1**. The mechanical system consists of an elastic rod (1) with a circular variable cross-section, a protective casing (2) representing a fiberglass viscoelastic shell, and a perfectly rigid body (3) rigidly attached to both the rod and the casing. The left ends of the rod and the casing are rigidly clamped.

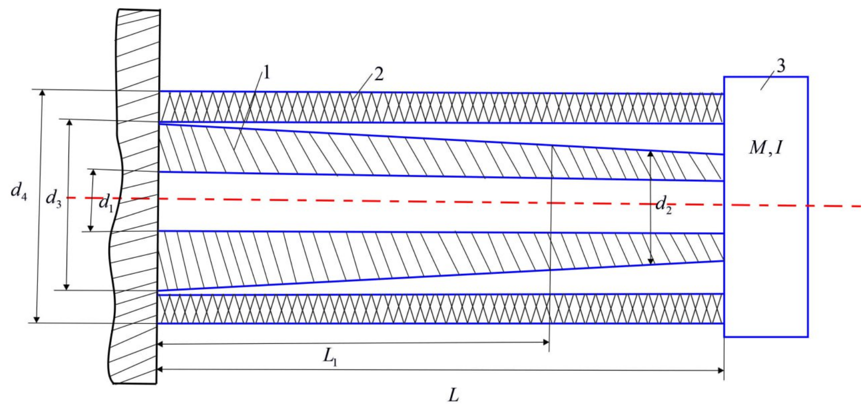

Figure 1. Computational scheme of a structurally heterogeneous structure.

The technical objective is to achieve a maximum reduction of the amplitude of the resonant angular vibrations of body (3) by varying, within physically feasible limits, the stiffness of the casing material, its dimensions, and its mass. The study focuses on the vibration frequencies and damping coefficients. In the general case, the equations of motion for a structurally inhomogeneous system, consisting of *N* elastic and *K* viscoelastic rods of length *l* with a circular cross-section of variable area, have the form:

$$\begin{gathered}\frac{\partial^2}{\partial x^2}(E_n I_n(x)\frac{\partial^2 u_i}{\partial x^2}) + \rho_n F_n(x)\frac{\partial^2 u_i}{\partial t^2} = 0,\\ \frac{\partial^2}{\partial x^2}(\widetilde{E}_k I_k(x)\frac{\partial^2 u_k}{\partial x^2}) + \rho_k F_k(x)\frac{\partial^2 u_k}{\partial t^2} = 0,\\ n = 1, 2, ..N; k = N+1, N+2, ...N+M,\end{gathered} \tag{6}$$

where $I_n(x), F_n(x)$ are the moment of inertia and the cross-sectional area of the *n*-th rod, ρ_n, E_n are the material density, and the Young's modulus of the *n*-th elastic rod, and $\widetilde{E}_k$ is the Volterra integral operator.

In addition to Equation (6), the displacements $u(x,t)$ must satisfy the boundary conditions: one end of the system (*x* = 0) is rigidly clamped, and a common mass *M* with a moment of inertia *I* is attached to the other end (*x* = *l*) for all rods. The boundary conditions can be written as follows:

$$\begin{gathered}x = 0 : u_k = 0, \frac{\partial u_k}{\partial x} = 0, k = 1, 2, ..., N+K,\\ x = l : \sum_{k=1}^{K} E_k \frac{\partial}{\partial x}(I_k(x)\frac{\partial^2 u_k}{\partial x^2}) + \sum_{n=N+1}^{K+N} \widetilde{E}_n \frac{\partial}{\partial x}(I_n(x)\frac{\partial^2 u_n}{\partial x^2}) -\\ -M\frac{\partial^2 u_1}{\partial t^2} = 0,\\ \sum_{k=1}^{N} E_k I_k(x)\frac{\partial^2 u_k}{\partial x^2} + \sum_{n=N+1}^{N+K} \widetilde{E}_n I_n(x)\frac{\partial^2 u_n}{\partial x^2} + I\frac{\partial^3 u_1}{\partial x \partial t^2} = 0,\\ u_1 = u_2 = ... = u_{N+K}, \frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} = ... = \frac{\partial u_{N+K}}{\partial x}.\end{gathered} \tag{7}$$

The solution of the boundary-value problem (6) and (7) is sought in the form:

$$u_n(x,t) = \vartheta_n(x)e^{i\omega t} \quad , \quad u_k(x,t) = \vartheta_k(x)e^{i\omega t}, \tag{8}$$

where $\vartheta_n(x)$ и $\vartheta(x)$ are the complex vibration modes, and $\omega = \omega_R + i\omega_I$ is the sought complex frequency.

