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Article

Nonlinear free vibration of conical beams using He's frequency formula: Educational implications

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Abstract: This study presents an analytical solution for the nonlinear free vibration of conical beams using He's frequency formula. This solution aims to efficiently address the challenges posed by their axially varying cross-sectional dimensions and associated nonlinear mechanical behaviors. Conical beams, with their customized stiffness and mass distribution, find wide application in aerospace, civil engineering, and micro-electromechanical systems (MEMS), where precise vibration analysis is imperative for ensuring structural stability and performance. The frequency formula, rooted in residual minimization, is employed to derive the frequency-amplitude relationship of conical beams. This method avoids complex iterative procedures and reduces computational complexity compared to traditional methods like Aboodh Transform-based variational iteration method (ATVIM) or homotopy perturbation. The validation of the method against ATVIM and numerical solutions (4th-order Runge–Kutta method) confirms its accuracy, with close agreement across moderate to large amplitudes and a frequency relative error of less than 5%. Beyond its practical utility in engineering design—enabling rapid parametric analysis for resonance avoidance—the study also highlights educational implications, as the conical beam case study bridges abstract nonlinear dynamics theory with real-world applications, aiding students in understanding frequency-amplitude coupling and method selection. This work demonstrates that He's frequency formula offers a robust, accessible framework for analyzing conical beam vibrations, linking theoretical nonlinear dynamics, engineering practice, and educational value.

Keywords: conical beams; nonlinear free vibration; He's frequency formula; residual minimization; vibration analysis; educational implications; MEMS applications; resonance avoidance

1. Introduction

Nonlinear vibrations are ubiquitous in both daily life and engineering scenarios. Their most distinctive trait lies in the complex relationship between frequency and amplitude, as explored in studies such as Khudoynazarov's work on the longitudinal-radial vibrations of a viscoelastic cylindrical three-layer structure, where the intricate interplay between these factors significantly impacts the system's behavior [1]. In the context of structures exhibiting nonlinear dynamic behavior, the precise determination of periodicity is paramount for ensuring operational stability and safety. This assertion is particularly salient in the context of specialized components, such as conical beams, which find extensive application across a diverse array of engineering disciplines [2,3].

In the domain of smart architectures, conical beams assume a pivotal role. For instance, in innovative building designs inspired by natural forms, such as the clover-shaped fractal architectures for sustainable buildings proposed by Liu et al. [4], conical beams can be employed to enhance structural stability while also enabling unique aesthetic and functional features. These smart architectures frequently necessitate precise regulation of structural vibrations to ensure comfort and integrity. Consequently, the accurate analysis of conical beam vibrations is imperative.

Conical nano/micro beams, characterized by their distinct properties, can now be fabricated with high precision through the use of advanced 3D printing technology, as evidenced by research on 3D printed concrete masonry for wall structures [5, 6]. This level of fabrication precision enables the creation of highly customized conical nanobeams. These materials find applications in MEMS (Micro-Electro-Mechanical Systems) [7, 8], where precise control of their vibration properties is paramount. In MEMS graphene resonators [9], conical nanobeams can be utilized to tune the resonance frequencies, thereby enabling more sensitive and accurate sensing and actuation. Furthermore, in piezoelectric biosensors based on ultrasensitive MEMS systems [10], conical nanobeams contribute to optimizing the device's performance by effectively managing vibration-related parameters, which is crucial for the accurate detection of biological substances.

