Sound & Vibration 2026, 60(2), 3837.
doi: https://doi.org/10.59400/sv3837

Article

Bifurcation in compressible moving fluids and suppressing atmospheric turbulence of aircraft-flow amplifies sound

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CITATION

Qian Z. Bifurcation in compressible moving fluids and suppressing atmospheric turbulence of aircraft-flow amplifies sound. Sound & Vibration. 2026; 60(2): 3837. https://doi.org/10.59400/sv3837

ARTICLE INFO

Received: 19 December 2025
Revised: 20 January 2026
Accepted: 12 March 2026
Available online: 25 March 2026

Abstract: Quasi-accumulation solutions for acoustic waves in a compressible moving fluid are obtained by applying the Lagrange parameter variation method to solve the Lighthill equation. The results demonstrate that nonlinear interactions lead first to period-doubling, followed by odd multiple half-period bifurcations, with all-order sub-harmonics subsequently generated. The amplitudes of these sub-harmonics depend not only on the acoustic Mach number but also on the Mach number of the flow. The latter result indicates that the acoustic wave has been amplified by the momentum of the flow. Furthermore, the relationship between the amplification gain of sub-harmonics and flow velocity is a polynomial function of flow-sound Mach number ratio M/m. If the kinetic energy gained through momentum amplification exceeds the energy loss due to the acoustic attenuation, a chain-reaction of the period-doubling followed by the odd multiple half-period bifurcation can be sustained. As the order number of the approximation m_1 increases, the number of degrees of freedom in the flow increases infinitely and the leading terms of the amplitudes for the generated sub-harmonics which are proportional to $\left(Mm\frac{k}{2\alpha}\right)^{m_1}$ approach to infinity, where M and m, k and α are Mach number for flow and sound, the wave-number and absorption coefficient, respectively. The obtained results also indicate that the appropriate control parameter for transitioning from bifurcation to chaos should be $\frac{k}{\alpha}$ instead of Reynolds number Re. This paper also demonstrates that in the moving fluid sound waves can be amplified through nonlinear interactions, particularly, the first sub-harmonic generates, thereby explaining the experimental results we discovered decades ago. Finally, a potential strategy for suppressing aircraft-induced atmospheric turbulence is proposed based on the present theoretical findings.

Keywords: bifurcation; mach number; reynolds number; chaos; sub-harmonics; flow amplifies sound; turbulence

1. Introduction

According to the theory of nonlinear sound propagation, a single-frequency wave traveling through a static medium will continuously generate all harmonic waves. However, due to the dissipation effects of the medium, the energy of the sound wave will eventually be attenuated and ultimately decay. In a previous publication [1], the author investigated nonlinear sound propagation in ideal media and obtained the so-called accumulation solutions. The results demonstrated that a sound wave propagating in a flowing medium undergoes period-doubling bifurcations, during which the amplitudes of successive sub-harmonic orders increase with propagation distance. Throughout the period-doubling bifurcation process, the sub-harmonics continuously generate. Since the medium concerned therein is ideal and the accumulation solutions obtained by Qian [1] increase with distance, thus, they do not converge at infinity.

However, in real media such as water, dissipation effects like viscous dissipation cause the sound wave to attenuate, preventing divergence at infinity. In this paper, we incorporate the viscous stress into the stress tensor and substitute it into the relevant equations from reference [1], conducting a further study on the bifurcation problem in a disturbed fluid. The results indicate that a sound wave propagating in a moving fluid can be amplified, and the amplitude of the generated sub-harmonics depends on acoustical absorption coefficient as well as on the Mach numbers of both the sound and the flow. When discussing chaos, one inevitably associates it with turbulence. What are the defining criteria of turbulence, and what theoretical scenarios are used to describe the transition from laminar flow to turbulence? To address this question, we first recall Landau's conjecture on turbulence [2]: as a system evolves, the number of degrees of freedom approaches infinity, and the amplitudes of the physical quantities corresponding to the degrees of freedom approach to infinity with the increase of time *t*, that is, an instability factor $e^{\gamma t}$ appears where $\gamma > 0$. On the other hand, according to chaos theory [3–8], as the control parameters reach certain critical values, a system that initially exhibits a predictable behavior can suddenly become chaotic and random. Through iterative computations, Feigenbaum [9, 10] obtained the critical control parameters for the population growth function to transition to chaos, but did not determine the instability factor. Therefore, it cannot be concluded that the system transitions to chaos and forms turbulence. More relevant publications in recent years are reported by Cui et al. and other scholars [11–15], as well as in studies by Jakobsen and related works [16–19]. In this paper, we apply the variable parameter method to solve the Lighthill equation, and the results indicate that the nonlinear interactions cause the period-doubling, followed by odd multiple half-period bifurcations, generating sub-harmonics of all orders of which the leading terms of amplitudes for these generated sub-harmonics are proportional to $\left(Mm\frac{k}{2\alpha}\right)^{m_1}$. In addition, this paper proves that the appropriate control parameter for transitioning from bifurcation to chaos should be $\frac{k}{\alpha}$ instead of Reynolds number *Re*, where *M* and *m* are Mach number for flow and sound, m_1 is the order number of the perturbation approximation, k and α are the wave-number and absorption coefficient, respectively. As $m_1$increases, the degrees of freedom of the system increase to infinity, but on the other hand, due to $\frac{k}{\alpha} >> 1$, $\left(\frac{k}{2\alpha}\right)^{m_1} >> 1$ is equivalent to the instability factor in Landau conjecture, although it does not depend on *t*. These results demonstrate that the bifurcation framework presented in this article provides an effective description of the transition from laminar flow to turbulence. This paper also demonstrates that the moving fluid can amplify sound waves through nonlinear interactions, particularly, in the moving fluid the first sub-harmonic can generate, thereby explaining the experimental results we discovered decades ago.

Finally, a potential; strategy for suppressing aircraft-induced atmospheric turbulence is proposed based on the present theoretical findings.

2. Theory

As is well known, the one-dimensional form of stress tensor in viscous media should be written as

$$T_{ij} = T^{(l)} + T^{(s)}, \tag{1}$$

where

$$T^{(l)} = \rho v^2 \tag{2}$$

is the Reynolds stress, $T^{(s)}$ is the stress that can denoted by

$$T^{(s)} = p - C_0^2\rho + (\mu_b + \frac{4}{3}\mu)\frac{\partial v}{\partial x}, \tag{3}$$

where p is the pressure, ρ is the density, C_0 is the sound velocity, μ and μ_b are the shear viscosity coefficient and the volume viscosity coefficient, respectively. v is the fluid particle velocity,

$$v = V_0 + v_0 cos2(\tau - \sigma) \tag{4}$$

Substituting Equation (4) into (2), Lighthill equation yields (**Appendix A**)

$$\frac{\partial^2\rho}{\partial\tau^2} - \frac{\partial^2\rho}{\partial\sigma^2} = \varepsilon\left\{g_1(\sigma,\tau)\frac{\partial^2\rho}{\partial\sigma^2} + g_2(\sigma,\tau)\frac{\partial\rho}{\partial\sigma} - g_3(\sigma,\tau)\rho\right\} \tag{5}$$

where

$$\varepsilon = Mm,\ M = \frac{V_0}{C_0}, m = \frac{v_0}{C_0} \tag{6}$$

and

$$\left.\begin{aligned} g_1(\sigma,\tau) &= \frac{M}{m} + \frac{m}{2M} + 2cos2(\tau-\sigma) + \frac{m}{2M}cos4(\tau-\sigma) \\ g_2(\sigma,\tau) &= 8sin2(\tau-\sigma) + 4\frac{m}{M}sin4(\tau-\sigma) \\ g_3(\sigma,\tau) &= 8\left[cos2(\tau-\sigma) + \frac{m}{M}cos4(\tau-\sigma)\right] \end{aligned}\right\} \tag{7}$$

where M and m are the Mach numbers for moving fluid and excite sound, respectively. Equation (5) characterizes the interaction between sound waves and the flow. It is well known that the viscosity term in the stress tensor $T^{(s)}$ affects sound propagation by adding sound absorption to the amplitude of sound waves [20–22] (cf. Equation (10) below). Furthermore, the remaining two terms in Equation (3) are associated exclusively with the nonlinear coefficients and are therefore neglected in the present analysis.

Due to the consideration of viscous absorption, there will be no long-term problem, so we can use the normal perturbation method to get the solution of Equation (5) which can be written as

$$\rho = \rho^{(0)} + \varepsilon\rho^{(1)} + \varepsilon^2\rho^{(2)} + ... = \sum\nolimits_{m_1} \varepsilon^{m_1}\rho^{(m_1)} \tag{8}$$

Substituting (8) into (5) yields

$$\begin{aligned} \varepsilon^0 :\ & \frac{\partial^2\rho^{(0)}}{\partial\tau^2} - \frac{\partial^2\rho^{(0)}}{\partial\sigma^2} = 0 \\ \varepsilon^{m_1} :\ & \frac{\partial^2\rho^{(m_1)}}{\partial\tau^2} - \frac{\partial^2\rho^{(m_1)}}{\partial\sigma^2} = g_1(\sigma,\tau)\frac{\partial^2\rho^{(m_1-1)}}{\partial\sigma^2} + g_2(\sigma,\tau)\frac{\partial\rho^{(m_1-1)}}{\partial\sigma} - g_3(\sigma,\tau)\rho^{(m_1-1)} \end{aligned} \tag{9}$$

and

$$\rho^{(0)} = e^{-\frac{\alpha}{k}\sigma} A_0 cos(\tau - \sigma) \tag{10}$$

In the following, we choose the cosine wave in Equation (10) as the zeroth order solution (the case of sine wave can be treated similarly). Substituting the cosine solution in Equation (10) into the second equation of Equation (9) yields approximately

$$\frac{\partial^2 \rho^{(1)}}{\partial \tau^2} - \frac{\partial^2 \rho^{(1)}}{\partial \sigma^2} \approx \sum\nolimits_{n=0}^{2} A_{2n+1}^{(1)} e^{-(2n+1)\frac{\alpha}{k}\sigma} cos[(2n+1)(\tau - \sigma)], \tag{11}$$

during the calculation, the approximation $\alpha << k$ was applied. In Equation (11), $A_{2n+1}^{(1)}$ should be (**Appendix B**)

$$\begin{gathered} A_1^{(1)} = -1^2 \left(\tfrac{M}{m} + \tfrac{m}{2M} + 1\right) A_0, \\ A_3^{(1)} = -3^2 \left(\tfrac{M}{m} + \tfrac{m}{4M}\right) A_0,\ A_5^{(1)} = -5^2 (\tfrac{m}{4M} A_0). \end{gathered} \tag{12}$$

By using the Lagrange parameter variation method, the solution of Equation (11) are obtained as follows (**Appendix C**)

$$\rho^{(1)}(\tau, \sigma) \approx \sum\nolimits_{n=0}^{2} A_{2n+1}^{(1)} e^{-(2n+1)\frac{\alpha}{k}\sigma} \frac{1}{(2n+1)^2} \{\frac{k}{2\alpha} sin[(2n+1)(\tau - \sigma)]\} \tag{13}$$

and we call it quasi-accumulation solution for $m_1 = 1$. Similarly, the solution for $m_1 = 2$ can be obtained as follows

$$\frac{\partial^2 \rho^{(2)}}{\partial \tau^2} - \frac{\partial^2 \rho^{(2)}}{\partial \sigma^2} = g_1(\sigma, \tau) \frac{\partial^2 \rho^{(1)}}{\partial \sigma^2} + g_2(\sigma, \tau) \frac{\partial \rho^{(1)}}{\partial \sigma} - g_3(\sigma, \tau) \rho^{(1)}. \tag{14}$$

By substituting (7) into (14) and utilizing the sum difference angle relationship of the trigonometric function, it can be seen that there are five quasi accumulation terms at the right end of Equation (14), that is

$$\frac{\partial^2 \rho^{(2)}}{\partial \tau^2} - \frac{\partial^2 \rho^{(2)}}{\partial \sigma^2} \approx \sum\nolimits_{n=0}^{4} e^{-(2n+1)\frac{\alpha}{k}\sigma} \{A_{2n+1}^{(2)} \left(\frac{k}{2\alpha}\right) sin(2n+1)(\tau - \sigma)\} \tag{15}$$

where the coefficients $A_{2n+1}^{(2)}$ satisfy (**Appendix D**)

$$\begin{gathered} A_1^{(2)} = -(\frac{M}{m} + \frac{m}{2M} - 1) A_1^{(1)} - \frac{1}{9} A_3^{(1)} - \frac{m}{2M} [-\frac{5}{3^2} A_3^{(1)} + \frac{13}{5^2} A_5^{(1)}] \\ A_3^{(2)} = -9 A_1^{(1)} + \frac{5m}{2M} A_1^{(1)} - (\frac{M}{m} + \frac{m}{2M}) A_3^{(1)} - \frac{3^2}{5^2} A_5^{(1)} \\ A_5^{(2)} = -\frac{13m}{2M} A_1^{(1)} - \frac{5^2}{3^2} A_3^{(1)} - (\frac{M}{m} + \frac{m}{2M}) A_5^{(1)} \\ A_7^{(2)} = -\frac{m}{2M} [\frac{29}{3^2} A_3^{(1)}] - [\frac{49}{5^2} A_5^{(1)}] \\ A_9^{(2)} = -\frac{m}{2M} [\frac{53}{5^2} A_5^{(1)}] \end{gathered} \tag{16}$$

By means of the Lagrange parameter variation method, the solution of (15) are obtained as follows (**Appendix E**)

$$\rho^{(2)} = \sum\nolimits_{n=0}^{4} \frac{1}{2(2n+1)^2} A_{2n+1}^{(2)} \left(\frac{k}{2\alpha}\right)^2 e^{-(2n+1)\frac{\alpha}{k}\sigma} cos[(2n+1)(\tau-\sigma)] \quad (17)$$

The term of quasi-accumulation solution $(k/2\alpha)^2$ appears in Equation (17). In addition, the Equation (16) shows us that $A_{2n+1}^{(1)}$ and $A_{2n+1}^{(2)}$ depend on both the Mach number of sound and the Mach number of flow. The results for $m_1 > 2$ can be obtained similarly.