Substituting (8) into (6) and the boundary conditions (7) leads to a homogeneous boundary-value problem:

$$\begin{gathered}E_n\frac{d^2}{dx^2}(I_n(x)\frac{d^2\vartheta_n}{dx^2}) - \omega^2\rho_n F_n(x)\vartheta_n = 0, n = 1, 2, .., N,\\ \widetilde{E}_k\frac{d^2}{dx^2}(I_k(x)\frac{d^2\vartheta_k}{dx^2}) - \omega^2\rho_k F_k(x)\vartheta_k = 0, k = N+1, N+2, .., N+K,\\ x = 0 : \vartheta_j = 0, \frac{d\vartheta_j}{dx} = 0, j = 1, 2, ..N+K;\\ x = l : \sum_{n=1}^{N} E_n\frac{d}{dx}(I_n(x)\frac{d^2\vartheta_n}{dx^2}) +\\ + \sum_{k=N+1}^{N+K} \widetilde{E}_k\frac{d}{dx}(I_k(x)\frac{d^2\vartheta_k}{dx^2}) + \omega^2 M\vartheta_1 = 0,\\ \sum_{n=1}^{N} E_n I_n(x)\frac{d^2\vartheta_n}{dx^2} + \sum_{k=N+1}^{N+K} \widetilde{E}_k I_k(x)\frac{d^2\vartheta_k}{dx^2} - \omega^2 I\frac{d\vartheta_1}{dx} = 0,\\ \vartheta_1 = \vartheta_1 = ... = \vartheta_{N+K}, \frac{d\vartheta_1}{dx} = \frac{d\vartheta_2}{dx} = ... = \frac{d\vartheta_{N+K}}{dx}.\end{gathered} \tag{9}$$

The operators $\widetilde{E}_k$ depend on the quantity ω, i.e., the boundary-value problem (7)–(9) represents an eigenvalue problem with a nonlinearly entering complex parameter. For the structure (**Figure 1**), accordingly have the equations of motion.

$$\begin{gathered} E_1 \tfrac{d^2}{dx^2}(I_1(x)\tfrac{d^2\vartheta_1}{dx^2}) - \omega^2\rho_1 F_1(x)\vartheta_1 = 0, \\ \widetilde{E}_2 \tfrac{d^2}{dx^2}(I_2(x)\tfrac{d^2\vartheta_k}{dx^2}) - \omega^2\rho_2 F_2(x)\vartheta_2 = 0; \\ x = 0 : \vartheta_1 = 0, \vartheta_2 = 0, \tfrac{d\vartheta_1}{dx} = 0, \tfrac{d\vartheta_2}{dx} = 0; \end{gathered} \tag{10}$$

$$\begin{gathered} x = l : E_1 \tfrac{d}{dx}(I_1(x)\tfrac{d^2\vartheta_1}{dx^2}) + \widetilde{E}_2 \tfrac{d}{dx}(I_2(x)\tfrac{d^2\vartheta_2}{dx^2}) + \omega^2 M\vartheta_1 = 0, \\ E_1 I_1(x)\tfrac{d^2\vartheta_1}{dx^2} + \widetilde{E}_2 I_2(x)\tfrac{d^2\vartheta_2}{dx^2} - \omega^2 I\tfrac{d\vartheta_1}{dx} = 0, \\ \vartheta_1 = \vartheta_2, \tfrac{d\vartheta_1}{dx} = \tfrac{d\vartheta_2}{dx}. \end{gathered} \tag{11}$$

By replacing the operator $\widetilde{E}_2\,(\omega_R)$ with a complex modulus of elasticity $\bar{E}_2(\omega_R)$, bring the equations of motion to a form convenient for numerical integration:

$$\frac{d\chi_n}{dx} = F_n(x, \chi_1, \chi_2, \chi_3, ..., \chi_N). \tag{12}$$

As the new variables of the problem, choose the quantities ϑ_1 and ϑ_2, as well as their derivatives up to the third order, inclusive:

$$\begin{gathered} \chi_1 = \vartheta_1, \chi_2 = \tfrac{d\vartheta_1}{dx}, \chi_3 = \tfrac{d^2\vartheta_1}{dx^2}, \chi_4 = \tfrac{d^3\vartheta_1}{dx^3}, \\ \chi_5 = \vartheta_2, \chi_6 = \tfrac{d\vartheta_2}{dx}, \chi_7 = \tfrac{d^2\vartheta_2}{dx^2}, \chi_8 = \tfrac{d^3\vartheta_2}{dx^3}. \end{gathered} \tag{13}$$

The resulting system of equations will take the form:

$$\begin{gathered} \tfrac{d\chi_1}{dx} = \chi_2, \tfrac{d\chi_2}{dx} = \chi_3, \tfrac{d\chi_3}{dx} = \chi_4, \\ \tfrac{d\chi_4}{dx} = -2\tfrac{I_1'(x)}{I_1(x)}\chi_4 - \tfrac{\chi_1''(x)}{I_1(x)}\chi_3 + \tfrac{\rho_1\omega^2}{E_1}\tfrac{F_1(x)}{I_1(x)}\chi_1, \\ \tfrac{d\chi_5}{dx} = \chi_6, \tfrac{d\chi_6}{dx} = \chi_7, \tfrac{d\chi_7}{dx} = \chi_8, \\ \tfrac{d\chi_8}{dx} = -2\tfrac{I_2'(x)}{I_2(x)}\chi_3 - \tfrac{I_2''(x)}{I_2(x)}\chi_7 + \tfrac{\rho_1\omega^2}{E_1}\tfrac{F_2(x)}{I_2(x)}\chi_5. \end{gathered} \tag{14}$$