Zuo, Luo, and Zeng's study on the gecko—inspired fractal buffer for passenger elevators [11] also offers valuable insights into the application of innovative designs to control vibrations in engineering systems. Inspired by the remarkable adhesion and shock—absorption capabilities of geckos' feet, which rely on hierarchical and self—similar structures at the micro and nano-scales to interact with surfaces, the proposed fractal buffer is designed to enhance the safety and comfort of elevator rides. Passenger elevator safety is of utmost importance, and the buffer system plays a critical role in absorbing kinetic energy during sudden stops or abnormal operations. Traditional buffers may have limitations in efficiently handling complex dynamic forces. The gecko—inspired fractal buffer, with its hierarchical structure mimicking the gecko's footpad, has the potential to better adapt to different impact scenarios. By distributing and dissipating energy across multiple scales of the fractal structure, it can provide more effective shock absorption compared to conventional designs. This not only improves the safety of passengers in case of elevator malfunctions but also reduces wear and tear on elevator components, thereby increasing the overall lifespan and reliability of the elevator system. This research showcases how nature-inspired designs can lead to advancements in engineering applications related to vibration control and energy absorption, paralleling the importance of accurately analyzing and controlling vibrations in conical beam structures for various engineering uses.

Conical beams are distinguished by their axially varying cross-sectional dimensions, which range from linearly tapered forms, such as trapezoidal sections, to non-linearly varying shapes, including parabolic or hyperbolic profiles. These beams offer distinct mechanical advantages. Through the strategic manipulation of stiffness and mass distributions through gradual cross-sectional changes, these beams achieve a more uniform stress distribution, a reduction in stress concentration, and an enhancement in load-bearing efficiency when compared to uniform-section beams. These properties render them indispensable in a variety of fields, including aerospace (e.g., aircraft engine blades), civil engineering (e.g., seismic-resistant bridge designs), and mechanical systems, where the objectives of avoiding resonance and ensuring structural integrity are of the utmost importance.

However, the variable stiffness and mass distribution of conical beams complicates their free vibration analysis, as their natural frequencies and modes exhibit greater non-linearity than those of uniform beams. The resolution of nonlinear oscillation problems has historically necessitated sophisticated analytical or computational methodologies, many of which are constrained by rigid assumptions (e.g., small-amplitude approximations) or encumbered by arduous iterative procedures, such as the homotopy perturbation method [12–15].

Among the emerging techniques for addressing nonlinear oscillators, He's frequency formula [16–20] stands out for its simplicity and efficiency. This method, still under development with ongoing refinements to enhance accuracy, provides a straightforward pathway to estimate frequencies by minimizing residual functions and leveraging trial solutions—eliminating the need for cumbersome iterations. Its potential to rapidly derive analytical solutions for nonlinear systems makes it a promising tool for engineering applications, where quick yet reliable predictions of vibrational behavior are essential.

Against this backdrop, this study focuses on solving the free vibration problem of conical beams using He's frequency formula. The objectives are threefold: (1) to review and analyze the mathematical foundations of He's frequency formula; (2) to apply this formula to derive analytical solutions for the nonlinear free vibration of conical beams; and (3) to validate the effectiveness of the proposed approach through comparisons with numerical solutions, thereby demonstrating its utility in engineering design and analysis.

By achieving these goals, this work aims to contribute a practical analytical tool for efficiently assessing the vibrational characteristics of conical beams, bridging the gap between theoretical nonlinear dynamics and real-world engineering needs.

Moreover, beyond its practical engineering applications, this study holds significant educational value in the teaching of nonlinear dynamics. Nonlinear vibration, with its intricate mathematical formulations and abstract concepts such as residual minimization, often poses challenges for students in bridging the gap between theory and real-world scenarios. By focusing on the specific case of conical beams and utilizing He's frequency formula as an analytical tool, this work provides a tangible and relatable example that can enhance students' understanding. It allows learners to observe how theoretical methods (like residual minimization and trial solution selection) are applied to solve actual engineering problems, making abstract nonlinear dynamics concepts more concrete. Through comparing this method with other techniques such as ATVIM and numerical simulations, students can gain insights into the trade-offs between accuracy, efficiency, and complexity in engineering analysis, fostering their ability to select appropriate methods for different practical situations. This integration of theoretical derivation, practical application, and comparative analysis not only enriches the content of nonlinear dynamics education but also helps cultivate students' problem-solving skills and engineering thinking, thereby bridging the gap between academic teaching and industrial practice.