3. Discussion

3.1. Period-doubling followed by the odd multiple half-period bifurcation

From the results above-mentioned, it is also evident that the sound propagation in this type of medium is a period-doubling followed by the odd multiple half-period bifurcation process. The original excitation wave is $v_0 e^{-2\frac{\alpha}{k}\sigma} cos2(\tau-\sigma)$ and the sub-harmonics are the combination of $e^{-(2n+1)\frac{\alpha}{k}\sigma} sin[(2n+1)(\tau-\sigma)]$ and $e^{-(2n+1)\frac{\alpha}{k}\sigma} cos[(2n+1)(\tau-\sigma)]$, $n = 0, 1, 2, ..., 2m_1$. And **Figure 1** describes the bifurcation process:

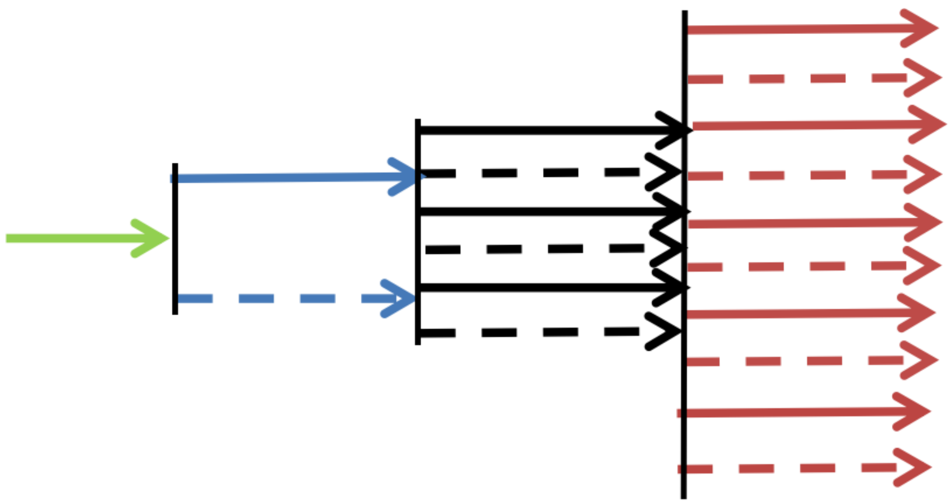

Figure 1. Bifurcation diagram.

The green thick arrow indicates the excitation wave $v_0 e^{-2\frac{\alpha}{k}\sigma} cos2(\tau-\sigma)$, while a pair of the blue (solid and dashed) arrows indicate the generated first sub-harmonic pair for $m_1 = 0$, the three pairs of black arrows indicate the generated sub-harmonic pairs for $m_1 = 1$, the five pairs of red arrows indicate the generated sub-harmonic pairs for $m_1 = 2$ and the bifurcation diagram for $m_1 > 2$ can be obtained similarly, where the solid arrows represent the cosine terms, while the dashed arrows represent the sine terms.

In the **Figure 1**, the green arrow represents the original excitation wave, while a pair of the blue (solid and dashed) arrows indicate the generated sub-harmonic pair for $m_1 = 0$, and its phase factor is $(\tau-\sigma)$; when $m_1 = 1$, the three pairs of black arrows indicate the generated sub-harmonic pairs, of which the phase factors are $(\tau-\sigma)$,

$3(\tau-\sigma)$ and $5(\tau-\sigma)$, respectively; when $m_1 = 2$, the five pairs of red arrows indicate the generated sub-harmonic pairs, of which the phase factors are 1, 3, 5, 7 and 9 times of the $(\tau-\sigma)$, respectively. the bifurcation diagram for *m* > 2 can be obtained.similarly, where the solid arrows represent the cosine terms, while the dashed arrows represent the sine terms.

From the bifurcation process, we can see that the bifurcation indicated by the first sub-harmonic pair represents period doubling, and the subsequent bifurcations differ from the period doubling bifurcation, so we refer to them as odd multiple half-period bifurcations. When $m_1 >> 1$, the numbers of the degree of freedom in the system are much greater than 1 and their representative quantity is traveling waves with the leading-term amplitude proportional to $\left[Mm\frac{k}{2\alpha}\right]^{m_1}$, which is equivalent to the instability factor in Landau conjecture, but not $e^{\gamma t}$. If the fluid sound absorption coefficient is caused by shear viscosity, then the absorption coefficient is classical absorption, that is

$$\alpha_s = \frac{2}{3}\frac{\omega^2}{\rho_0 c^3}\eta$$

It is easy to prove that the control parameters of the system should be

$$M^2\frac{k}{\alpha_s} = \frac{V_0^2}{c^2}\frac{\omega/c}{\frac{2}{3}\frac{\omega^2}{\rho_0 c^3}\eta} = \frac{3V_0(V_0/\omega)}{2\nu} = \frac{3}{2}\text{Re}$$

This suggests that the system's control parameter can be identified as the Reynolds number only when the sound absorption coefficient follows the classical absorption law. However, in most natural media, the sound absorption does not follow the classical absorption law. For instance, the sound absorption mechanism in atmosphere comes from the contributions of viscosity, thermal conduction, and thermal relaxation [20–22]. Therefore, based on the results of the above consideration and Equations (13) and (17), as well as comparisons with reference [23], the author believes: when studying bifurcation phenomena in fluids, the appropriate control parameter should be $\frac{k}{\alpha}$ instead of Reynolds number Re.

3.2. Flow amplifies sound

From the solutions of Equations (12), (13) and (16), (17), it can be seen that the amplitude of sub-harmonics generated by the nonlinear interaction of sound waves is directly proportional to two factors, one of which is

$$A_{2n+1}^{(m_1)}\left(\frac{k}{2\alpha}\right)^{m_1}$$

that were called quasi-accumulation solutions. When the frequency of sound wave is 100 Hz, k/α (for water) can reach the numerical range of 10^9 (Nepers)-1. Therefore, these terms in (13) and (17) are much larger than other terms. Another one is proportional to $A_{2n+1}^{(m_1)}[Mm]^{m_1}$ or dependent of two Mach numbers, which means that the interaction between sound and flow leads to the acoustic wave amplification in the moving fluid.

Amplification gain (Appendix F)

Define the gain of flow amplification sound

$$\beta_{2n+1}^{(m_1)} = \frac{\varepsilon^{m_1} A_{2n+1}^{(m_1)}}{\varepsilon^{m_1} A_{2n+1}^{(m_1)}\Big|_{M=0}}$$

The relationship between the amplification gain of the five sub-harmonics for $m_1 = 2$ and the flow-sound Mach number ratio was *M/m* calculated in **Appendix B**, that is

$$\left.\begin{aligned} \beta_1^{(2)} &= \frac{4}{5}\{[(M/m)^2 + \frac{1}{2} - M/m][(M/m)^2 + \frac{1}{2} + M/m)] \\ &+ M/m[(\frac{1}{4} + M/m)] + \frac{1}{2}[-5(\frac{1}{4} + \frac{M}{m}) + \frac{13}{4}]\} \\ \beta_3^{(2)} &= \frac{8}{9}[16M^3/m^3 - \frac{37}{4}M^2/m^2 - \frac{13}{2}M/m - \frac{9}{8}] \\ \beta_5^{(2)} &= \frac{8}{51}\{\frac{131}{4}(\frac{M}{m})^2 + \frac{51}{4}\frac{M}{m} + \frac{51}{8}\} = \frac{262}{51}(\frac{M}{m})^2 + 2\frac{M}{m} + 1 \\ \beta_7^{(2)} &= \frac{[\frac{29}{8} + (\frac{29}{2} + \frac{49}{4})M/m)]m^4 A_0}{\frac{29}{8}m^4 A_0} = 1 + (4 + \frac{98}{29})M/m \\ \beta_7^{(2)} &= 1 \end{aligned}\right\} \quad (18)$$

From (18) we can see that $\beta_1^{(2)}$ is the fourth degree polynomial of M/m, $\beta_3^{(2)}$ is the third degree polynomial of M/m, $\beta_5^{(2)}$ is the second degree polynomial of M/m, $\beta_7^{(2)}$ is the first degree polynomial of M/m and $\beta_9^{(2)}$ is independent of M/m. Numerical calculations were performed on Equation (18) to obtain the relationship between the amplification gains of the five sub-harmonics and the flow-sound Mach number ratio M/n. The results are shown in **Figure 2**,

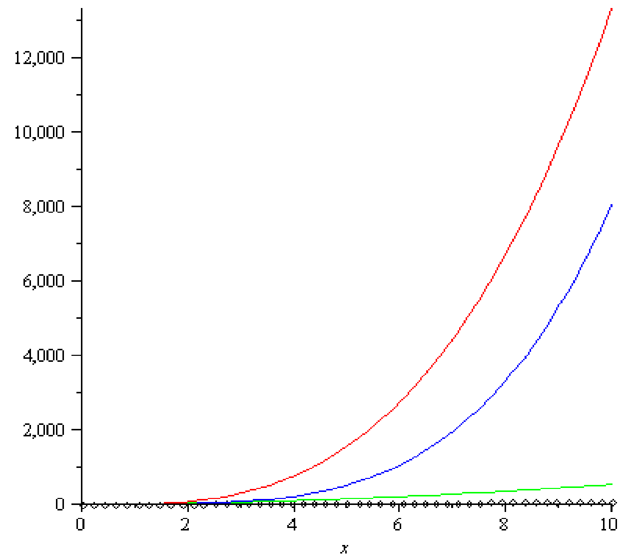

Figure 2. The relationship between the amplification gains of the five sub-harmonics and the Mach number ratio *M/m* for $m_1 = 2$.
Note: the horizontal axis in the figure is x = *M/m*, and the vertical axis is the amplification gain: red line: $\beta_3^{(2)}$, blue line: $\beta_1^{(2)}$, green line: $\beta_5^{(2)}$, dotted line: $\beta_7^{(2)}$, $\beta_9^{(2)} = 1$ (not shown in the figure).

When $M = 0$ and $M = 3m$, it can be calculated from Equation (18) that the amplitude of the first sub-harmonic in the quasi-accumulation solution is amplified by nearly 56 times due to the appearance of flow. Reviewing the experimental results of reference [24, 25] (**Figures 3** and **4**), the first sub-harmonic can be only observed in disturbed water, but not in stationary water. This paper gives it a powerful explanation.

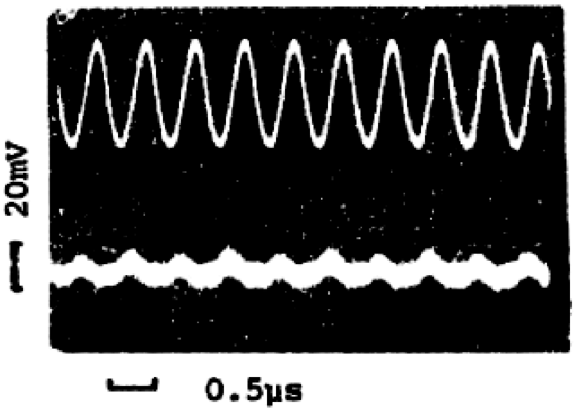

Figure 3. The signals in static medium.
Note: Fundamental frequency signal (upper); First sub-harmonic signal (lower); From **Figure 2** in Shao and Qian [24].

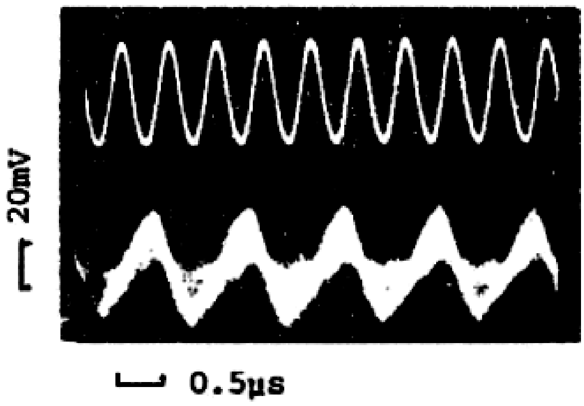

Figure 4. The signals in disturbed medium.
Note: Fundamental frequency signal (upper); First sub-harmonic signal (lower); From **Figure 3** in Shao and Qian [24].

Now let's study the situation for $m_1 = 1$.

$$\left.\begin{aligned} \beta_1^{(1)} &= (M/m)^2 + M/m + \frac{1}{2} \\ \beta_3^{(1)} &= 4(\frac{1}{4} + M/m) \\ \beta_5^{(1)} &= 1 \end{aligned}\right\} \tag{19}$$

Numerical calculations show that $\beta_1^{(1)}$ is a quadratic parabola, is $\beta_3^{(1)}$ a straight line, $\beta_5^{(1)}$ is independent of flow, and not magnified. From this, we can further infer that the amplification gain of $\varepsilon^4 A_1^{(4)}$ is an 8th degree polynomial of *M/m*, the amplification gain of $\varepsilon^4 A_3^{(4)}$ is a 7th degree polynomial of *M/m*, the amplification gain of $\varepsilon^4 A_5^{(4)}$ is a 6th degree polynomial of *M/m*, $\cdots$, the amplification gain of $\varepsilon^4 A_{15}^{(4)}$ is a straight line, and the amplification gain of $\varepsilon^4 A_{17}^{(4)}$ is 1, which is independent of the flow. From the results above-mentioned, it can be seen that the relationship between the amplification gain of sub-harmonics and flow velocity is a polynomial of *M/m*, of which the first sub-harmonic has the highest degree being 2.