The boundary conditions of the problem can be expressed in terms of the new variables as follows:

$$\begin{gathered} x = 0 : \chi_1 = 0, \chi_2 = 0, \chi_5 = 0, \chi_6 = 0, \\ x = l : E_1 I_1'(x)\chi_3 + E_1 I_1(x)\chi_4 + \bar{E}_2(\omega_R) I_2'(x)\chi_7 + \\ + \bar{E}_2(\omega_R) I_2(x)\chi_8 + M\omega^2\chi_1 = 0, \\ E_1 I_1(x)\chi_3 + \bar{E}_2(\omega_R) I_2(x)\chi_7 - I\omega^2\chi_2 = 0, \\ \chi_1 - \chi_5 = 0, \chi_2 - \chi_6 = 0. \end{gathered} \tag{15}$$

This system can be written in vector–matrix form:

$$\overrightarrow{x}' = A(x)\overrightarrow{x} + \overrightarrow{f}(x). \tag{16}$$

Here the matrix $A(x)$ is a square matrix of order *n* and has *n* linearly independent eigenvectors. In this case, the general solution of Equation (16) can be expressed in the form:

$$\bar{\chi}_k = \sum_{j=1}^{n} \xi_{kj} e^{\lambda j x} \overrightarrow{e}_j, k = 1, \ldots, n.$$

where $\overrightarrow{f}(x) = (f_1(x), f_2(x), ..., f_n(x))^T$ is a given column vector, λj are the eigenvalues of the matrix, T denotes the transpose operation, and $\overrightarrow{\chi}(x) = (\chi_1(x), \chi_2(x), \ldots, \chi_n(x))^T$ is the vector function to be determined. A prime indicates differentiation with respect to *x*. In the case where the system of differential equations has a matrix with constant coefficients (A = const), the solution of the Cauchy problem takes the form:

$$\overrightarrow{\chi}(x) = e^{A(x-x_0)} \overrightarrow{\chi}(x_0) + e^{Ax} \int_{x_0}^{x} e^{-At} \overrightarrow{f}(t) dt$$

where $e^{A(x-x_0)} = E + A(x-x_0) + A^2(x-x_0)^2/2! + A^3(x-x_0)^3/3! + \ldots E$ is the identity matrix.

In matrix form, system (14) and the boundary conditions (15) can be represented as [5]:

$$\left(A - \omega^2 B\right) \bar{\xi} = 0, (A - \omega^2 B) \cdot \bar{\xi} = 0 \quad (17)$$

where *A* and *B* are square matrices, and $\bar{\xi}$ is the column vector of unknowns.

System of Equation (17) represents a generalized eigenvalue problem for determining the eigenvalues ω^2 and the eigenvectors $\bar{\xi}$. For this system to have a nontrivial solution, it is necessary and sufficient that its determinant be equal to zero:

$$\left|A - \omega^2 B\right| = 0. \quad (18)$$

By solving Equation (18) using Müller's method, we obtain the frequency spectrum $\omega\ , \ldots, \omega_{k-s'} \omega_1, \ldots, \omega_{K-S/}$. The corresponding eigenvector spectrum $\bar{\xi}'_1, \ldots, \bar{\xi}'_{k-s'}$ is determined using Müller's method in combination with Godunov's orthogonal sweep method. Müller's method generalizes the secant method: instead of a straight line through two points, it constructs a parabola through three points, which allows finding both real and complex roots. At each step of iterative refinement, the values are adjusted using the orthogonal sweep method [20], and the boundary value problem (14) and (15) is solved.

3. Results and discussion

3.1. Free vibrations of a structurally inhomogeneous rod structure

To justify the reliability of the calculation results for free vibrations, a test problem is solved. A steel rod with a constant rectangular cross-section and constant curvature is considered, with one end rigidly fixed and the other end free. The rod dimensions (in mm) are: length—210, cross-section—3.2 × 18.2, radius of curvature —57.8. The smaller side of the cross-section is parallel to the normal to the axis line. The natural flexural–longitudinal vibrations are investigated [5]. It is necessary to find such values for which the system of equations with boundary conditions (9) and (10) has non-zero solutions. This problem was solved numerically using Godunov's orthogonal sweep method for the values $E = 19.6 \cdot 10^4$ MPa, $\rho = 8$ g/sm^3. The first three free vibration frequencies are presented in **Table 1** for comparison with experimental results [6].

Table 1. Comparison of theoretical and experimental results.

Frequency	Frequency number	Calculation	Experiment
	1	75.8	75
ω/2π [гц]	2	212	204
	3	713	680

From **Table 1**, it can be seen that the difference between the experimental and calculated results is greater for higher frequencies, but overall does not exceed 5%. In the calculations, the relaxation cores of the viscoelastic elements were chosen in the form of:

$$R_n(t) = A_n \exp(-\beta_n t) t^{\alpha_n - 1} \ (n = 1, 2), \tag{19}$$

where $A,\ \beta,\ \alpha$ are the core parameters. The viscosity of the elements was chosen so that their creep deformation under a quasi-static process would constitute a small fraction of the total (~12%). For this case, the core parameters are as follows: $A = 0.01,\ \alpha = 0.1,\ \ \beta = 0.05$.