2. Theoretical foundations

2.1. Theoretical foundations of He's frequency formula

In order to address the issue of nonlinear oscillation problems in an efficient manner, the frequency formula [16–20] offers a streamlined analytical framework that is rooted in the principle of minimizing residuals to derive periodic solutions. This principle was first elucidated by He and Liu in 2022 [21] and subsequently improved upon by He et al. in 2025 [22]. This section provides a detailed exposition of the mathematical underpinnings and key derivations established by He and Liu.

Consider a general nonlinear oscillator governed by the equation:

$$u'' + f(u) = 0 \quad u(0) = A \quad u'(0) = 0 \tag{1}$$

where u denotes displacement, u'' is the second derivative with respect to time, and $f(u)$ represents a nonlinear function characterizing the restoring force. For systems with periodic solutions, a simplified standard form can be expressed as:

$$u'' + (\frac{f(u)}{u})u = 0, \quad \frac{f(u)}{u} > 0 \tag{2}$$

According to He and Liu [21], He's frequency formula is derived through a systematic process of defining residuals, selecting trial solutions, and enforcing the condition of minimum residual to optimize frequency estimation.

We define a residual function as:

$$R(t) = \widetilde{u}'' + f(\widetilde{u}) \tag{3}$$

where $\widetilde{u}$ is a trial solution,

$$\widetilde{u}(t) = Acos\omega t \tag{4}$$

In order to achieve the best frequency recognition, this requires the minimum residual.

$$R(\widetilde{u}) = -\omega^2 Acos\omega t + f(\widetilde{u}) = -\omega^2\widetilde{u} + f(\widetilde{u}) \to min \tag{5}$$

It means that:

$$\frac{dR(\widetilde{u})}{d\widetilde{u}} = -\omega^2 + \frac{df(\widetilde{u})}{d\widetilde{u}} = 0 \tag{6}$$

According to some previous publications [21,22], we chose $\widetilde{u} = \frac{1}{2}A$, so we have:

$$\omega^2 = \left.\frac{df}{du}\right|_{\widetilde{u}\,=\,\frac{1}{2}A} \tag{7}$$

If we write:

$$\frac{df}{du} \approx \frac{f(\widetilde{u}) - f(0)}{\widetilde{u} - 0} = \frac{f(\widetilde{u})}{\widetilde{u}} \tag{8}$$

We, therefore, obtain a simple frequency formulation.

$$\omega^2 = \frac{f(\widetilde{u})}{\widetilde{u}} \tag{9}$$

The formula's strength lies in its simplicity: it avoids complex iterative procedures by leveraging trial solutions and residual minimization, enabling rapid derivation of frequency-amplitude relationships. This efficiency, in conjunction with its capacity to adapt to moderately and strongly nonlinear systems, renders it especially valuable for engineering applications that demand rapid yet precise estimates of vibrational characteristics. A notable illustration of this application is the analysis of conical beams, a subject that will be explored in subsequent sections.

It is important to note that He's frequency formulation is characterized as a point solution. This concept, as elucidated in related studies, underscores its capacity to yield precise outcomes for particular conditions without necessitating extensive solutions across the entire problem domain [23,24].

The point solution concept is applied to reaction-diffusion problems in porous catalysts [23]. The primary objective of this approach is the extraction of critical information at specific points, such as localized reaction rates or concentration gradients, rather than the resolution of the entire spatial-temporal distribution. This ensures efficiency without compromising accuracy for key parameters. In a similar vein, Liu and He demonstrate that point solutions can rapidly predict specific responses (e.g., amperometric current at a given time or reactant concentration) in reaction kinetics, thereby highlighting their utility for targeted, real-time analysis [24].

In a similar manner, the frequency formula under consideration functions as a point solution in the context of nonlinear vibration analysis. It directly computes the frequency for a particular amplitude or structural parameter (e.g., the taper rate of conical beams) through simplified calculations, thereby providing critical vibrational characteristics in an efficient manner. This approach is consistent with the engineering community's need for rapid and dependable estimates in the domains of design and optimization processes.