3.3. Atmospheric turbulence suppression

When an aircraft is in flight, its plate and shell structures may begin to vibrate and radiate waves under certain airflow conditions. This vibration generates three types of waves: two longitudinal waves and one transverse wave, the latter of which experiences significant attenuation. The transverse wave is a viscous wave produced by the in-plane vibration of a flat plate [2]. Its propagation direction is perpendicular to the flow velocity, while the direction of particle velocity is parallel to the direction of flow velocity. On the other hand, one of two longitudinal waves propagates perpendicular to the direction of the flow velocity and does not produce flow-amplification-sound, while the other longitudinal wave propagates along the direction of the flow velocity and does produce flow-amplification-sound. The interaction between high-speed airflow and these waves leads to a cascade of bifurcation process. During this evolution, the wave amplitudes are amplified by the airflow, potentially developing into turbulence and thereby posing a significant hazard to the aircraft. To address this issue and enhance flight safety, we propose two measures. The first measure is to temporarily reduce the aircraft's flight speed, thereby directly mitigating the problem. Alternatively, from (13) and (17), it can be observed that the leading-term amplitude for the sub-harmonics is proportional to

$$A_{2n+1}^{(m_1)}\left(\frac{k}{2\alpha}\right)^{m_1} \propto \left[Mm\left(\frac{k}{2\alpha}\right)\right]^{m_1}$$

It is worth noting that in reference [23] the author studied the formation process of turbulence in cylindrical tubes, and the results showed that the *n*-order energy density in the tube is proportional to $(1/2\alpha R)^n$, where *R* is the radius of the tube. Comparing the results of them, it can be seen that both depend on the energy dissipation of the medium. As is well known, the sound absorption mechanism in the atmosphere involves three contributors: viscosity, thermal conduction, and thermal relaxation [20–22]. Thus, the second measure involves effectively increasing the absorption coefficient of the surrounding atmosphere. For example, raising the temperature around the aircraft can enhance both the viscosity and thermal conduction of the medium, thereby increasing the absorption coefficient. Additionally, adjusting environmental parameters such as temperature and humidity can shift the frequency position of the relaxation absorption peak where thermal relaxation predominates, further enhancing the absorption coefficient in that frequency range.

4. Conclusion

In this paper, the author applies the Lagrange parameter variation method to solve the differential equation that describes the interaction between sound waves and the flow, and obtains the highest growth terms (leading terms) in their quasi-accumulation solutions which are proportional to $[Mm\left(\frac{k}{\alpha}\right)]^{m_1}$. As m_1 increases, it tends towards infinity. The results indicate that a wave propagating in a moving fluid produces the period-doubling, followed by odd multiple half-period bifurcations of the sub-harmonics. The Mach number of the moving fluid generates amplification to the sound wave, and the process of the bifurcation driven by the nonlinear interaction between moving fluid and sound wave can proceed continuously. This process leads to a continuous increase in system's the number of degrees of freedom, ultimately approaching infinity. The obtained results also indicate that the appropriate control parameter for transitioning from bifurcation to chaos should be $\frac{k}{\alpha}$ instead of Reynolds number *Re*. As a result of the interaction between the flow and sound waves, the relationship between the amplification gain of the flow and the flow-sound Mach ratio is polynomial of *M/m*, of which the first sub-harmonic has the highest degree being 2.

Based on the findings presented in this article, we propose two suggestions to mitigate the aircraft's intense vertical movements: The first is to temporarily reduce the flight speed, which decrease the amplification effect of sub-harmonics and thereby helps alleviate the aircraft's movement. The second is to increase the sound absorption coefficient of the surrounding medium within the relevant frequency band, thereby reducing the amplitudes of the sub-harmonics and further stabilizes the aircraft.

Funding: This work received no external funding.

Institutional review board statement: Not applicable.

Informed consent statement: Not applicable.

Data availability statement: All data are presented in the work.

Conflict of interest: The author declares no conflict of interest.

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Appendix A

Lighthill equation

$$\frac{1}{C_0^2}\frac{\partial^2\rho}{\partial t^2}-\nabla^2\rho=\frac{1}{C_0^2}\frac{\partial^2 T_{ij}}{\partial x_i\partial x_j} \tag{A1}$$

$$T_{11}=\rho v^2, \tau=\omega t, \sigma=kx \tag{A2}$$

$$v=V_0+v_0cos2(\tau-\sigma) \tag{A3}$$

Substituting (A3) into the first expression of (A2) yields

$$T_{11}^{(1)}=\rho v^2=\rho\left\{V_0^2+\frac{1}{2}v_0^2+2V_0v_0cos2(\tau-\sigma)+\frac{1}{2}v_0^2cos4(\tau-\sigma)\right\} \tag{A4}$$

and

$$\frac{\partial^2\rho}{\partial\tau^2}-\frac{\partial^2\rho}{\partial\sigma^2}=\frac{1}{C_0^2}\frac{\partial^2 T_{11}}{\partial\sigma^2} \tag{A5}$$

$$\frac{1}{C_0^2}\frac{\partial^2 T_{11}}{\partial\sigma^2}=\frac{1}{C_0^2}\frac{\partial^2(\rho v^2)}{\partial\sigma^2}=\frac{1}{C_0^2}\{v^2\frac{\partial^2\rho}{\partial\sigma^2}+2\frac{\partial v^2}{\partial\sigma}\frac{\partial\rho}{\partial\sigma}+\rho\frac{\partial^2 v^2}{\partial\sigma^2}\} \tag{A6}$$

$$\begin{aligned}\frac{\partial v^2}{\partial\sigma} &= \frac{\partial}{\partial\sigma}\left\{V_0^2+\tfrac{1}{2}v_0^2+2V_0v_0cos2(\tau-\sigma)+\tfrac{1}{2}v_0^2cos4(\tau-\sigma)\right\}\\ &= 4V_0v_0sin2(\tau-\sigma)+2v_0^2sin4(\tau-\sigma)\\ \frac{\partial^2 v^2}{\partial\sigma^2} &= \frac{\partial^2}{\partial\sigma^2}\left\{V_0^2+\tfrac{1}{2}v_0^2+2V_0v_0cos2(\tau-\sigma)+\tfrac{1}{2}v_0^2cos4(\tau-\sigma)\right\}\\ &= -8V_0v_0cos2(\tau-\sigma)-8v_0^2cos2(\tau-\sigma)\end{aligned} \tag{A7}$$

Substituting them into (A6) to get

$$\begin{aligned}\frac{1}{C_0^2}\frac{\partial^2 T_{11}}{\partial\sigma^2} = Mm\{&[\tfrac{M}{m}+\tfrac{1}{2}\tfrac{m}{M}+2cos2(\tau-\sigma)+\tfrac{1}{2}\tfrac{m}{M}cos4(\tau-\sigma)]\frac{\partial^2\rho}{\partial\sigma^2}\\ &+4[2sin2(\tau-\sigma)+\tfrac{m}{M}sin4(\tau-\sigma)]\frac{\partial\rho}{\partial\sigma}\\ &-8[cos2(\tau-\sigma)+\tfrac{m}{M}cos4(\tau-\sigma)]\rho\ \}\end{aligned} \tag{A8}$$

Finally, the expressions (5)–(7) in the text can be obtained from (A8).

Appendix B

Due to acoustical absorption, Equations (4) and (7) should be modified as

$$v=V_0+v_0e^{-2\frac{\alpha}{k}\sigma}cos2(\tau-\sigma) \tag{A9}$$

$$\left.\begin{aligned}g_1(\sigma,\tau)&=\frac{M}{m}+\frac{m}{2M}+2e^{-2\frac{\alpha}{k}\sigma}cos2(\tau-\sigma)+\frac{m}{2M}e^{-4\frac{\alpha}{k}\sigma}cos4(\tau-\sigma)\\ g_2(\sigma,\tau)&=8e^{-2\frac{\alpha}{k}\sigma}sin2(\tau-\sigma)+4\frac{m}{M}e^{-4\frac{\alpha}{k}\sigma}sin4(\tau-\sigma)\\ g_3(\sigma,\tau)&=8\left[e^{-2\frac{\alpha}{k}\sigma}cos2(\tau-\sigma)+\frac{m}{M}e^{-4\frac{\alpha}{k}\sigma}cos4(\tau-\sigma)\right]\end{aligned}\right\} \tag{A10}$$

Substituting (A10) into (11) in the text to get

$$\left.\begin{aligned}&\frac{\partial^2\rho^{(1)}}{\partial\tau^2}-\frac{\partial^2\rho^{(1)}}{\partial\sigma^2}=g_1(\sigma,\tau)\frac{\partial^2\rho^{(0)}}{\partial\sigma^2}+g_2(\sigma,\tau)\frac{\partial\rho^{(0)}}{\partial\sigma}-g_3(\sigma,\tau)\rho^{(0)}\\ &\approx\{[\frac{M}{m}+\frac{m}{2M}+2e^{-2\frac{\alpha}{k}\sigma}cos2(\tau-\sigma)+\frac{m}{2M}e^{-4\frac{\alpha}{k}\sigma}cos4(\tau-\sigma)][-A_0e^{-\frac{\alpha}{k}\sigma}cos(\tau-\sigma)]\\ &+[8e^{-2\frac{\alpha}{k}\sigma}sin2(\tau-\sigma)+4\frac{m}{M}e^{-4\frac{\alpha}{k}\sigma}sin4(\tau-\sigma)]A_0e^{-\frac{\alpha}{k}\sigma}sin(\tau-\sigma)\\ &-8\left[e^{-2\frac{\alpha}{k}\sigma}cos2(\tau-\sigma)+\frac{m}{M}e^{-4\frac{\alpha}{k}\sigma}cos4(\tau-\sigma)\right][A_0e^{-\frac{\alpha}{k}\sigma}cos(\tau-\sigma)]\}\end{aligned}\right\} \tag{A11}$$

By using the trigonometry relationship and approximation $\alpha << k$, following expression can be approximately obtained

$$\begin{aligned}\frac{\partial^2\rho^{(1)}}{\partial\tau^2}-\frac{\partial^2\rho^{(1)}}{\partial\sigma^2}&\approx-(\frac{M}{m}+\frac{m}{2M}+1)A_0e^{-\frac{\alpha}{k}\sigma}cos(\tau-\sigma)-9(\frac{m}{4M}+1)e^{-3\frac{\alpha}{k}\sigma}A_0[+cos3(\tau-\sigma)]\\ &\quad-\frac{25m}{4M}A_0e^{-5\frac{\alpha}{k}\sigma}cos5(\tau-\sigma)\\ &=\{A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}cos(\tau-\sigma)+A_3^{(1)}e^{-3\frac{\alpha}{k}\sigma}cos3(\tau-\sigma)+A_5^{(1)}e^{-5\frac{\alpha}{k}\sigma}cos5(\tau-\sigma)\}\end{aligned}$$

where

$$A_1^{(1)}=-A_0(\frac{M}{m}+\frac{m}{2M}+1),\ A_3^{(1)}=-3^2A_0(\frac{m}{4M}+1),\ A_5^{(1)}=-5^2A_0\frac{m}{4M}$$

This is exactly the results of (11) and (12) in the text.

Appendix C

$$\frac{\partial^2\rho^{(1)}}{\partial\tau^2}-\frac{\partial^2\rho^{(1)}}{\partial\sigma^2}=-\sum_{n=0}^{2}A_{2n+1}^{(1)}e^{-(2n+1)\frac{\alpha}{k}\sigma}cos[(2n+1)(\tau-\sigma)]$$
$$=-\frac{1}{2}\sum_{n=0}^{2}A_{2n+1}^{(1)}[e^{i(2n+1)(\tau-\sigma)}+e^{-i(2n+1)(\tau-\sigma)}]e^{-(2n+1)\frac{\alpha}{k}\sigma}=-\frac{1}{2}\sum_{n=0}^{2}A_{2n+1}^{(1)}e^{i(2n+1)(\tau-\sigma)}+\text{c.c.} \tag{A12}$$
$$=-\frac{1}{2}A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}e^{i(\tau-\sigma)}+c.c.-\frac{1}{2}A_3^{(1)}e^{-3\frac{\alpha}{k}\sigma}e^{3j(\tau-\sigma)}+c.c.-\frac{1}{2}A_5^{(1)}e^{-5\frac{\alpha}{k}\sigma}e^{5j(\tau-\sigma)}+c.c.$$

(A12) can be written as three non-homogeneous equations, that is

$$\frac{\partial^2\rho^{(1,1)}}{\partial\tau^2}-\frac{\partial^2\rho^{(1,1)}}{\partial\sigma^2}=-\frac{1}{2}A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}e^{i(\tau-\sigma)}+c.c.$$
$$\frac{\partial^2\rho^{(1,3)}}{\partial\tau^2}-\frac{\partial^2\rho^{(1,3)}}{\partial\sigma^2}=-\frac{1}{2}A_3^{(1)}e^{-3\frac{\alpha}{k}\sigma}e^{i3(\tau-\sigma)}+c.c. \tag{A13}$$
$$\frac{\partial^2\rho^{(1,5)}}{\partial\tau^2}-\frac{\partial^2\rho^{(1,5)}}{\partial\sigma^2}=-\frac{1}{2}A_5^{(1)}e^{-5\frac{\alpha}{k}\sigma}e^{i5(\tau-\sigma)}+c.c.$$

of which the homogeneous equation and the solution can be

$$\frac{\partial^2\bar{\rho}^{(1,l)}}{\partial\tau^2}-\frac{\partial^2\bar{\rho}^{(1,l)}}{\partial\sigma^2}=0$$
$$\bar{\rho}^{(1,l)}=Be^{il(\tau-\sigma)} \tag{A14}$$
$$l=2n+1$$

and the spatial accumulation solutions of their non-homogeneous equations can be written as

$$\rho^{(1,l)}=B_1(\sigma)e^{il(\tau-\sigma)}+B_2(\sigma)e^{-il(\tau-\sigma)} \tag{A15}$$

$$\frac{\partial\rho^{(1,l)}}{\partial\sigma}=\frac{\partial}{\partial\sigma}[B_1(\sigma)e^{il(\tau-\sigma)}+B_2(\sigma)e^{-il(\tau-\sigma)}]=\frac{\partial B_1(\sigma)}{\partial\sigma}e^{il(\tau-\sigma)}-ilB_1(\sigma)e^{il(\tau-\sigma)}$$
$$+\frac{\partial B_2(\sigma)}{\partial\sigma}e^{-il(\tau-\sigma)}+ilB_2(\sigma)e^{-il(\tau-\sigma)} \tag{A16}$$