In the calculations, the instantaneous modulus of elasticity E_2, the relaxation core parameters A, α, β, and the diameter d_4 of the fiberglass shell are varied while keeping its thickness and the values of other parameters fixed. **Figure 2** shows, respectively, the dependencies of the lowest five frequencies and their corresponding damping coefficients on the value of E_2 for the following values of the problem parameters:

$$d_1 = 3.0 \cdot 10^{-2}\ \text{m}, d_2 = 6.0 \cdot 10^{-2}\ \text{m}, d_3 = 5.0 \cdot 10^{-2}\ \text{m},$$

$$d_4 = 6.6 \cdot 10^{-2}\ \text{m}, l_1 = 1.0\ \text{m}, l = 1.0\ \text{m}, M = 2.8\ \text{kg},$$
$$I = \frac{0.02\ \text{kg}}{\text{m}^2}, A = 0.0022, \alpha = 0.05, \beta = 0.05.$$

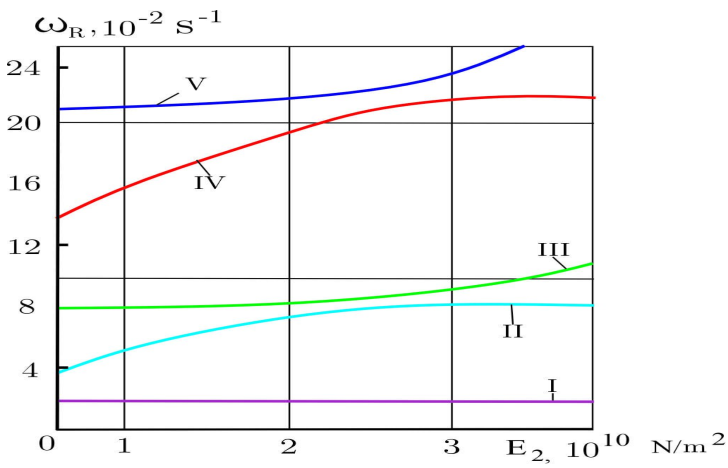

Figure 2. Dependence of the real part of the five complex frequencies on the instantaneous modulus of elasticity E_2.

From **Figure 3**, one can observe the convergence of the frequency pairs II–III and IV–V at $E_2 \approx 2.1 \cdot \frac{10^{10} N}{\text{m}^2}$, the damping coefficients corresponding to these frequencies (at the same value of E_2) intersect with each other. The damping coefficient $\delta = -\omega_I$, directly related to the logarithmic decrement of decay, serves as a measure of energy dissipation in the process. The most practically significant aspect in solving the posed problem is the minimum value of the damping coefficient (the determining damping coefficient) $\delta = min(-\omega_{Ik})$ for the considered Eigenfrequencies, where k denotes the number of the vibration mode under consideration. The minimum value of the damping coefficient for a converging pair of frequencies provides information about the decay rate of the vibration mode of elastic rod 1 that is damped more slowly and, therefore, is the determining mode. **Figure 3** shows the damping coefficients generalized in this way for the second and third vibration modes of rod 1; in the region of maximum frequency convergence, a pronounced generalized maximum of the dissipative properties is observed.

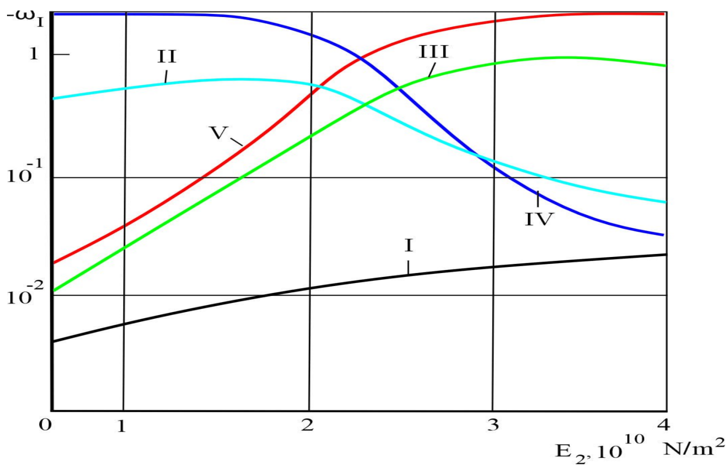

Figure 3. Dependence of the imaginary part of the five complex frequencies on the instantaneous modulus of elasticity E_2.

Within the given range of variation, the damping coefficient changes by more than an order of magnitude; the first vibration mode, meanwhile, is damped to a significantly lesser extent. Due to the difficulty of accurately determining the relaxation core parameters of the material, the calculations were repeated for various values of these parameters, covering a wide range of viscoelastic properties (the values of the integral terms in expression (2) varied from 3.5% to 30%). The qualitative picture of the process, as shown in **Figure 3**, was preserved. Changing the relaxation core parameters, or, equivalently, the viscoelastic properties of the material, shifts the curves vertically. In other words, the absolute values of the damping coefficients change slightly, while the positions of their maxima relative to the magnitude of E_2 remain unchanged.