2.2. Theoretical foundations of the homotopy perturbation method

Homotopy perturbation method [25–27] for obtaining approximate solutions to nonlinear differential equations was proposed by He in 1988. The fundamental concept is as follows:

Consider a nonlinear differential equation given by:

$$A(u) - f(r) = 0, r \in \Omega \tag{10}$$

with the boundary conditions:

$$B(u, \frac{\partial u}{\partial n}) = 0, r \in \Gamma \tag{11}$$

Here, A represents a general differential operator. Typically, A can be decomposed into a linear part L and a nonlinear part N. The function $f(r)$ is a known analytic function, B is a boundary operator, Γ is the boundary of the domain Ω, and $\frac{\partial}{\partial n}$ denotes differentiation along the normal vector to Γ.

A homotopy $H(r,p) : \Omega \times [0,1] \rightarrow \mathbb{R}$ is constructed, satisfying:

$$H(u,p) = (1-p)[L(u) - L(u_0)] + p[A(u) - f(r)] = 0, r \in \Omega \tag{12}$$

where $p \in [0,1]$ is an embedding parameter, and u_0 is an initial approximation satisfying the boundary conditions. From this Equation (12), it follows that:

$$H(u,0) = L(u) - L(u_0) = 0, H(u,1) = A(u) - f(u) = 0 \tag{13}$$

The process of p varying from 0 to 1 corresponds to the transformation of $u(r,p)$ from $u_0(r)$ to $u(r)$. Consequently, $H(u,p)$ changes from $L(u)-L(u_0)$ to $A(u)-f(r)$, that is, $L(u) - L(u_0)$ and $A(u) - f(r)$ are homotopic.

In the homotopy perturbation method (HPM), the embedding parameter p is treated as a small parameter. The solution u is assumed to be expressible as a power series in p:

$$u = u_0 + pu_1 + p^2u_2 + ... \tag{14}$$

Setting $p = 1$ yields the approximate solution to the original equation:

$$u = lim_{p\rightarrow 1}u = u_0 + u_1 + u_2 + ... \tag{15}$$

The expansion with respect to the parameter p pertains to perturbation techniques. Hence, this synthesis of homotopy theory and perturbation methods is termed the homotopy perturbation method.

3. Structural characteristics and vibration model of conical beams

Conical beams, characterized by axially varying cross-sectional dimensions (ranging from linearly tapered forms like trapezoidal sections to non-linearly varying shapes such as parabolic or hyperbolic profiles), possess unique mechanical advantages that make them highly suitable for MEMS (Micro-Electro-Mechanical Systems). The variable stiffness and mass distribution of these materials, achieved through gradual cross-sectional changes, enable tailored dynamic behaviors, a critical aspect of MEMS, where precise control over vibration properties, resonance frequencies, and actuation responses is paramount. Conical nanobeams have been demonstrated to be a viable solution for tuning the resonance frequencies of MEMS graphene resonators and piezoelectric biosensors. These sensors are based on ultra-sensitive MEMS systems, and the use of conical nanobeams allows for the optimization of performance by adapting to specific operational demands.

However, the establishment of mathematical models for conical beams in MEMS is a relatively complex undertaking. This complexity arises from their variable cross-sectional geometry, which introduces nonlinearities in their governing equations. For instance, there is the coupling between frequency and amplitude in vibration analysis, and the intricate relationships between structural parameters (such as taper rate) and mechanical responses. The presence of these nonlinearities poses a significant challenge to the derivation of accurate and efficient analytical solutions using conventional methods.

The variational approach [28] offers a valuable solution. This approach involves formulating a variational principle tailored to MEMS using the semi-inverse method, resulting in a principle of least action. This variational principle can be applied to address the complexities of conical beams in MEMS by providing a systematic framework to derive governing equations and solve for key parameters (e.g., natural frequencies, pull-in voltage in actuators). By leveraging the variational approach, researchers and engineers can more accurately and efficiently model the nonlinear behaviors of conical beams in MEMS, facilitating optimized design and enhanced reliability of these microscale systems.