What we are seeking is only one unknown function, but in (A15) there are two unknown functions so that we can propose following constraint condition, that is

$$\frac{\partial B_1(\sigma)}{\partial\sigma}e^{il(\tau-\sigma)}+\frac{\partial B_2(\sigma)}{\partial\sigma}e^{-il(\tau-\sigma)}=0 \tag{A17}$$

Substituting it into (A16) yields

$$\frac{\partial\rho^{(1,l)}}{\partial\sigma}=-ilB_1(\sigma)e^{il(\tau-\sigma)}+ilB_2(\sigma)e^{-il(\tau-\sigma)} \tag{A18}$$

and

$$\frac{\partial^2\rho^{(1,l)}}{\partial\sigma^2}=\frac{\partial^2}{\partial\sigma^2}[B_1(\sigma)e^{il(\tau-\sigma)}+B_2(\sigma)e^{-il(\tau-\sigma)}]$$
$$=\frac{\partial}{\partial\sigma}[-ilB_1(\sigma)e^{il(\tau-\sigma)}+ilB_2(\sigma)e^{-il(\tau-\sigma)}]=-l^2B_1(\sigma)e^{il(\tau-\sigma)}-l^2B_2(\sigma)e^{-il(\tau-\sigma)} \tag{A19}$$
$$-il\frac{\partial B_1(\sigma)}{\partial\sigma}e^{il(\tau-\sigma)}+il\frac{\partial B_2(\sigma)}{\partial\sigma}e^{-il(\tau-\sigma)}$$

Similarly

$$\begin{aligned}\frac{\partial^2\rho^{(1,l)}}{\partial\tau^2} &= \frac{\partial^2}{\partial\tau^2}[B_1(\sigma)e^{il(\tau-\sigma)} + B_2(\sigma)e^{-il(\tau-\sigma)}] \\ &= -l^2[B_1(\sigma)e^{il(\tau-\sigma)} + B_2(\sigma)e^{-il(\tau-\sigma)}]\end{aligned} \tag{A20}$$

Substituting (A19) and (A20) into (A12) to get

$$\begin{aligned}\frac{\partial^2\rho^{(1,l)}}{\partial\tau^2} - \frac{\partial^2\rho^{(1,l)}}{\partial\sigma^2} &= -l^2B_1(\sigma)e^{il(\tau-\sigma)} - lB_2(\sigma)e^{-il(\tau-\sigma)} \\ &- il\frac{\partial B_1(\sigma)}{\partial\sigma}e^{il(\tau-\sigma)} + il\frac{\partial B_2(\sigma)}{\partial\sigma}e^{-il(\tau-\sigma)} \\ &- \{[-l^2B_1(\sigma)e^{il(\tau-\sigma)} - l^2B_2(\sigma)e^{-il(\tau-\sigma)}]\} \\ &= -il\frac{\partial B_1(\sigma)}{\partial\sigma}e^{il(\tau-\sigma)} + il\frac{\partial B_2(\sigma)}{\partial\sigma}e^{-il(\tau-\sigma)}\end{aligned} \tag{A21}$$

Combine (A21) with (A12) one has

$$\frac{\partial B_1(\sigma)}{\partial\sigma}e^{il(\tau-\sigma)} - \frac{\partial B_2(\sigma)}{\partial\sigma}e^{-il(\tau-\sigma)} = -\frac{1}{2il}A_l^{(1)}e^{-l\frac{\alpha}{k}\sigma}e^{il(\tau-\sigma)} \tag{A22}$$

By using (A17) and (A22), we can obtain

$$\frac{\partial B_1(\sigma)}{\partial\sigma}e^{il(\tau-\sigma)} = -\frac{1}{4il}A_l^{(1)}e^{-l\frac{\alpha}{k}\sigma}e^{il(\tau-\sigma)} \tag{A23}$$

$$\frac{\partial B_2(\sigma)}{\partial\sigma}e^{-il(\tau-\sigma)} = \frac{1}{2il}A_l^{(1)}e^{-l\frac{\alpha}{k}\sigma+il(\tau-\sigma)} \tag{A24}$$

Integrate them to get

$$B_1(\sigma) = -\frac{1}{4il}A_l^{(1)}\int e^{-l\frac{\alpha}{k}\sigma}d\sigma = A_l^{(1)}\frac{1}{4il^2}\frac{k}{\alpha} \tag{A25}$$

and

$$\begin{aligned}B_2(\sigma) &= \tfrac{1}{2il}A_l\int e^{-l\frac{\alpha}{k}\sigma+i2l(\tau-\sigma)}d\sigma = \tfrac{1}{2il}A_l^{(1)}e^{-l\frac{\alpha}{k}\sigma+i2l(\tau-\sigma)} \\ &\approx \tfrac{1}{2il}lA_{2n+1}^{(1)}e^{-l\frac{\alpha}{k}\sigma+i2l(\tau-\sigma)} = \tfrac{1}{4l^2}A_l^{(1)}e^{-l\frac{\alpha}{k}\sigma+i2l(\tau-\sigma)}\end{aligned} \tag{A26}$$

$$\begin{aligned}\rho^{(1,l)} &= B_1(\sigma)e^{il(\tau-\sigma)} + B_2(\sigma)e^{-il(\tau-\sigma)} \\ &= A_l^{(1)}\tfrac{1}{4il^2}\tfrac{k}{\alpha}e^{-l\frac{\alpha}{k}\sigma}e^{il(\tau-\sigma)} + \tfrac{1}{4l^2}A_l^{(1)}e^{-l\frac{\alpha}{k}\sigma+il(\tau-\sigma)} \\ &= A_l\tfrac{1}{4l^2}e^{-l\frac{\alpha}{k}\sigma}e^{i2l(\tau-\sigma)}(1-i\tfrac{k}{\alpha})\end{aligned} \tag{A27}$$

of which the conjugate value is

$$\rho^{(1,l)*} = A_l^{(1)}\frac{1}{4l^2}e^{-l\frac{\alpha}{k}\sigma}e^{-il(\tau-\sigma)}(1+i\frac{k}{\alpha}) \tag{A28}$$

After adding the two terms and then summing them up and taking an approximation $\frac{k}{\alpha} >> 1$ one has

$$\begin{aligned}\rho^{(1)} = \sum_{n=0}^{2}[\rho^{(1,2n+1)} + \rho^{(1,2n+1)*}] &\approx \sum_{n=0}^{2}A_{2n+1}^{(1)}\tfrac{1}{4(2n+1)^2}e^{-(2n+1)\frac{\alpha}{k}\sigma}e^{i(2n+1)(\tau-\sigma)}(-i\tfrac{k}{\alpha}) \\ &+ \sum_{n=0}^{2}A_{2n+1}^{(1)}\tfrac{1}{4(2n+1)^2}e^{-(2n+1)\frac{\alpha}{k}\sigma}e^{-i(2n+1)(\tau-\sigma)}(i\tfrac{k}{\alpha}) \\ &= \sum_{n=0}^{2}A_{2n+1}^{(1)}\tfrac{1}{(2n+1)^2}e^{-(2n+1)\frac{\alpha}{k}\sigma}\{\tfrac{k}{2\alpha}sin[(2n+1)(\tau-\sigma)]\}\end{aligned} \tag{A29}$$

which is Equation (13) in the main text.

Appendix D

$$\varepsilon^2:\frac{\partial^2\rho^{(2)}}{\partial\tau^2}-\frac{\partial^2\rho^{(2)}}{\partial\sigma^2}=g_1(\sigma,\tau)\frac{\partial^2\rho^{(1)}}{\partial\sigma^2}+g_2(\sigma,\tau)\frac{\partial\rho^{(1)}}{\partial\sigma}-g_3(\sigma,\tau)\rho^{(1)} \tag{A30}$$

where

$$\left.\begin{aligned} g_1(\sigma,\tau)&=\frac{M}{m}+\frac{m}{2M}+2cos2(\tau-\sigma)+\frac{m}{2M}cos4(\tau-\sigma)\\ g_2(\sigma,\tau)&=8sin2(\tau-\sigma)+4\frac{m}{M}sin4(\tau-\sigma)\\ g_3(\sigma,\tau)&=8\left[cos2(\tau-\sigma)+\frac{m}{M}cos4(\tau-\sigma)\right]\end{aligned}\right\} \tag{A31}$$

In **Appendix C** following results were obtained, that is

$$\begin{aligned}\rho^{(1)}=\textstyle\sum_{n=0}^{2}A_{2n+1}^{(1)}\frac{1}{4(2n+1)^2}e^{-(2n+1)\frac{\alpha}{k}\sigma}\{2cos[(2n+1)(\tau-\sigma)]\\ +2\frac{k}{\alpha}sin[(2n+1)(\tau-\sigma)]\}\approx\frac{k}{\alpha}\textstyle\sum_{n=0}^{2}A_{2n+1}^{(1)}\frac{1}{2(2n+1)^2}e^{-(2n+1)\frac{\alpha}{k}\sigma}sin[(2n+1)(\tau-\sigma)]\}\end{aligned} \tag{A32}$$

Let's calculate the relevant quantities under approximate condition $\alpha<<k$. In this and the following calculations, summation signs will be temporarily omitted until the calculation is completed, at which point they will be restored.

$$\begin{aligned}\frac{\partial\rho^{(1)}}{\partial\sigma}&\approx\frac{k}{\alpha}\frac{e^{-(2n+1)\frac{\alpha}{k}\sigma}\partial\{A_{2n+1}^{(1)}\frac{1}{2(2n+1)^2}sin[(2n+1)(\tau-\sigma)]\}}{\partial\sigma}\\ &=-\frac{k}{\alpha}A_{2n+1}^{(1)}\frac{1}{2(2n+1)}e^{-(2n+1)\frac{\alpha}{k}\sigma}cos[(2n+1)(\tau-\sigma)]\end{aligned} \tag{A33}$$

$$\frac{\partial^2\rho^{(1)}}{\partial\sigma^2}\approx-A_{2n+1}^{(1)}\frac{k}{2\alpha}e^{-(2n+1)\frac{\alpha}{k}\sigma}sin[(2n+1)(\tau-\sigma)]=-A_{2n+1}^{(1)}\frac{k}{2\alpha}e^{-(2n+1)\frac{\alpha}{k}\sigma}s_{2n+1} \tag{A34}$$

For the convenience of writing, define symbols

$$sin(nx)=s_n,cos(nx)=c_n$$

then one has

$$\begin{aligned}g_1(\sigma,\tau)\frac{\partial^2\rho^{(1)}}{\partial\sigma^2}&=[\frac{M}{m}+\frac{m}{2M}+2cos2(\tau-\sigma)+\frac{m}{2M}cos4(\tau-\sigma)]\cdot(-A_{2n+1}^{(1)}\frac{k}{2\alpha}e^{-(2n+1)\frac{\alpha}{k}\sigma}s_{2n+1})\\ &=-A_{2n+1}^{(1)}e^{-(2n+1)\frac{\alpha}{k}\sigma}\frac{k}{2\alpha}[\frac{M}{m}+\frac{m}{2M}+2cos2(\tau-\sigma)+\frac{m}{2M}cos4(\tau-\sigma)]\cdot(s_{2n+1})\\ &=-A_{2n+1}^{(1)}e^{-(2n+1)\frac{\alpha}{k}\sigma}\frac{k}{2\alpha}\{(\frac{M}{m}+\frac{m}{2M})s_{2n+1}+2c_2s_{2n+1}+\frac{m}{2M}c_4s_{2n+1}\}\end{aligned}$$

$$\begin{aligned}g_2(\sigma,\tau)\frac{\partial\rho^{(1)}}{\partial\sigma}&=\left[8sin2(\tau-\sigma)+4\frac{m}{M}sin4(\tau-\sigma)\right]\left\{e^{-(2n+1)\frac{\alpha}{k}\sigma}\left[-A_{2n+1}^{(1)}\frac{1}{2(2n+1)}\frac{k}{\alpha}cos(2n+1)(\tau-\sigma)\right]\right\}\\ &=A_{2n+1}^{(1)}\frac{1}{2(2n+1)}e^{-(2n+1)\frac{\alpha}{k}\sigma}\left\{\left(-\frac{8k}{\alpha}s_2c_{2n+1}\right)+4\frac{m}{M}\left(-\frac{k}{\alpha}s_4c_{2n+1}\right)\right\}\end{aligned}$$

$$\begin{aligned}g_3(\sigma,\tau)\rho^{(1)}&=8\left[cos2(\tau-\sigma)+\frac{m}{M}cos4(\tau-\sigma)\right]\cdot\\ &\{A_{2n+1}^{(1)}\frac{1}{2(2n+1)^2}e^{-(2n+1)\frac{\alpha}{k}\sigma}[\frac{k}{\alpha}sin(2n+1)(\tau-\sigma)]\}\\ &=A_{2n+1}^{(1)}\frac{8}{2(2n+1)^2}e^{-(2n+1)\frac{\alpha}{k}\sigma}[(\frac{k}{\alpha}c_2s_{2n+1})+\frac{m}{M}(\frac{k}{\alpha}c_4s_{2n+1})]\end{aligned}$$