In order to identify the dependence of the dissipative properties of the structure on the geometry of the casing, the diameter of the casing d_2 was varied at the found optimal value of E_2, while keeping its thickness unchanged. As expected, the maximum of the determining damping coefficients corresponded to those values of the diameter d_4 for which the optimal value of E_2 was established, namely $d_4 = 6.6 \cdot 10^{-2}$ m. Changing the thickness of the casing reveals a weak dependence of the damping coefficients on this parameter: as the casing thickness increases up to a certain value, the determining damping coefficients rise, and then change insignificantly, slightly decreasing with further thickening of the casing. This indicates that the damping characteristics of the structurally non-uniform system are influenced primarily not by the amount of viscoelastic material, but by the presence of closely spaced natural frequencies in the system. To achieve maximum damping of vibrations, it is therefore necessary to select the stiffness of the casing such that its fundamental frequencies are close to the natural frequencies of rod 1 that need to be damped. The decay rate of free vibrations can be increased by choosing fiberglass composites with an optimal modulus E_2, which depends, in particular, on the reinforcement scheme, type of filler, degree of filling, material of the fiberglass casing, and also by selecting an optimal casing design.

3.2. Forced vibrations of a structurally inhomogeneous rod structure

Let us consider the forced vibrations of a structurally inhomogeneous rod-like structure (**Figure 1**), subjected at the right end to a harmonic transverse force Qe^{-ipt} (the quantities Q and p represent the amplitude and frequency of the external excitation, respectively). The equations of motion of the system are similar to (1); the vibrations are assumed to be steady-state, so the lower limit of integration in the expressions of the Volterra operator is taken as $-\infty$. This allows the hereditary integral relations to yield the analytical expressions (5) exactly:

$$\widetilde{E}_2\varphi = E_{02}\left[1 - \Gamma^c_{E2}(\omega_R) - i\Gamma^s_{E2}(\omega_R)\right]\varphi. \tag{20}$$

The effect of the transverse force on the system is taken into account in the boundary conditions at the right end of integration, which are represented as nonhomogeneous:

$$\begin{gathered} x = l: E_1\frac{\partial}{\partial x}(I_1(x)\frac{\partial^2 u_1}{\partial x^2}) + \widetilde{E}_2\frac{\partial}{\partial x}(I_2(x)\frac{\partial^2 u_2}{\partial x^2}) - M\frac{\partial^2 u_1}{\partial t^2} = Qe^{-ipt}, \\ E_1 I_1(x)\frac{\partial^2 u_1}{\partial x^2} + \widetilde{E}_2 I_2(x)\frac{\partial^2 u_2}{\partial x^2} + I\frac{\partial^3 u_1}{\partial x \partial t^2} = 0, \\ u_1 = u_2, \frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x}. \end{gathered} \tag{21}$$

The solution to the problem is sought in the form:

$$u_1(x,t) = a_1(x)e^{ipt}, u_2(x,t) = a_2(x)e^{ipt}, \tag{22}$$

where $a_1(x), a_2(x)$ are the amplitudes of the forced vibrations.

The numerical implementation is carried out using the orthogonal sweep method. In a specific interval covering the five main resonance amplitude values, the amplitude-frequency characteristics of the structure are determined; the amplitude Q in the calculations is taken as unity. We examine the dependence of the resonance value of the rotation angle $A = max\alpha$ of the rigid body 3 about the *x*-axis on the system parameters: by varying the modulus E_2, the stiffness of the structure is adjusted while other mechanical parameters are kept constant.

Figures 4–6 show the amplitude-frequency characteristics for different values of the instantaneous modulus of elasticity E_2.

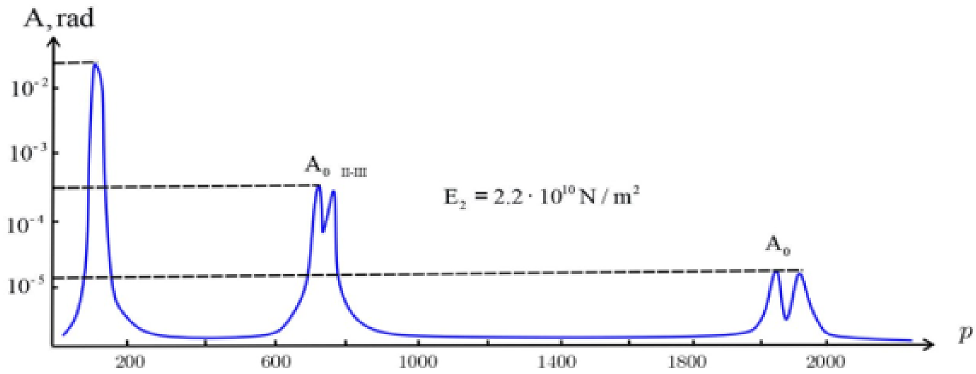

Figure 4. Amplitude–frequency characteristics for different values of the instantaneous modulus of elasticity for $E_2 = 2.2 \cdot 10^{10}$ N/m^2.