3.1. Structural features of conical beams

The cross-sectional profile of a conical beam changes continuously along its axial direction, with two primary categories of variation based on the rate of geometric change: linearly varying cross-sections and nonlinearly varying cross-sections. Linearly varying cross-sections include trapezoidal profiles, where the width or height of the cross-section decreases or increases at a constant rate from one end to the other, resulting in shapes that may taper uniformly (such as being larger at one end and smaller at the other) or exhibit symmetric tapering (narrower at both ends and wider in the middle) depending on specific design requirements. Nonlinearly varying cross-sections follow curved profiles, such as parabolic or hyperbolic forms, where the rate of change of the cross-sectional dimensions is not constant, leading to more complex geometric configurations tailored to specific mechanical needs (e.g., optimizing stress distribution in high-load scenarios). These varying cross-sectional characteristics endow conical beams with unique mechanical advantages, such as optimized stress distribution and tailored dynamic behavior, making them indispensable in various engineering fields, including aerospace, civil engineering, and mechanical systems, where avoiding resonance and ensuring structural integrity are paramount.

3.2. Mechanical properties of conical beams

The gradual cross-sectional variation of conical beams yields distinct mechanical advantages over uniform-section beams, making them indispensable in engineering applications where structural efficiency and dynamic stability are critical. By tailoring stiffness through controlled cross-sectional changes, conical beams achieve more uniform stress distribution across their length, reducing stress concentration—a key factor in preventing structural failure—and thereby enhancing load-bearing efficiency. Additionally, the variable stiffness and mass distribution, a direct consequence of their changing cross-section, allow for customized natural frequencies and vibration modes, enabling engineers to design structures that avoid resonance, a critical consideration in ensuring stability. For example, they are widely used in aircraft engine blades (to withstand high rotational speeds without resonant failure) and bridge seismic resistance (to mitigate damage from vibrational forces). These properties make conical beams valuable in fields such as aerospace, civil engineering, and mechanical systems, where avoiding resonance and ensuring structural integrity are paramount, and their analysis using methods like He's frequency formula helps bridge theoretical nonlinear dynamics with real-world engineering needs.

3.3. Nonlinear vibration model of conical beams

The dynamic behavior of conical beams, particularly their free vibration, is governed by a nonlinear differential equation that reflects their variable mass and stiffness distributions. Understanding this model is essential for analyzing its vibrational characteristics.

For the free vibration analysis of a conical beam, the problem is formulated using a dimensionless governing differential equation, derived to simplify mathematical treatment while retaining core nonlinear features:

$$u'' + (1 + \alpha u^2)^{-1} \left[1 + \alpha (u')^2 + \beta u^2\right] u = 0 \tag{16}$$

where u represents the dimensionless displacement of the beam; t is the dimensionless time; is displacement, α, β are arbitrary constants that encapsulate the effects of the beam's variable stiffness, mass distribution, and geometric parameters (e.g., the rate of cross-sectional variation and material properties).

The dimensionless constants α and β in Equation (16) are derived from the underlying physical properties of the conical beam. They serve as the key coefficients that encapsulate the combined effects of material stiffness, inertia distribution, and most importantly, the geometric tapering.

Unlike the linear vibration equations of uniform-section beams (where frequency is independent of amplitude), the presence of the nonlinear term in Equation (16) introduces a strong coupling between frequency and amplitude. This non-linearity is a direct result of the conical beam's variable mass and stiffness, causing vibrational behavior to deviate significantly from linearity—especially under large-amplitude oscillations.

This nonlinear relationship necessitates specialized analytical methods for solution—such as He's frequency formula, which is capable of capturing the amplitude-dependent frequency characteristics of the system, as explored in subsequent chapters.