Add up the three results above to obtain

$$\begin{aligned} & g_1(\sigma,\tau)\tfrac{\partial^2\rho^{(1)}}{\partial\sigma^2}+g_2(\sigma,\tau)\tfrac{\partial\rho^{(1)}}{\partial\sigma}-g_3(\sigma,\tau)\rho^{(1)} \\ &=[-A^{(1)}_{2n+1}\tfrac{k}{2\alpha}e^{-(2n+1)\frac{\alpha}{k}\sigma}][(\tfrac{M}{m}+\tfrac{m}{2M})(s_{2n+1})+2c_2s_{2n+1}+\tfrac{m}{2M}c_4s_{2n+1}] \\ &+A^{(1)}_{2n+1}\tfrac{1}{2n+1}\tfrac{k}{2\alpha}e^{-(2n+1)\frac{\alpha}{k}\sigma}[(-8s_2c_{2n+1})+4\tfrac{m}{M}(-s_4c_{2n+1})] \\ &-A^{(1)}_{2n+1}\tfrac{8}{(2n+1)^2}\tfrac{k}{2\alpha}e^{-(2n+1)\frac{\alpha}{k}\sigma}[c_2s_{2n+1}+\tfrac{m}{M}(c_4s_{2n+1})] \\ &=-A^{(1)}_{2n+1}\tfrac{k}{2\alpha}e^{-(2n+1)\frac{\alpha}{k}\sigma}[(\tfrac{M}{m}+\tfrac{m}{2M})s_{2n+1}+2c_2s_{2n+1}+\tfrac{m}{2M}(c_4s_{2n+1})] \\ &+A^{(1)}_{2n+1}\tfrac{1}{(2n+1)}\tfrac{k}{2\alpha}e^{-(2n+1)\frac{\alpha}{k}\sigma}[-8s_2c_{2n+1}+4\tfrac{m}{M}(-s_4c_{2n+1})] \\ &-A^{(1)}_{2n+1}\tfrac{8}{(2n+1)^2}\tfrac{k}{2\alpha}e^{-(2n+1)\frac{\alpha}{k}\sigma}[c_2s_{2n+1}+\tfrac{m}{M}(c_4s_{2n+1})] \end{aligned} \tag{A35}$$

Using the sum difference relationship in trigonometry (A35) becomes

$$\begin{aligned} &=-A^{(1)}_{2n+1}\tfrac{k}{2\alpha}e^{-(2n+1)\frac{\alpha}{k}\sigma}[(\tfrac{M}{m}+\tfrac{m}{2M})s_{2n+1}+2c_2s_{2n+1}+\tfrac{m}{2M}(c_4s_{2n+1})] \\ &+A^{(1)}_{2n+1}\tfrac{1}{(2n+1)}\tfrac{k}{2\alpha}e^{-(2n+1)\frac{\alpha}{k}\sigma}[-8s_2c_{2n+1}+4\tfrac{m}{M}(-s_4c_{2n+1})] \\ &-A^{(1)}_{2n+1}\tfrac{8}{(2n+1)^2}\tfrac{k}{2\alpha}e^{-(2n+1)\frac{\alpha}{k}\sigma}[c_2s_{2n+1}+\tfrac{m}{M}(c_4s_{2n+1})] \end{aligned}$$

$$\begin{aligned} & g_1(\sigma,\tau)\tfrac{\partial^2\rho^{(1)}}{\partial\sigma^2}+g_2(\sigma,\tau)\tfrac{\partial\rho^{(1)}}{\partial\sigma}-g_3(\sigma,\tau)\rho^{(1)} \\ &=[-A^{(1)}_{2n+1}\tfrac{k}{2\alpha}e^{-(2n+1)\frac{\alpha}{k}\sigma}]\{(\tfrac{M}{m}+\tfrac{m}{2M})(s_{2n+1})+2[\tfrac{1}{2}(s_{2n+3}+s_{2n-1})] \\ &+\tfrac{m}{2M}\cdot[\tfrac{1}{2}(s_{2n+5}+s_{2n-3})]\} \\ &+A^{(1)}_{2n+1}\tfrac{1}{(2n+1)}\tfrac{k}{2\alpha}e^{-(2n+1)\frac{\alpha}{k}\sigma}\{-8\cdot\tfrac{1}{2}(s_{2n+3}-s_{2n-1})+4\tfrac{m}{M}[-\tfrac{1}{2}(s_{2n+5}-s_{2n-3})]\} \\ &-A^{(1)}_{2n+1}\tfrac{8}{(2n+1)^2}\tfrac{k}{2\alpha}e^{-(2n+1)\frac{\alpha}{k}\sigma}\{[\tfrac{1}{2}(s_{2n+3}+s_{2n-1})+\tfrac{m}{M}[\tfrac{1}{2}(s_{2n+5}+s_{2n-3})]\} \\ &+\text{“}cos\text{”}terms << \text{“}sin\text{”}terms \end{aligned} \tag{A36}$$

By restoring the summation signs the Equation (A36) can be written as the following seven terms:

The first term

$$\begin{aligned} &\textstyle\sum_{n=0}^{2}A^{(1)}_{2n+1}\tfrac{k}{2\alpha}e^{-(2n+1)\frac{\alpha}{k}\sigma}\{-(\tfrac{M}{m}+\tfrac{m}{2M})s_{2n+1}\}= \\ &-(\tfrac{M}{m}+\tfrac{m}{2M})\tfrac{k}{2\alpha}\{A^{(1)}_1e^{-\frac{\alpha}{k}\sigma}s_1+A^{(1)}_3e^{-\frac{3\alpha}{k}\sigma}s_3+A^{(1)}_5e^{-\frac{5\alpha}{k}\sigma}s_5\} \end{aligned}$$

The second term

$$\begin{aligned} &-\sum_{n=0}^{2}A^{(1)}_{2n+1}e^{-(2n+1)\frac{\alpha}{k}\sigma}\tfrac{k}{2\alpha}(s_{2n+3}+s_{2n-1})=-A^{(1)}_1e^{-\frac{\alpha}{k}\sigma}\tfrac{k}{2\alpha}(s_3-s_1)- \\ &A^{(1)}_3e^{-\frac{3\alpha}{k}\sigma}\tfrac{k}{2\alpha}(s_5+s_1)-A^{(1)}_5e^{-\frac{5\alpha}{k}\sigma}\tfrac{k}{2\alpha}(s_7+s_3) \\ &=-A^{(1)}_1e^{-\frac{\alpha}{k}\sigma}\tfrac{k}{2\alpha}s_3-A^{(1)}_1\tfrac{k}{2\alpha}e^{-\frac{\alpha}{k}\sigma}(-s_1)-A^{(1)}_3e^{-\frac{3\alpha}{k}\sigma}\tfrac{k}{2\alpha}s_1-A^{(1)}_3e^{-\frac{3\alpha}{k}\sigma}\tfrac{k}{2\alpha}s_5- \\ &A^{(1)}_5e^{-\frac{5\alpha}{k}\sigma}\tfrac{k}{2\alpha}s_3-A^{(1)}_5e^{-\frac{5\alpha}{k}\sigma}\tfrac{k}{2\alpha}s_7 \\ &\approx A^{(1)}_1\tfrac{k}{2\alpha}e^{-\frac{\alpha}{k}\sigma}s_1-A^{(1)}_3e^{-\frac{\alpha}{k}\sigma}\tfrac{k}{2\alpha}s_1-A^{(1)}_1e^{-\frac{3\alpha}{k}\sigma}\tfrac{k}{2\alpha}s_3-A^{(1)}_5e^{-\frac{3\alpha}{k}\sigma}\tfrac{k}{2\alpha}s_3-A^{(1)}_3e^{-\frac{5\alpha}{k}\sigma}\tfrac{k}{2\alpha}s_5-A^{(1)}_5e^{-\frac{7\alpha}{k}\sigma}\tfrac{k}{2\alpha}s_7 \\ &\approx\tfrac{k}{2\alpha}[A^{(1)}_1-A^{(1)}_3]e^{-\frac{\alpha}{k}\sigma}s_1-\tfrac{k}{2\alpha}[A^{(1)}_1+A^{(1)}_5]e^{-\frac{3\alpha}{k}\sigma}s_3-A^{(1)}_3e^{-\frac{5\alpha}{k}\sigma}\tfrac{k}{2\alpha}s_5-A^{(1)}_5e^{-\frac{7\alpha}{k}\sigma}\tfrac{k}{2\alpha}s_7 \end{aligned}$$

The third term

$$-\sum_{n=0}^{2}A_{2n+1}^{(1)}e^{-(2n+1)\frac{\alpha}{k}\sigma}\frac{k}{2\alpha}\{\frac{m}{2M}[(s_{2n+5}+s_{2n-3})]\}=-\frac{m}{2M}\frac{k}{2\alpha}\{[A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}(s_5-s_3)]$$
$$+[A_3^{(1)}e^{-\frac{3\alpha}{k}\sigma}(s_7-s_1)]+[A_5^{(1)}e^{-\frac{5\alpha}{k}\sigma}(s_9+s_1)]\}$$
$$=-\frac{m}{2M}\frac{k}{2\alpha}\{[A_5^{(1)}e^{-\frac{5\alpha}{k}\sigma}s_1]+[A_3^{(1)}e^{-\frac{3\alpha}{k}\sigma}(-s_1)]+[A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}(-s_3)]$$
$$+[A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}s_5]+[A_3^{(1)}e^{-\frac{3\alpha}{k}\sigma}s_7]+[A_5^{(1)}e^{-\frac{5\alpha}{k}\sigma}s_9]\}$$
$$\approx-\frac{m}{2M}\frac{k}{2\alpha}\{[A_5^{(1)}-A_3^{(1)}]e^{-\frac{\alpha}{k}\sigma}s_1-A_1^{(1)}e^{-\frac{3\alpha}{k}\sigma}s_3+A_1^{(1)}e^{-\frac{5\alpha}{k}\sigma}s_5+A_3^{(1)}e^{-\frac{7\alpha}{k}\sigma}s_7+A_5^{(1)}e^{-\frac{9\alpha}{k}\sigma}s_9\}$$

The fourth term

$$\sum_{n=0}^{2}A_{2n+1}^{(1)}\frac{1}{2(2n+1)}e^{-(2n+1)\frac{\alpha}{k}\sigma}\{-\frac{8k}{2\alpha}(s_{2n+3}-s_{2n-1})$$
$$=A_1^{(1)}\frac{k}{2\alpha}e^{-\frac{\alpha}{k}\sigma}\{-\frac{4}{1}(s_3+s_1)\}+A_3^{(1)}\frac{4}{3}(\frac{k}{2\alpha})e^{-\frac{3\alpha}{k}\sigma}\{-(s_5-s_1)\}+A_5^{(1)}\frac{4}{5}(\frac{k}{2\alpha})e^{-\frac{5\alpha}{k}\sigma}\{-(s_7-s_3)\}$$
$$=A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}\frac{k}{2\alpha}\frac{4}{1}\{-s_3\}+A_1^{(1)}\frac{k}{2\alpha}e^{-\frac{\alpha}{k}\sigma}\frac{4}{1}\{-s_1\}-A_3^{(1)}\frac{4}{3}(\frac{k}{2\alpha})e^{-\frac{3\alpha}{k}\sigma}s_5+A_3^{(1)}\frac{4}{3}\frac{k}{2\alpha}e^{-\frac{3\alpha}{k}\sigma}\{s_1\}$$
$$+A_5^{(1)}\frac{4}{5}\frac{k}{2\alpha}e^{-\frac{5\alpha}{k}\sigma}\{-s_7\}+A_5^{(1)}\frac{4}{5}\frac{k}{2\alpha}e^{-\frac{5\alpha}{k}\sigma}\{s_3\}$$

$$=A_3^{(1)}\frac{4}{3}\frac{k}{2\alpha}e^{-\frac{3\alpha}{k}\sigma}\{s_1\}+A_1^{(1)}\frac{k}{2\alpha}e^{-\frac{\alpha}{k}\sigma}\frac{4}{1}\{-s_1\}+A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}\frac{k}{2\alpha}\frac{4}{1}\{-s_3\}$$
$$+A_5^{(1)}\frac{4}{5}\frac{k}{2\alpha}e^{-\frac{5\alpha}{k}\sigma}s_3-A_3^{(1)}\frac{4}{3}(\frac{k}{2\alpha})e^{-\frac{3\alpha}{k}\sigma}s_5++A_5^{(1)}\frac{4}{5}\frac{k}{2\alpha}e^{-\frac{5\alpha}{k}\sigma}\{-s_7\}$$
$$=\frac{k}{2\alpha}\{A_3^{(1)}\frac{4}{3}e^{-\frac{3\alpha}{k}\sigma}s_1-A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}\frac{4}{1}s_1-A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}\frac{4}{1}s_3$$
$$+A_5^{(1)}\frac{4}{5}e^{-\frac{5\alpha}{k}\sigma}s_3-A_3^{(1)}\frac{4}{3}e^{-\frac{3\alpha}{k}\sigma}s_5-A_5^{(1)}\frac{4}{5}e^{-\frac{5\alpha}{k}\sigma}s_7$$
$$\approx\frac{k}{2\alpha}\{[-A_1^{(1)}\frac{4}{1}+A_3^{(1)}\frac{4}{3}](e^{-\frac{\alpha}{k}\sigma}s_1)\text{-}[A_1^{(1)}\frac{4}{1}-A_5^{(1)}\frac{4}{5}](e^{-\frac{3\alpha}{k}\sigma}s_3)-A_3^{(1)}\frac{4}{3}(e^{-\frac{5\alpha}{k}\sigma}s_5)-A_5^{(1)}\frac{4}{5}e^{-\frac{7\alpha}{k}\sigma}s_7\}$$