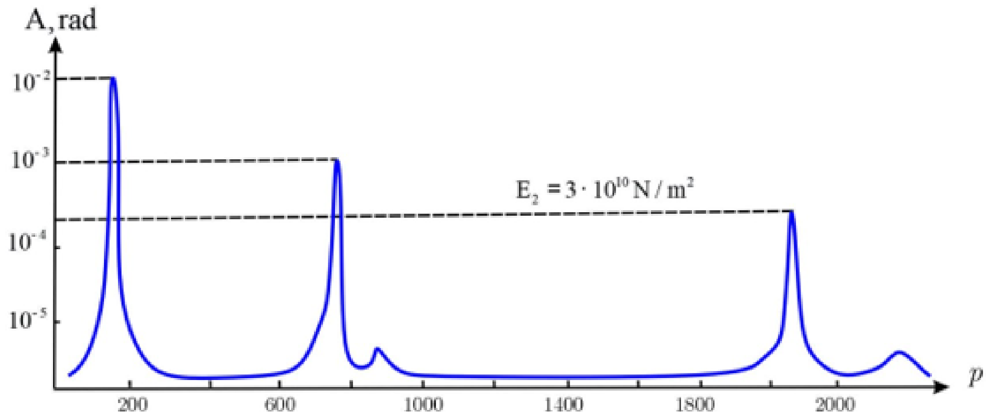

Figure 5. Amplitude–frequency characteristics for different values of the instantaneous modulus of elasticity for $E_2 = 3 \cdot 10^{10}$ N/m^2.

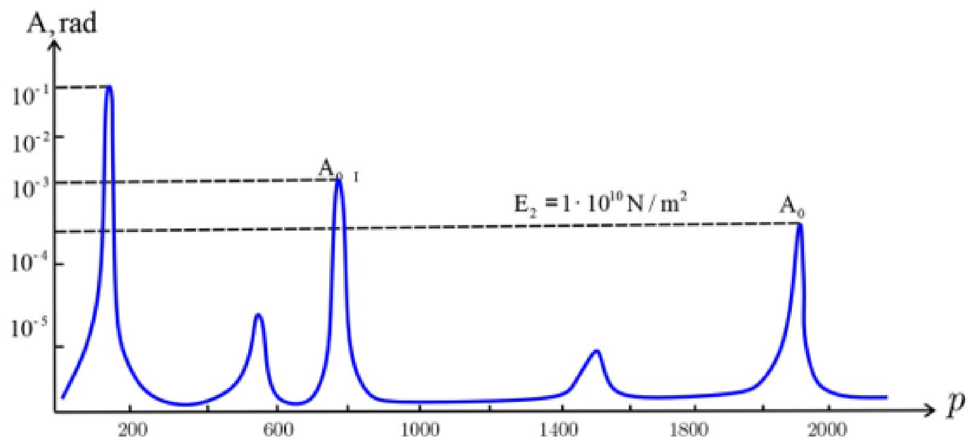

Figure 6. Amplitude-frequency characteristics for different values of the instantaneous modulus of elasticity for $E_2 = 1 \cdot 10^{10}$ N/m^2.

Figure 7 shows the dependence of the resonance values $A_{R.\max}$ on frequency for a given modulus of elasticity $E_2 = 1 \cdot 10^{10}$ N/m^2. The maximum resonance values of the forced vibration amplitude quantitatively characterize the intensity of the dissipative processes in the system, which is higher the lower the peaks of the resonance maximum amplitude.

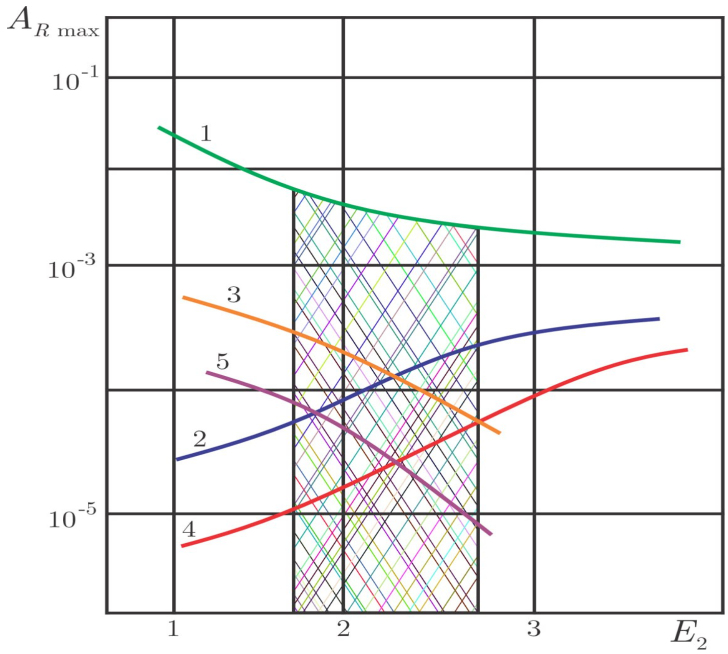

Figure 7. Dependence of the resonance amplitude values on the instantaneous modulus of elasticity for $E_2 = 1 \cdot 10^{10}$ N/m^2.

A comparison of the obtained results with the solutions of the free damped vibrations problem for a heterogeneous rod structure shows that the resonance peaks correspond to excitation frequencies close to the natural frequencies; pairs of the main natural frequencies correspond to closely spaced resonance peaks. For each such pair, the concept of the determining resonance amplitude $A_Q = \underbrace{max\,|A_k|}_{k}$ can be introduced, where k is the number of the resonance peak. Depending on the value of E_2, the amplitude A_Q is determined by either the first or the second resonance peak of the pair; the maximum value of A_Q corresponds to the value of E_2 at which the lower frequencies of the system are closest.