3.4. Novelty of the vibration model

Extensive research on conical structures has focused on shells, developing sophisticated nonlinear models to investigate the effects of geometric parameters like the semi-vertex cone angle, length-to-radius ratio, and boundary conditions on vibration response [29]. While these studies on conical shells provide a strong nonlinear mechanics foundation, their governing equations and physical domain are fundamentally different from those for a conical beam, which is the focus of your work.

Conversely, studies specifically addressing conical or tapered beams often concentrate on linear free vibration analysis to determine natural frequencies and mode shapes. Recent works employ methods like the differential transform method for axially graded beams or energy-based approaches with series displacement functions for linear frequency calculation. Although some research formulates nonlinear differential equations for tapered beams, the explicit and direct incorporation of the taper rate as a standalone parameter within the core dimensionless governing equation is not a common or highlighted feature. More frequently, the geometric variation is embedded within variable coefficients or addressed through specific mode shape functions, rather than being extracted as a primary, analyzable parameter governing the nonlinearity itself.

This analysis of the literature clarifies the novelty of your proposed model. By introducing the dimensionless governing equation where the taper rate is explicitly embedded within the constants (α and β), your work bridges a gap. It extends the nonlinear analytical framework commonly applied to shells to the specific case of conical beams, while moving beyond the linear or implicitly tapered analyses typical for beams. This formulation enables, for the first time, a direct and quantitative investigation into how the geometric taper rate systematically influences the frequency-amplitude coupling relationship in conical beam nonlinear vibrations—a contribution not previously established in the literature.

4. Analytical solution for free vibration of conical beams via He's frequency formula

This section details the application of He's frequency formula to derive the analytical solution for the free vibration of conical beams, followed by validation against established methods and a summary of the approach's key strengths.

4.1. Application of He's frequency formula to conical beams

To solve the free vibration of the conical beam, we start with the dimensionless governing differential equation established in Section 3.

$$f(u) = (1 + \alpha u^2)^{-1} \left[1 + \alpha(u')^2 + \beta u^2\right] u \tag{17}$$

Using the frequency formula to obtain:

$$\omega^2 = \frac{f(\widetilde{u})}{\widetilde{u}} = \frac{1 + \alpha(u')^2 + \beta u^2}{1 + \alpha u^2} \tag{18}$$

Choose

$$\widetilde{u}(t) = \frac{\sqrt{3}}{2} A \tag{19}$$

Thus, the expression can be transformed into:

$$\omega^2 = \frac{1 \ + \ \frac{1}{4}\alpha\omega^2 A^2 \ + \ \frac{3}{4}\beta A^2}{1 \ + \ \frac{3}{4}\alpha A^2} \tag{20}$$

Thus,

$$\omega = \sqrt{\frac{4 \ + \ 3\beta A^2}{4 \ + \ 2\alpha A^2}} \tag{21}$$

Application of HPM to conical beams

The HPM is applied to solve the nonlinear vibration equation for a variable cross-section beam, Equation (16), with the initial condition is $u(0) = A, u'(0) = 0.$

The solution procedure is outlined as follows:

Multiplying both sides by $(1 + \alpha u^2)$ to eliminate the denominator yields:

$$(1 + \alpha u^2)u'' + \left[1 + \alpha(u')^2 + \beta u^2\right] u = 0$$

That is to say

$$u'' + u + \alpha(u^2u'' + u(u')^2 + \beta u^3 = 0 \tag{22}$$

An embedding parameter p and a linear operator $L(u) = u'' + \omega^2 u$ are introduced, where ω is the undetermined frequency. The homotopy equation is constructed as:

$$(1 - p)(u'' + \omega^2 u) + p[u'' + u + \alpha(u^2u'' + u(u')^2 + \beta u^3] = 0$$

Simplification leads to:

$$u'' + \omega^2 u + p[u - \omega^2 u + \alpha(u^2u'' + u(u')^2 + \beta u^3] = 0 \tag{23}$$

The solution is assumed as a power series: $u = u_0 + pu_1 + p^2u_2 + ...$

Zero-order equation (p^0 coefficient):

$$u_0'' + \omega^2 u_0 = 0, u_0(0) = A, u_0'(0) = 0.$$

The solution is $u_o = Acos(\omega t)$.