The fifth term

$$-\sum_{n=0}^{2}A_{2n+1}^{(1)}\frac{1}{(2n+1)}e^{-(2n+1)\frac{\alpha}{k}\sigma}\frac{m}{M}[\frac{k}{\alpha}(s_{2n+5}-s_{2n-3})]$$
$$=-\frac{k}{\alpha}\frac{m}{M}\{[A_1^{(1)}\frac{1}{1}e^{-\frac{\alpha}{k}\sigma}(s_5+s_3)]+[A_3^{(1)}\frac{1}{3}e^{-\frac{3\alpha}{k}\sigma}(s_7+s_1)]+[A_5^{(1)}\frac{1}{5}e^{-\frac{5\alpha}{k}\sigma}(s_9-s_1)]\}$$
$$=-\frac{k}{\alpha}\frac{m}{M}[A_1^{(1)}\frac{1}{1}e^{-\frac{\alpha}{k}\sigma}s_5]-\frac{k}{\alpha}\frac{m}{M}[A_1^{(1)}\frac{1}{1}e^{-\frac{\alpha}{k}\sigma}s_3]-\frac{k}{\alpha}\frac{m}{M}[A_3^{(1)}\frac{1}{3}e^{-\frac{3\alpha}{k}\sigma}s_7]$$
$$-\frac{k}{\alpha}\frac{m}{M}[A_3^{(1)}\frac{1}{3}e^{-\frac{3\alpha}{k}\sigma}s_1]-\frac{k}{\alpha}\frac{m}{M}[A_5^{(1)}\frac{1}{5}e^{-\frac{5\alpha}{k}\sigma}s_9]+(\frac{k}{\alpha}\frac{m}{M})[A_5^{(1)}\frac{1}{5}e^{-\frac{5\alpha}{k}\sigma}s_1]$$
$$=-\frac{k}{\alpha}\frac{m}{M}\{[A_3^{(1)}\frac{1}{3}e^{-\frac{3\alpha}{k}\sigma}-A_5^{(1)}\frac{1}{5}e^{-\frac{5\alpha}{k}\sigma}]s_1+[A_1^{(1)}\frac{1}{1}e^{-\frac{\alpha}{k}\sigma}s_3]$$
$$+[A_1^{(1)}\frac{1}{1}e^{-\frac{\alpha}{k}\sigma}s_5]+[A_3^{(1)}\frac{1}{3}e^{-\frac{3\alpha}{k}\sigma}s_7]+[A_5^{(1)}\frac{1}{5}e^{-\frac{5\alpha}{k}\sigma}s_9]\}$$
$$\approx-\frac{k}{2\alpha}\frac{m}{M}\{[A_3^{(1)}\frac{2}{3}-A_5^{(1)}\frac{2}{5}]e^{-\frac{\alpha}{k}\sigma}s_1+2A_1^{(1)}e^{-\frac{3\alpha}{k}\sigma}s_3+2A_1^{(1)}e^{-\frac{5\alpha}{k}\sigma}s_5+A_3^{(1)}\frac{2}{3}e^{-\frac{7\alpha}{k}\sigma}s_7+A_5^{(1)}\frac{2}{5}e^{-\frac{9\alpha}{k}\sigma}s_9\}$$

The The sixth term

$$-\frac{k}{2\alpha}\sum_{n=0}^{2}A_{2n+1}^{(1)}\frac{8}{2(2n+1)^2}e^{-(2n+1)\frac{\alpha}{k}\sigma}[(s_{2n+3}+s_{2n-1})]$$
$$=-\frac{k}{2\alpha}\cdot4\{A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}[(s_3-s_1)]+A_3^{(1)}\frac{1}{9}e^{-\frac{3\alpha}{k}\sigma}[(s_5+s_1)]+A_5^{(1)}\frac{1}{25}e^{-\frac{5\alpha}{k}\sigma}[(s_7+s_3)]\}$$
$$=-\frac{k}{2\alpha}\cdot4\{A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}s_3+A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}(-s_1)+A_3^{(1)}\frac{1}{9}e^{-\frac{3\alpha}{k}\sigma}s_5+A_3^{(1)}\frac{1}{9}e^{-\frac{3\alpha}{k}\sigma}s_1+A_5^{(1)}\frac{1}{25}e^{-\frac{5\alpha}{k}\sigma}s_7+A_5^{(1)}\frac{1}{25}e^{-\frac{5\alpha}{k}\sigma}s_3\}$$
$$=-\frac{k}{2\alpha}\cdot4\{A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}(-s_1)+A_3^{(1)}\frac{1}{9}e^{-\frac{3\alpha}{k}\sigma}s_1+A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}s_3+A_5^{(1)}\frac{1}{25}e^{-\frac{5\alpha}{k}\sigma}s_3+A_3^{(1)}\frac{1}{9}e^{-\frac{3\alpha}{k}\sigma}s_5+A_5^{(1)}\frac{1}{25}e^{-\frac{5\alpha}{k}\sigma}s_7\}$$
$$=-\frac{k}{2\alpha}\cdot4\{(-A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}+A_3^{(1)}\frac{1}{9}e^{-\frac{3\alpha}{k}\sigma})s_1+(A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}+A_5^{(1)}\frac{1}{25}e^{-\frac{5\alpha}{k}\sigma})s_3+A_3^{(1)}\frac{1}{9}e^{-\frac{3\alpha}{k}\sigma}s_5+A_5^{(1)}\frac{1}{25}e^{-\frac{5\alpha}{k}\sigma}s_7$$
$$\approx-\frac{2k}{\alpha}\{[-A_1^{(1)}\frac{1}{1^2}+A_3^{(1)}\frac{1}{3^2}]e^{-\frac{\alpha}{k}\sigma}s_1+[A_1^{(1)}\frac{1}{1^2}+A_5^{(1)}\frac{1}{5^2}]e^{-\frac{3\alpha}{k}\sigma}s_3+[A_3^{(1)}\frac{1}{3^2}e^{-\frac{5\alpha}{k}\sigma}s_5]+[A_5^{(1)}\frac{1}{5^2}e^{-\frac{7\alpha}{k}\sigma}s_7]\}$$

The seventh term

$$-\frac{k}{2\alpha}\sum_{n=0}^{2}A_{2n+1}^{(1)}\frac{1}{(2n+1)^2}e^{-(2n+1)\frac{\alpha}{k}\sigma}\{(s_{2n+5}+s_{2n-3})\}$$

$$=-\frac{k}{2\alpha}\frac{4m}{M}\{A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}\{(s_5-s_3)+A_3^{(1)}\frac{1}{3^2}e^{-\frac{3\alpha}{k}\sigma}\{(s_7-s_1)+A_{2n+1}^{(1)}\frac{1}{5^2}e^{-\frac{5\alpha}{k}\sigma}\{(s_9+s_1)\}$$

$$=-\frac{k}{2\alpha}\frac{4m}{M}\{A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}s_5-A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}s_3+A_3^{(1)}\frac{1}{3^2}e^{-\frac{3\alpha}{k}\sigma}s_7-A_3^{(1)}\frac{1}{3^2}e^{-\frac{3\alpha}{k}\sigma}s_1+A_5^{(1)}\frac{1}{5^2}e^{-\frac{5\alpha}{k}\sigma}s_9+A_5^{(1)}\frac{1}{5^2}e^{-\frac{5\alpha}{k}\sigma}s_1\}$$

$$=-\frac{k}{2\alpha}\frac{4m}{M}\{-A_3^{(1)}\frac{1}{3^2}e^{-\frac{3\alpha}{k}\sigma}s_1+A_5^{(1)}\frac{1}{5^2}e^{-\frac{5\alpha}{k}\sigma}s_1-A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}s_3+A_1^{(1)}e^{-\frac{\alpha}{k}\sigma}s_5+A_3^{(1)}\frac{1}{3^2}e^{-\frac{3\alpha}{k}\sigma}s_7+A_5^{(1)}\frac{1}{5^2}e^{-\frac{5\alpha}{k}\sigma}s_9\}$$

$$\approx-\frac{4m}{M}\frac{k}{\alpha}\{[-A_3^{(1)}\frac{1}{3^2}+A_5^{(1)}\frac{1}{5^2}]e^{-\frac{\alpha}{k}\sigma}s_1-A_1^{(1)}\frac{1}{1^2}e^{-\frac{3\alpha}{k}\sigma}s_3+A_1^{(1)}\frac{1}{1^2}e^{-\frac{5\alpha}{k}\sigma}s_5+A_3^{(1)}\frac{1}{3^2}e^{-\frac{7\alpha}{k}\sigma}s_7+A_5^{(1)}\frac{1}{5^2}e^{-\frac{9\alpha}{k}\sigma}s_9\}$$

In above results, only the "sin" terms proportional to $\frac{k}{\alpha}$ are written out, while other terms such as the "cos" terms are ignored!

Adding above seven items together we obtain

$$g_1(\sigma,\tau)\frac{\partial^2\rho^{(1)}}{\partial\sigma^2}+g_2(\sigma,\tau)\frac{\partial\rho^{(1)}}{\partial\sigma}-g_3(\sigma,\tau)\rho^{(1)}=\sum_{n=0}^{4}A_{2n+1}^{(2)}\frac{k}{2\alpha}e^{-\frac{\alpha}{k}\sigma}s_{2n+1} \tag{A37}$$

First sub-harmonic term: $A_1^{(2)}\frac{k}{2\alpha}\,e^{-\frac{\alpha}{k}\sigma}s_1$

$$\frac{k}{2\alpha}\{-(\frac{M}{m}+\frac{m}{2M})A_1^{(1)}+[A_1^{(1)}-A_3^{(1)}]-\frac{m}{2M}\{[A_5^{(1)}-A_3^{(1)}]-8[A_1^{(1)}\frac{1}{2\cdot 1}-A_3^{(1)}\frac{1}{2\cdot 3}]$$
$$-\frac{m}{2M}4[A_3^{(1)}\frac{1}{3}-A_5^{(1)}\frac{1}{5}]-4[-A_1^{(1)}\frac{1}{1^2}+A_3^{(1)}\frac{1}{3^2}]-\frac{m}{2M}[A_5^{(1)}\frac{8}{5^2}-A_3^{(1)}\frac{8}{3^2}]\}e^{-\frac{\alpha}{k}\sigma}s_1$$
$$=\frac{k}{2\alpha}\{-(\frac{M}{m}+\frac{m}{2M})A_1^{(1)}+[A_1^{(1)}-A_3^{(1)}]-8[A_1^{(1)}\frac{1}{2\cdot 1}-A_3^{(1)}\frac{1}{2\cdot 3}]-4[-A_1^{(1)}\frac{1}{1^2}+A_3^{(1)}\frac{1}{3^2}]$$
$$-\frac{m}{2M}4[A_3^{(1)}\frac{1}{3}-A_5^{(1)}\frac{1}{5}]-\frac{m}{2M}[A_5^{(1)}-A_3^{(1)}]-\frac{m}{2M}[A_5^{(1)}\frac{8}{5^2}-A_3^{(1)}\frac{8}{3^2}]\}e^{-\frac{\alpha}{k}\sigma}s_1$$
$$=\frac{k}{2\alpha}\{-(\frac{M}{m}+\frac{m}{2M})A_1^{(1)}+A_1^{(1)}-\frac{1}{9}A_3^{(1)}-\frac{m}{2M}[\frac{1}{3}A_3^{(1)}-\frac{8}{3^2}A_3^{(1)}-\frac{4}{5}A_5^{(1)}+A_5^{(1)}+\frac{8}{5^2}A_5^{(1)}]\}e^{-\frac{\alpha}{k}\sigma}s_1$$
$$=\frac{k}{2\alpha}\{-(\frac{M}{m}+\frac{m}{2M}-1)A_1^{(1)}-\frac{1}{9}A_3^{(1)}-\frac{m}{2M}[-\frac{5}{3^2}A_3^{(1)}+\frac{13}{5^2}A_5^{(1)}]\}e^{-\frac{\alpha}{k}\sigma}s_1=A_1^{(2)}\frac{k}{2\alpha}e^{-\frac{\alpha}{k}\sigma}s_1$$

Second sub-harmonic term: $A_3^{(2)}\frac{k}{2\alpha}\,e^{-\frac{3\alpha}{k}\sigma}s_3$

$$\frac{k}{2\alpha}\{-(\frac{M}{m}+\frac{m}{2M})A_3^{(1)}-[A_1^{(1)}+A_5^{(1)}]-\frac{m}{2M}[-A_1^{(1)}]-\ 8[A_1^{(1)}\frac{1}{2\cdot 1}$$
$$-A_5^{(1)}\frac{1}{2\cdot 5}]-\frac{2m}{M}[A_1^{(1)}\frac{1}{1}]-4[A_1^{(1)}\frac{1}{1^2}+A_5^{(1)}\frac{1}{5^2}]-\frac{m}{M}[-A_1^{(1)}\frac{4}{1^2}]\}e^{-\frac{3\alpha}{k}\sigma}s_3$$
$$\frac{k}{2\alpha}\{-(\frac{M}{m}+\frac{m}{2M})A_3^{(1)}-[A_1^{(1)}+A_5^{(1)}]-\ 8A_1^{(1)}\frac{1}{2\cdot 1}-4[A_1^{(1)}\frac{1}{1^2}+A_5^{(1)}\frac{1}{5^2}]+A_5^{(1)}\frac{8}{2\cdot 5}$$
$$-\frac{m}{2M}[-A_1^{(1)}]-\frac{2m}{M}[A_1^{(1)}\frac{1}{1}]-\frac{m}{M}[-A_1^{(1)}\frac{4}{1^2}]\}e^{-\frac{3\alpha}{k}\sigma}s_3$$
$$=\frac{k}{2\alpha}\{-9A_1^{(1)}+\frac{5m}{2M}A_1^{(1)}-(\frac{M}{m}+\frac{m}{2M})A_3^{(1)}-\frac{3^2}{5^2}A_5^{(1)}\}e^{-\frac{3\alpha}{k}\sigma}s_3=A_3^{(2)}\frac{k}{2\alpha}e^{-\frac{3\alpha}{k}\sigma}s_3$$

Third sub-harmonic term: $A_5^{(2)}\frac{k}{2\alpha}\,e^{-\frac{5\alpha}{k}\sigma}s_5$

$$\frac{k}{2\alpha}\{-(\frac{M}{m}+\frac{m}{2M})A_5^{(1)}-A_3^{(1)}-\frac{m}{2M}A_1^{(1)}-\frac{8k}{2\alpha}[\frac{1}{2\cdot3}A_3^{(1)}]-\frac{k}{\alpha}\frac{m}{M}[A_1^{(1)}\frac{1}{1}]-\frac{2k}{\alpha}[A_3^{(1)}\frac{1}{3^2}]$$
$$-\frac{m}{M}\frac{k}{\alpha}[A_1^{(1)}\frac{2}{1^2}]\}e^{-\frac{5\alpha}{k}\sigma}s_5$$
$$=\frac{k}{2\alpha}\{-(\frac{M}{m}+\frac{m}{2M})A_5^{(1)}-A_3^{(1)}-\frac{m}{2M}A_1^{(1)}-8[\frac{1}{2\cdot3}A_3^{(1)}]-\frac{m}{2M}(4A_1^{(1)})-[A_3^{(1)}\frac{4}{3^2}]-\frac{m}{2M}[8A_1^{(1)}]\}e^{-\frac{5\alpha}{k}\sigma}s_5$$
$$=\frac{k}{2\alpha}\{-\frac{m}{2M}(13A_1^{(1)})-\frac{5^2}{3^2}A_3^{(1)}-(\frac{M}{m}+\frac{m}{2M})A_5^{(1)}\}e^{-\frac{5\alpha}{k}\sigma}s_5=A_5^{(2)}\frac{k}{2\alpha}e^{-\frac{5\alpha}{k}\sigma}s_5$$