Thus, the obtained results for the considered structurally inhomogeneous viscoelastic structure are fully consistent with the solutions of the free damped vibrations problem and confirm the sharp increase in the intensity of dissipative processes when the main frequencies in heterogeneous viscoelastic systems approach each other.

3.3. Vibrations of the structurally inhomogeneous structure with an additional intermediate ring mass

We further modify the degree of inhomogeneity of the considered mechanical structure by introducing an additional intermediate ring mass 4 on the protective shells 2 (**Figure 8**).

Figures 9 (solid line) and **10** show the obtained dependencies of the main frequencies ω_R and damping coefficients ω_I on the value of the instantaneous modulus of elasticity of the shell E_2 (without the intermediate mass).

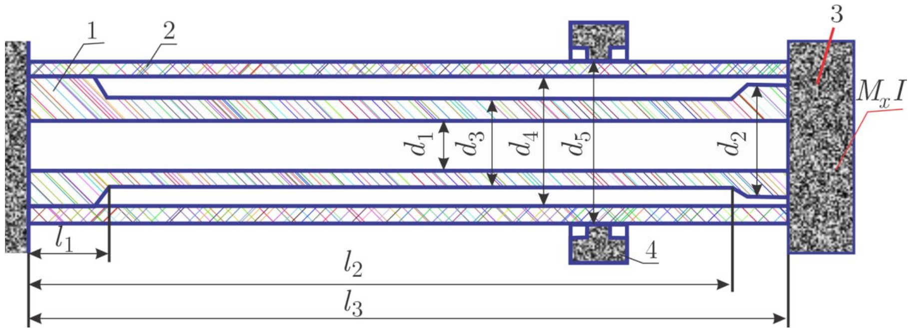

Figure 8. Schematic of the structurally non-uniform structure with an intermediate ring mass on the protective casing.

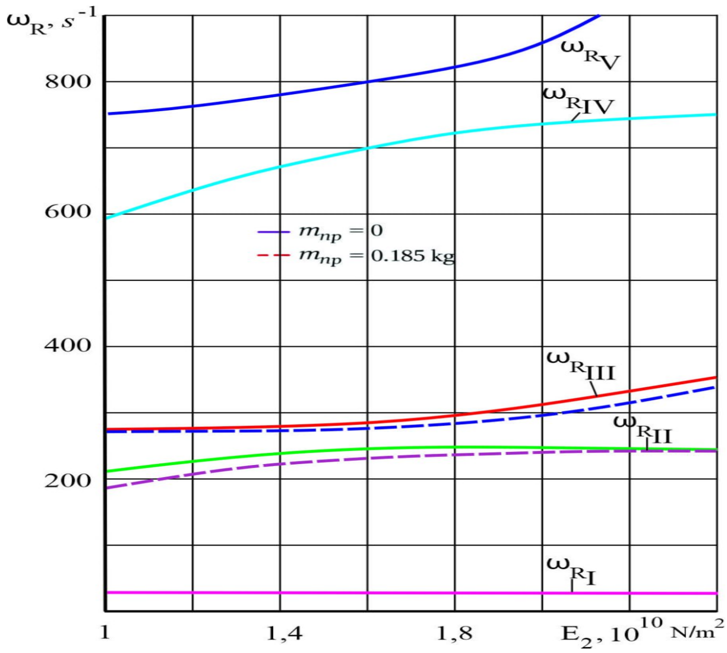

Figure 9. Dependence of the main frequencies on the instantaneous modulus of elasticity E_2.

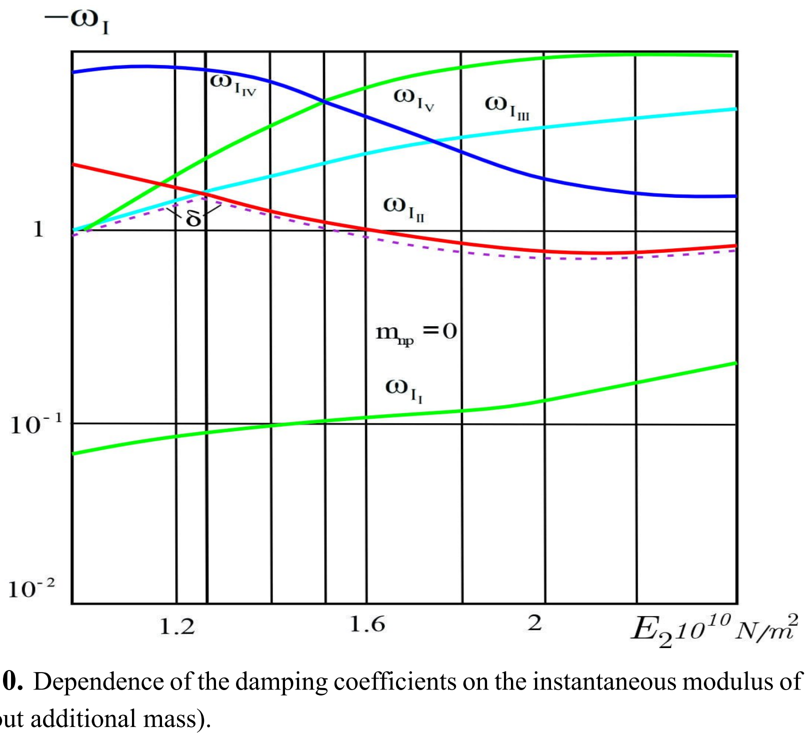

Figure 10. Dependence of the damping coefficients on the instantaneous modulus of elasticity E_2 (without additional mass).