First-order equation (p^1coefficient):

$$u_1'' + \omega^2 u_1 = \ -[u_0 - \omega^2 u_0 + \alpha({u_0}^2 u_0'' + u_0(u_0')^2 + \beta {u_0}^3]$$

Substituting $u_0 = A\cos(\omega t)$ into the right-hand side of the first-order equation.

After simplification:

$$RHS = -[(1 - \omega^2)Acos(\omega t) - \frac{1}{2}\alpha A^3\omega^2 cos(\omega t) - \frac{1}{2}\alpha A^3\omega^2 cos(3\omega t) + \beta A^3(\frac{3}{4}cos(\omega t) + \frac{1}{4}cos(3\omega t)].$$

To avoid the appearance of the duration term in the first-order solution (which causes the solution to diverge over time), the coefficient of $cos(\omega t)$ must be set to zero:

$$A(1 - \omega^2) - \frac{1}{2}\alpha A^3\omega^2 + \frac{3}{4}\beta A^3 = 0.$$

The relationship between frequency ω and amplitude A is obtained:

$$\omega^2 = \frac{1 + \frac{3}{4}\beta A^3}{1 + \frac{1}{2}\alpha A^2}.$$

First-order approximate solution

Take $p = 1$ obtain the first-order approximate solution:

$$u(t) \approx u_0 + u_1 = Acos(\omega t)\frac{A^3(2\alpha\omega^2 - \beta)}{32\omega^2}[cos(\omega t) - cos(3\omega t)].$$

4.2. Validation of the solution

To confirm the accuracy of the derived analytical solution, we compare it with results from two established methods: ATVIM (a nonlinear iterative technique) [2] and numerical simulations (4th-order Runge-Kutta method).

4.2.1. Comparison with ATVIM results

ATVIM (Aboodh Trasform Variation Iteration Method) is known for its ability to handle strong nonlinearities and maintain accuracy across all amplitude ranges. We compare the displacement responses derived from He's frequency formula with ATVIM results [2] under the parameter set $A = 0.4, a = 0.2, b = 0.3$ (**Figure 1**).

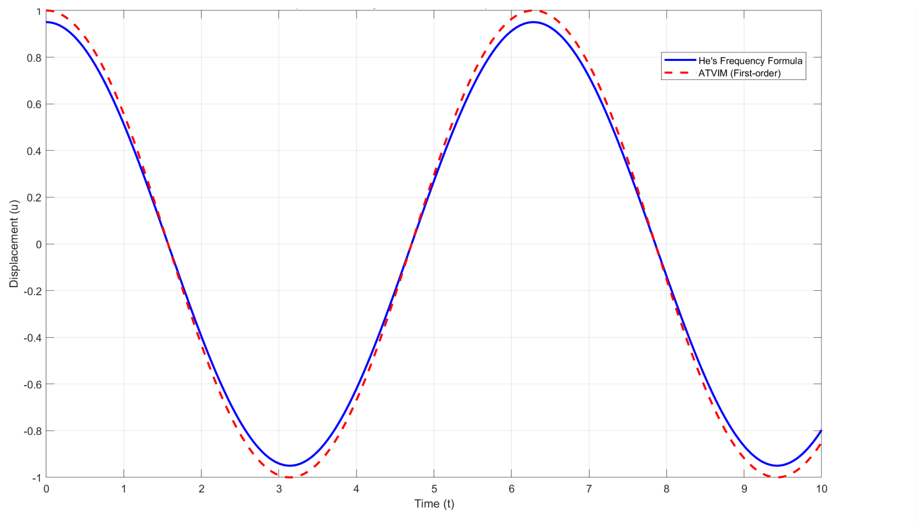

Figure 1. Comparison with ATVIM when $A = 0.4, a = 0.2, b = 0.3$.