Fourth sub-harmonic term: $A_7^{(2)}\frac{k}{2\alpha}\,e^{-\frac{7\alpha}{k}\sigma}s_7$

$$\frac{k}{2\alpha}\{-A_5^{(1)}-\frac{m}{2M}A_3^{(1)}-[A_5^{(1)}\frac{8}{2\cdot5}]-\frac{m}{M}[\frac{2}{3}A_3^{(1)}]-[A_5^{(1)}\frac{4}{5^2}]-\frac{m}{2M}\frac{k}{2\alpha}[A_3^{(1)}\frac{8}{3^2}]\}e^{-\frac{7\alpha}{k}\sigma}s_7$$
$$=\frac{k}{2\alpha}\{-[\frac{49}{5^2}A_5^{(1)}]-\frac{m}{2M}[A_3^{(1)}\frac{29}{3^2}]\}e^{-\frac{7\alpha}{k}\sigma}s_7=A_7^{(2)}\frac{k}{2\alpha}e^{-\frac{7\alpha}{k}\sigma}s_7$$

Fifth sub-harmonic term: $A_9^{(2)}\frac{k}{2\alpha}e^{-\frac{9\alpha}{k}\sigma}s_9$

$$\frac{k}{2\alpha}\{-\frac{m}{2M}A_5^{(1)}-\frac{m}{2M}[\frac{4}{5}A_5^{(1)}]-\frac{m}{2M}[\frac{8}{5^2}A_5^{(1)}]\}e^{-\frac{9\alpha}{k}\sigma}s_9$$
$$=-\frac{k}{2\alpha}\frac{m}{2M}[\frac{53}{5^2}A_5^{(1)}]e^{-\frac{9\alpha}{k}\sigma}s_9=A_9^{(2)}\frac{k}{2\alpha}e^{-\frac{9\alpha}{k}\sigma}s_9$$

Substituting (A37) into (A30) to get following differential equation

$$\begin{aligned}\frac{\partial^2\rho^{(2)}}{\partial\tau^2}-\frac{\partial^2\rho^{(2)}}{\partial\sigma^2}&=g_1(\sigma,\tau)\frac{\partial^2\rho^{(1)}}{\partial\sigma^2}+g_2(\sigma,\tau)\frac{\partial\rho^{(1)}}{\partial\sigma}-g_3(\sigma,\tau)\rho^{(1)}\\&=\sum_{n=0}^{4}A_{2n+1}^{(2)}\frac{k}{2\alpha}e^{-(2n+1)\frac{\alpha}{k}\sigma}sin[(2n+1)(\tau-\sigma)]\end{aligned}\tag{A38}$$

where

$$\begin{aligned}A_1^{(2)}&=-(\frac{M}{m}+\frac{m}{2M}-1)A_1^{(1)}-\frac{1}{9}A_3^{(1)}-\frac{m}{2M}[-\frac{5}{3^2}A_3^{(1)}+\frac{13}{5^2}A_5^{(1)}]\\A_3^{(2)}&=-9A_1^{(1)}+\frac{5m}{2M}A_1^{(1)}-(\frac{M}{m}+\frac{m}{2M})A_3^{(1)}-\frac{3^2}{5^2}A_5^{(1)}\\A_5^{(2)}&=-\frac{13m}{2M}A_1^{(1)}-\frac{5^2}{3^2}A_3^{(1)}-(\frac{M}{m}+\frac{m}{2M})A_5^{(1)}\\A_7^{(2)}&=-\frac{m}{2M}[\frac{29}{3^2}A_3^{(1)}]-[\frac{49}{5^2}A_5^{(1)}]\\A_9^{(2)}&=-\frac{m}{2M}[\frac{53}{5^2}A_5^{(1)}]\end{aligned}\tag{A39}$$

(A38) is just the Equation (15) in the main text.

Tables related to sum and difference relationships in trigonometry:

$c_2c_{2n+1}=\frac{1}{2}[c_{2n-1}+c_{2n+3}]$, $c_2s_{2n+1}=\frac{1}{2}[s_{2n+3}+s_{2n-1}]$
$s_{2n+1}c_4=\frac{1}{2}[s_{2n+5}+s_{2n-3}]$, $c_4c_{2n+1}=\frac{1}{2}[c_{2n-3}+c_{2n+5}]$,
$s_2s_{2n+1}=\frac{1}{2}[c_{2n-1}+c_{2n+3}]$, $s_2c_{2n+1}=\frac{1}{2}[s_{2n+3}-s_{2n-1}]$
$s_4s_{2n+1}=\frac{1}{2}[c_{2n-3}-c_{2n+5}]$, $s_4c_{2n+1}=\frac{1}{2}[s_{2n+5}-s_{2n-3}]$

Appendix E

To find the quasi-accumulation solution of following differential equation

$$\begin{aligned}\frac{\partial^2\rho^{(2)}}{\partial\tau^2}-\frac{\partial^2\rho^{(2)}}{\partial\sigma^2}&\approx\sum_{n=0}^{4}e^{-(2n+1)\frac{\alpha}{k}\sigma}\{A_{2n+1}^{(2)}\left(\frac{k}{2\alpha}\right)sin(2n+1)(\tau-\sigma)\}\\&=\sum_{n=0}^{4}e^{-(2n+1)\frac{\alpha}{k}\sigma}A_{2n+1}^{(2)}\left(\frac{k}{2\alpha}\right)\frac{e^{i(2n+1)(\tau-\sigma)}}{2i}+c.c.\end{aligned} \tag{A40}$$

Let the homogeneous solution of Equation (A40) be

$$\bar{\rho}^{(2,2n+1)}=B^{(2)}e^{i(2n+1)(\tau-\sigma)},\ n=0,1,2,3,4,$$

The right-hand end of Equation (A40) has a total of five terms, and according to the algorithm in **Appendix C**, their quasi cumulative solutions can be written as

$$\rho^{(2,2n+1)}=B_1^{(2)}(\sigma)e^{i(2n+1)(\tau-\sigma)}+B_2^{(2)}(\sigma)e^{-i(2n+1)(\tau-\sigma)} \tag{A41}$$

$$\begin{aligned}\frac{\partial\rho^{(2,2n+1)}}{\partial\sigma}&=\frac{\partial B_1^{(2)}(\sigma)}{\partial\sigma}e^{i(2n+1)(\tau-\sigma)}-i(2n+1)B_1^{(2)}(\sigma)e^{i(2n+1)(\tau-\sigma)}+\frac{\partial B_2^{(2)}(\sigma)}{\partial\sigma}e^{-i(2n+1)(\tau-\sigma)}\\&\quad+i(2n+1)B_2^{(2)}(\sigma)e^{i(2n+1)(\tau-\sigma)}\\&=\frac{\partial B_1^{(2)}(\sigma)}{\partial\sigma}e^{i(2n+1)(\tau-\sigma)}+\frac{\partial B_2^{(2)}(\sigma)}{\partial\sigma}e^{-i(2n+1)(\tau-\sigma)}-i(2n+1)B_1^{(2)}(\sigma)e^{i(2n+1)(\tau-\sigma)}+\\&\quad i(2n+1)B_2^{(2)}(\sigma)e^{i(2n+1)(\tau-\sigma)}\\&=-i(2n+1)B_1^{(2)}(\sigma)e^{i(2n+1)(\tau-\sigma)}+i(2n+1)B_2^{(2)}(\sigma)e^{i(2n+1)(\tau-\sigma)}\end{aligned}$$

$$\because\frac{\partial B_1^{(2)}(\sigma)}{\partial\sigma}e^{i(2n+1)(\tau-\sigma)}+\frac{\partial B_2^{(2)}(\sigma)}{\partial\sigma}e^{-i(2n+1)(\tau-\sigma)}=0$$

$$\begin{aligned}\frac{\partial^2\rho^{(2,2n+1)}}{\partial\sigma^2}&=-2i(2n+1)\frac{\partial B_1^{(2)}(\sigma)}{\partial\sigma}e^{i(2n+1)(\tau-\sigma)}+[-2i(2n+1)]^2B_1^{(2)}(\sigma)\\&\quad+2i(2n+1)\frac{\partial B_2^{(2)}(\sigma)}{\partial\sigma}e^{-i(2n+1)(\tau-\sigma)}+[2i(2n+1)]^2B_2^{(2)}(\sigma)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\rho^{(2)}}{\partial\tau^2}&=\frac{\partial^2B_1^{(2)}(\sigma)e^{i(2n+1)(\tau-\sigma)}}{\partial\tau^2}+\frac{\partial^2B_2^{(2)}(\sigma)e^{-i(2n+1)(\tau-\sigma)}}{\partial\tau^2}\\&=[i(2n+1)]^2B_1^{(2)}(\sigma)e^{i(2n+1)(\tau-\sigma)}+[-i(2n+1)]^2B_2^{(2)}(\sigma)e^{-i(2n+1)(\tau-\sigma)}\end{aligned}$$

Substituting this result into (A40) yields

$$e^{-(2n+1)\frac{\alpha}{k}\sigma}A_{2n+1}^{(2)}\left(\frac{k}{2\alpha}\right)\frac{e^{i(2n+1)(\tau-\sigma)}}{2i}$$

$$\begin{aligned}\frac{\partial^2\rho^{(2)}}{\partial\tau^2}-\frac{\partial^2\rho^{(2)}}{\partial\sigma^2}&=[i(2n+1)]^2B_1^{(2)}(\sigma)e^{i(2n+1)(\tau-\sigma)}+[-i(2n+1)]^2B_2^{(2)}(\sigma)e^{-i(2n+1)(\tau-\sigma)}\\&\quad-\{-2i(2n+1)\frac{\partial B_1^{(2)}(\sigma)}{\partial\sigma}e^{i(2n+1)(\tau-\sigma)}+[-2i(2n+1)]^2B_1^{(2)}(\sigma)\\&\quad+2i(2n+1)\frac{\partial B_2^{(2)}(\sigma)}{\partial\sigma}e^{-i(2n+1)(\tau-\sigma)}+[2i(2n+1)]^2B_2^{(2)}(\sigma)\}\\&=2i(2n+1)\frac{\partial B_1^{(2)}(\sigma)}{\partial\sigma}e^{i(2n+1)(\tau-\sigma)}-2i(2n+1)\frac{\partial B_2^{(2)}(\sigma)}{\partial\sigma}e^{-i(2n+1)(\tau-\sigma)}\end{aligned}$$

Simultaneous this result and equation (A40) to get

$$2i(2n+1)\frac{\partial B_1^{(2)}(\sigma)}{\partial\sigma}e^{i(2n+1)(\tau-\sigma)}-2i(2n+1)\frac{\partial B_2^{(2)}(\sigma)}{\partial\sigma}e^{-i(2n+1)(\tau-\sigma)}=e^{-(2n+1)\frac{\alpha}{k}\sigma}A_{2n+1}^{(2)}\left(\frac{k}{2\alpha}\right)\frac{e^{i(2n+1)(\tau-\sigma)}}{2i}$$

or

$$\begin{aligned}\frac{\partial B_1^{(2)}(\sigma)}{\partial\sigma}e^{i(2n+1)(\tau-\sigma)}-\frac{\partial B_2^{(2)}(\sigma)}{\partial\sigma}e^{-i(2n+1)(\tau-\sigma)}&=\frac{1}{2i(2n+1)}e^{-(2n+1)\frac{\alpha}{k}\sigma}A_{2n+1}^{(2)}\left(\frac{k}{2\alpha}\right)\frac{e^{i(2n+1)(\tau-\sigma)}}{2i}\\&=-\frac{1}{4(2n+1)}A_{2n+1}^{(2)}\left(\frac{k}{2\alpha}\right)e^{-(2n+1)\frac{\alpha}{k}\sigma+i(2n+1)(\tau-\sigma)}\end{aligned}\tag{A42}$$

$B_1^{(2)}(\sigma)$ and $B_2^{(2)}(\sigma)$ must satisfy the condition

$$\frac{\partial B_1^{(2)}(\sigma)}{\partial\sigma}e^{i(2n+1)(\tau-\sigma)}+\frac{\partial B_2^{(2)}(\sigma)}{\partial\sigma}e^{-i(2n+1)(\tau-\sigma)}=0\tag{A43}$$

(A43) plus (A42), one has

$$\frac{\partial B_1^{(2)}(\sigma)}{\partial\sigma}e^{i(2n+1)(\tau-\sigma)}=-\frac{1}{8(2n+1)}A_{2n+1}^{(2)}\left(\frac{k}{2\alpha}\right)e^{-(2n+1)\frac{\alpha}{k}\sigma+i(2n+1)(\tau-\sigma)}\Rightarrow$$

$$\frac{\partial B_1^{(2)}(\sigma)}{\partial\sigma}=-\frac{1}{8(2n+1)}A_{2n+1}^{(2)}\left(\frac{k}{2\alpha}\right)e^{-(2n+1)\frac{\alpha}{k}\sigma}$$

Integrating this result with respect to σ yields

$$\begin{aligned}B_1^{(2)}(\sigma)&=-\frac{1}{-8(2n+1)(2n+1)\frac{\alpha}{k}}A_{2n+1}^{(2)}\left(\frac{k}{2\alpha}\right)e^{-(2n+1)\frac{\alpha}{k}\sigma}\\&=\frac{1}{4(2n+1)^2}A_{2n+1}^{(2)}\left(\frac{k}{2\alpha}\right)^2e^{-(2n+1)\frac{\alpha}{k}\sigma}\end{aligned}\tag{A44}$$