The value of E_2 is varied across the entire physically realizable range $E_2 = 1 \div 10 \cdot 10^{10}$ N/m^2. Of greatest interest are the values of E_2 in the interval $1 \div 2 \cdot 10^{10}$ N/m^2, where the pairs of frequencies II–III and IV–V approach each other, and, consequently, the determining damping coefficients (indicated by dashed lines in **Figure 10**) reach their maximum.

For the purpose of optimizing the damping characteristics, structures are considered within a specified frequency range. For a given type of rod 1, an additional ring mass is attached to the protective shell, placing it at the vibration node of the shell mode belonging to the second converging pair of frequencies (ω_{R4},ω_{R5}). This allows, while preserving the vibration pattern of this pair, modification of the vibrations of the lower shell mode and the nearby vibration mode of elastic rod 1 (ω_{R2},ω_{R3}). Optimization is then possible at $E_2^I = E_2^{II} = E_2^{opt}$, where two vibration modes of elastic rod 1 are damped at the same value of the instantaneous modulus of elasticity of the protective fiberglass shell E_2. We take the following parameters:

$$l_1 = 0.082 \text{ m}, l_2 = 2.147 \text{ m}, l_3 = 2.219 \text{ m}, d_1 = 3 \text{ sm}, d_2 = 4 \text{ sm},$$
$$d_3 = 5 \text{ sm}, d_4 = 6 \text{ sm}, d_5 = 6.6 \text{ sm}, M = 2.75 \text{ kg}, I = 0.002 \text{ kg} \cdot \text{m}^2.$$

The values of the remaining parameters are the same as those adopted previously.

The calculations show that the maxima of the determining damping coefficients coincide along the E_2 coordinate (**Figure 9**) at the value of the additional mass $m_{pr} = 0.185$ kg; corresponding to the fundamental frequency in this case, indicated by the dashed line in **Figure 11**.

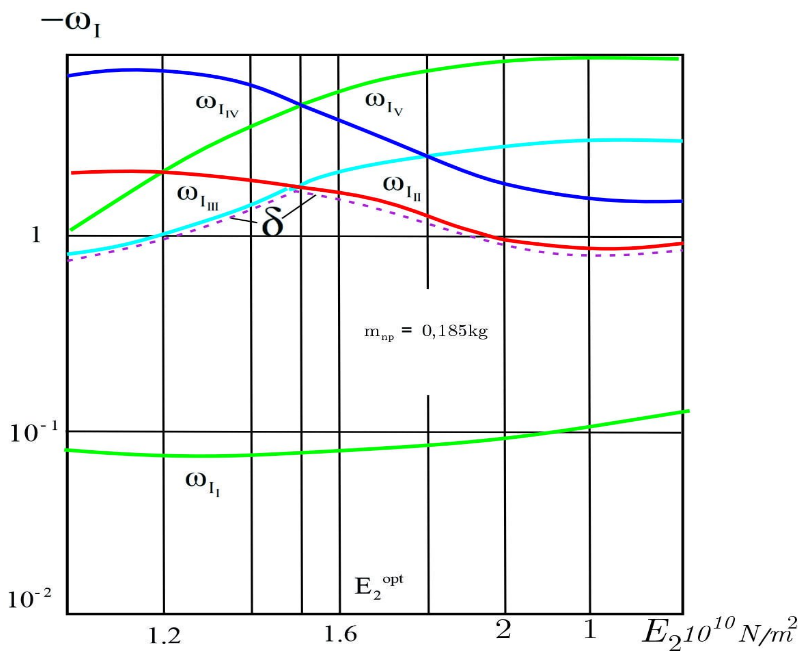

Figure 11. Dependence of the damping coefficients on the instantaneous modulus of elasticity E_2 (with the additional mass).

4. Conclusion

Thus, using the example of a simple structurally inhomogeneous structure, the fundamental possibility of significantly enhancing dissipative processes in dynamic systems and reducing the resonance amplitudes of the primary vibrations has been demonstrated by bringing the corresponding free frequencies closer together. Moreover, the role of rheology is manifested both in the damping of vibrations and in the mutually reinforcing interaction of vibrations of different modes, which substantially increases the overall dissipative properties of the system. This effect of interaction between different forms of motion in continuous bodies holds fundamental potential for the synthesis of structures with optimized dissipative properties and material efficiency, including structurally inhomogeneous engineering constructions, building components, damping compounds, materials and composites, as well as various vibration protection systems and devices.

Author contributions: The authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional review board statement: Not applicable.

Informed consent statement: Not applicable.

Data availability statement: Data is contained within the article.

Conflict of interest: The authors declare no conflict of interest.

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