(A43) minus (A42) to get

$$\frac{\partial B_2^{(2)}(\sigma)}{\partial\sigma}e^{-i(2n+1)(\tau-\sigma)}=-\frac{1}{8(2n+1)}A_{2n+1}^{(2)}\left(\frac{k}{2\alpha}\right)e^{-(2n+1)\frac{\alpha}{k}\sigma-i(2n+1)(\tau-\sigma)}\Rightarrow$$

$$\frac{\partial B_2^{(2)}(\sigma)}{\partial\sigma}=-\frac{1}{8(2n+1)}A_{2n+1}^{(2)}\left(\frac{k}{2\alpha}\right)e^{-(2n+1)\frac{\alpha}{k}\sigma-i2(2n+1)(\tau-\sigma)}$$

$$\Rightarrow B_2^{(2)}(\sigma)=\frac{1}{8(2n+1)}A_{2n+1}^{(2)}\left(\frac{k}{2\alpha}\right)e^{-(2n+1)\frac{\alpha}{k}\sigma-i2(2n+1)(\tau-\sigma)}\propto\frac{k}{2\alpha}$$

Due to $B_2^{(2)}(\sigma) << B_1^{(2)}(\sigma)$, we ignore $B_2^{(2)}(\sigma)$ and substitute the obtained result into (A41), add its conjugate value, and restore the summation sign to obtain the final result

$$\begin{aligned}\rho^{(2)}&=B_1^{(2)}(\sigma)e^{i(2n+1)(\tau-\sigma)}+c.c.\\&=\sum_{n=0}^{4}\frac{1}{4(2n+1)^2}A_{2n+1}^{(2)}\left(\frac{k}{2\alpha}\right)^2e^{-(2n+1)\frac{\alpha}{k}\sigma}[e^{i(2n+1)(\tau-\sigma)}+e^{-i(2n+1)(\tau-\sigma)}]\\&=\sum_{n=0}^{4}\frac{1}{2(2n+1)^2}A_{2n+1}^{(2)}\left(\frac{k}{2\alpha}\right)^2e^{-(2n+1)\frac{\alpha}{k}\sigma}cos[(2n+1)(\tau-\sigma)]\end{aligned}\tag{A45}$$

Appendix F

Flows amplify sound

$$\rho^{(2)} = \sum_{n=0}^{4} \frac{1}{2(2n+1)^2} A_{2n+1}^{(2)} \left(\frac{k}{2\alpha}\right)^2 e^{-(2n+1)\frac{\alpha}{k}\sigma} cos[(2n+1)(\tau-\sigma)]$$

where

$$\varepsilon^2 A_1^{(2)} = M^2 m^2 \{-(\tfrac{M}{m} + \tfrac{m}{2M} - 1)A_1^{(1)} - \tfrac{1}{9}A_3^{(1)} - \tfrac{m}{2M}[-\tfrac{5}{3^2}A_3^{(1)} + \tfrac{13}{5^2}A_5^{(1)}]\}$$

$$=(\tfrac{M}{m} + \tfrac{m}{2M} + 1)M^2 m^2 \{(\tfrac{M}{m} + \tfrac{m}{2M} - 1)A_0 - \tfrac{1}{9}[-3^2 A_0(\tfrac{m}{4M} + 1)] - ()^2 Mm\tfrac{m}{2M}[-\tfrac{5}{3^2}\{-3^2 A_0 Mm(\tfrac{m}{4M} + 1)\}$$

$$+\tfrac{13}{5^2}MmA_5^{(1)}]$$

$$=\{(M^2 + \tfrac{m^2}{2} - Mm)[(M^2 + \tfrac{m^2}{2} + Mm)] + Mm[(\tfrac{m^2}{4} + Mm)] -$$

$$\tfrac{m^2}{2}[5\left((\tfrac{m^2}{4} + Mm)\right) - 13\left(\tfrac{m^2}{4}\right)]\}A_0$$

$$=\{[(M/m)^2 + \tfrac{1}{2} - M/m][(M/m)^2 + \tfrac{1}{2} + M/m)] + M/m[(\tfrac{1}{4} + M/m)] + \tfrac{1}{2}[-5(\tfrac{1}{4} + \tfrac{M}{m}) + \tfrac{13}{4}]\}m^4 A_0$$

Define the gain of flow amplification sound

$$\beta_{2n+1}^{(m_1)} = \frac{\varepsilon^{m_1} A_{2n+1}^{(m_1)}}{\varepsilon^{m_1} A_{2n+1}^{(m_1)}\Big|_{M=0}}$$

then one has

$$\varepsilon^2 A_1^{(2)}\Big|_{M=0} = (\frac{1}{4} - \frac{5}{8} + \frac{13}{8})m^4 A_0 = \frac{5}{4}m^4 A_0$$

$$\varepsilon^2 A_1^{(2)} = \{[(M/m)^2 + \frac{1}{2} - M/m][(M/m)^2 + \frac{1}{2} + M/m)] + M/m[(\frac{1}{4} + M/m)] + \frac{1}{2}[-5(\frac{1}{4} + \frac{M}{m}) + \frac{13}{4}]\}m^4 A_0$$

$$\beta_1^{(2)} = \frac{\varepsilon^2 A_1^{(2)}}{\frac{5}{4}m^4 A_0} = \frac{4}{5}\{[(M/m)^2 + \frac{1}{2} - M/m][(M/m)^2 + \frac{1}{2} + M/m)]$$

$$+ M/m[(\frac{1}{4} + M/m)] + \frac{1}{2}[-5(\frac{1}{4} + \frac{M}{m}) + \frac{13}{4}]\}$$

$$\varepsilon^2 A_3^{(2)} = (Mm)^2\{-9A_1^{(1)} + \frac{5m}{2M}A_1^{(1)} - (\frac{M}{m} + \frac{m}{2M})A_3^{(1)} - \frac{3^2}{5^2}[-5^2 A_0 \frac{m}{4M}]\}$$

$$=\{-9(Mm)^2[-(\frac{M}{m} + \frac{m}{2M} + 1)] + \frac{5m}{2M}(Mm)^2[-(\frac{M}{m} + \frac{m}{2M} + 1)]$$

$$-(Mm)(\frac{M}{m} + \frac{m}{2M})(Mm)[-3^2(\frac{m}{4M} + 1)] - \frac{3^2}{5^2}[-5^2\frac{m}{4M}](Mm)^2\}A_0$$

$$=\{(9 - \frac{5}{2} + 9 + \frac{1}{2})M^3 m + (-9 - \frac{5}{2} + \frac{9}{4})M^2 m^2 + (-\frac{9}{2} - \frac{5}{2} + \frac{1}{2})Mm^3 - \frac{5m^4}{4} + \frac{m^4}{8}\}A_0$$

$$=16M^3 m - \frac{37}{4}M^2 m^2 - \frac{13}{2}Mm^3 - \frac{9}{8}m^4 = 16M^3/m^3 - \frac{37}{4}M^2/m^2 - \frac{13}{2}M/m - \frac{9}{8}\}m^4 A_0$$

$$\varepsilon^2 A_3^{(2)}\Big|_{M=0} = -\frac{9}{8}m^4 A_0,$$

$$\beta_3^{(2)} = \frac{\{16M^3/m^3 - \frac{37}{4}M^2/m^2 - \frac{13}{2}M/m - \frac{9}{8}\}m^4 A_0}{-\frac{9}{8}m^4 A_0} = \frac{8}{9}[16M^3/m^3 - \frac{37}{4}M^2/m^2 - \frac{13}{2}M/m - \frac{9}{8}]$$

$$\varepsilon^2 A_5^{(2)} = (Mm)^2\{-\frac{13m}{2M}[-A_0(\frac{M}{m}+\frac{m}{2M}+1)] - \frac{5^2}{3^2}[-3^2A_0(\frac{m}{4M}+1)] - (\frac{M}{m}+\frac{m}{2M})[-5^2A_0\frac{m}{4M}]\}$$

$$=\{-\frac{M^2}{m^2}\frac{13m}{2M}[-(\frac{M}{m}+\frac{m}{2M}+1)] + 5^2\frac{M^2}{m^2}[(\frac{m}{4M}+1)] + [(\frac{M}{m})^2+\frac{1}{2})[5^2\frac{1}{4}]\}m^4A_0$$

$$=\{\frac{13}{2}[(\frac{M}{m})^2+\frac{M}{m}+\frac{1}{2})] + 5^2[(\frac{M}{4m}+\frac{M^2}{m^2})] + \frac{5^2}{4}(\frac{M^2}{m^2}+\frac{1}{2})\}m^4A_0$$

$$=\{[\frac{13}{2}(\frac{M}{m})^2+\frac{13}{2}\frac{M}{m}+\frac{13}{4}] + (\frac{5^2M}{4m}+\frac{5^2M^2}{m^2}) + (\frac{5^2}{4}\frac{M^2}{m^2}+\frac{5^2}{8})\}m^4A_0$$

$$=\{(\frac{13}{2}+25+\frac{25}{4})(\frac{M}{m})^2 + (\frac{13}{2}+\frac{25}{4})\frac{M}{m} + \frac{13}{4}+\frac{25}{8}\}m^4A_0$$

$$=\{\frac{131}{4}(\frac{M}{m})^2 + \frac{51}{4}\frac{M}{m} + \frac{51}{8}\}m^4A_0$$

$$\varepsilon^2 A_5^{(2)}\Big|_{M=0} = \frac{51}{8}m^4A_0$$

$$\beta_5^{(2)} = \frac{\{\frac{131}{4}(\frac{M}{m})^2+\frac{51}{4}\frac{M}{m}+\frac{51}{8}\}m^4A_0}{\frac{51}{8}m^4A_0} = \frac{8}{51}\{\frac{131}{4}(\frac{M}{m})^2+\frac{51}{4}\frac{M}{m}+\frac{51}{8}\} = \frac{262}{51}(\frac{M}{m})^2 + 2\frac{M}{m} + 1$$

$$\varepsilon^2 A_7^{(2)} = \varepsilon^2\{-\frac{m}{2M}[\frac{29}{3^2}A_3^{(1)}] - [\frac{49}{5^2}A_5^{(1)}]\} = \varepsilon^2\{-\frac{m}{2M}\frac{29}{3^2}[-3^2A_0(\frac{m}{4M}+1)] - \frac{49}{5^2}[-5^2A_0\frac{m}{4M}]\}$$

$$= \{\frac{29}{2}m^2[(\frac{m^2}{4}+Mm)] + 49Mm[\frac{m^2}{4}]\}A_0 = \{\frac{29}{2}[\frac{1}{4}+(M/m)] + \frac{49}{4}M/m\}m^4A_0$$

$$=\{[\frac{29}{8}+(\frac{29}{2}+\frac{49}{4})M/m)]\}m^4A_0$$

$$\varepsilon^2 A_7^{(2)}\Big|_{M=0} = \frac{29}{8}m^4A_0$$

$$\beta_7^{(2)} = \frac{[\frac{29}{8}+(\frac{29}{2}+\frac{49}{4})M/m)]m^4A_0}{\frac{29}{8}m^4A_0} = 1 + (4+\frac{98}{29})M/m$$

$$\varepsilon^2 A_9^{(2)} = -\frac{m}{2M}(Mm)^2[\frac{53}{5^2}A_5^{(1)}] = -\frac{53}{5^2}(Mm)^2\frac{m}{2M}[-5^2A_0\frac{m}{4M}] = \frac{53}{8}m^4A_0$$

$$\beta_7^{(2)} = 1$$

Now let's study the situation $m_1 = 1$. Since

$$A_1^{(1)} = -A_0(\tfrac{M}{m}+\tfrac{m}{2M}+1),\ A_3^{(1)} = -3^2A_0(\tfrac{m}{4M}+1),\ A_5^{(1)} = -5^2A_0\tfrac{m}{4M}$$

$$\varepsilon A_1^{(1)} = -[(M/m)^2+\tfrac{1}{2}+M/m)]m^2A_0,\ \ \varepsilon A_1^{(1)}\Big|_{M=0} = -\tfrac{1}{2}m^2A_0,$$

$$\beta_1^{(1)} = \frac{-[(M/m)^2+\frac{1}{2}+M/m)]m^2A_0}{-\frac{1}{2}m^2A_0} = [(M/m)^2+M/m+\tfrac{1}{2})]$$

$$\varepsilon A_3^{(1)} = -3^2(\tfrac{1}{4}+M/m)m^2A_0,\ \ \varepsilon A_3^{(1)}\Big|_{M=0} = -\tfrac{9}{4}m^2A_0,\ \beta_3^{(1)} = \frac{-3^2(\frac{1}{4}+M/m)m^2A_0}{-\frac{9}{4}m^2A_0} = 4(\tfrac{1}{4}+M/m)$$

$$\varepsilon A_5^{(1)} = -5^2A_0Mm\tfrac{m}{4M} = -5^2\tfrac{1}{4}m^2A_0,\ \ \varepsilon A_5^{(1)}\Big|_{M=0} = -5^2\tfrac{1}{4}m^2A_0,\ \beta_5^{(1)} = \frac{-5^2\frac{1}{4}m^2A_0}{-5^2\frac{1}{4}m^2A_0} = 1$$

Obviously, $\beta_1^{(1)}$ is a quadratic parabola, $\beta_3^{(1)}$ is a straight line, independent of flow, and $\beta_5^{(1)}$ does not magnify. Thus, it can be inferred that the amplification gain of $\varepsilon^4 A_1^{(4)}$ is an 8th degree polynomial of M/m, the amplification gain of $\varepsilon^4 A_3^{(4)}$ is a 7th degree polynomial of M/m,, the amplification gain of $\varepsilon^4 A_5^{(4)}$ is a 6th degree polynomial of M/m, ..., the amplification gain of $\varepsilon^4 A_{15}^{(4)}$ is a straight line, and the amplification gain of $\varepsilon^4 A_{17}^{(4)}$ is 1 independent of the flow.