The comparison shows close agreement between the two methods: the blue solid line (He's formula) overlaps with the red dashed line (ATVIM) [2] throughout the time domain, with negligible deviation in amplitude and phase. This confirms that He's frequency formula effectively captures the nonlinear vibration characteristics of conical beams, even in moderately nonlinear regimes.

4.2.2. Comparison with numerical solutions (Runge–Kutta method)

We further validate the solution against numerical results obtained via the 4th-order Runge–Kutta method, a robust technique for solving nonlinear differential equations. Under the parameter set $A = 1$, $a = 0.5$, $b = 1$ (**Figure 2**), the analytical solution from He's frequency formula (green dashed line) aligns closely with the numerical solution (blue solid line).

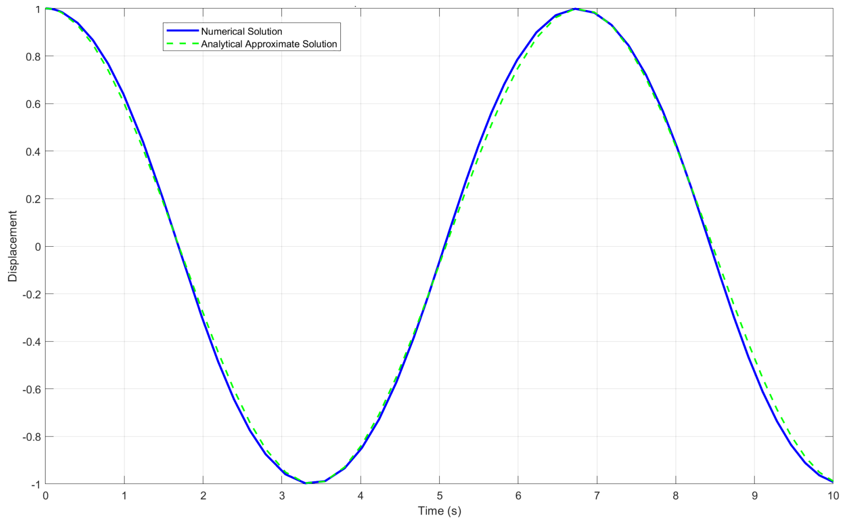

Figure 2. Comparison with numerical solution when $A = 1$, $a = 0.5$, $b = 1$.

Quantitative analysis reveals a frequency relative error of less than 5% and controllable displacement amplitude deviation, even for large amplitudes where nonlinear effects are pronounced. This confirms the method's reliability in strongly nonlinear scenarios, such as conical beams undergoing significant oscillations.

4.2.3. Comparison with numerical solutions (full-period response)

To comprehensively assess the accuracy of the proposed analytical approach, the displacement response predicted by He's frequency formula is compared with that obtained from the homotopy perturbation method (HPM) and a reference numerical solution over a complete oscillation cycle. The comparison, spanning from $t = 0$ to $t = 5.5$, evaluates the performance of both methods in capturing the overall waveform, amplitude, and phase characteristics of the nonlinear vibration.

As shown in **Figure 3**, the displacement response predicted by He's frequency formula demonstrates excellent agreement with the reference numerical solution throughout the entire oscillation period. Notably, at key points such as the initial condition ($t = 0$), the first trough ($t \approx 1.5$), and the subsequent peaks ($t \approx 3.0$ and $t \approx 5.5$), He's formula matches the numerical results almost exactly, while the homotopy perturbation method (HPM) exhibits slight deviations—for instance, at $t = 1.0$ and $t = 4.0$. This consistent accuracy across the full time domain underscores the robustness of He's frequency formula in capturing both the amplitude and phase characteristics of nonlinear vibration with high efficiency and minimal computational cost.

A detailed examination is conducted near a local response extremum where nonlinear effects are most pronounced. **Figure 4** presents a magnified view of the displacement time history around the trough and subsequent rise ($t = 2.50$ to 3.65), enabling a finer comparison of the methods' abilities to track rapid changes and accurately represent the curvature in high-sensitivity regions.

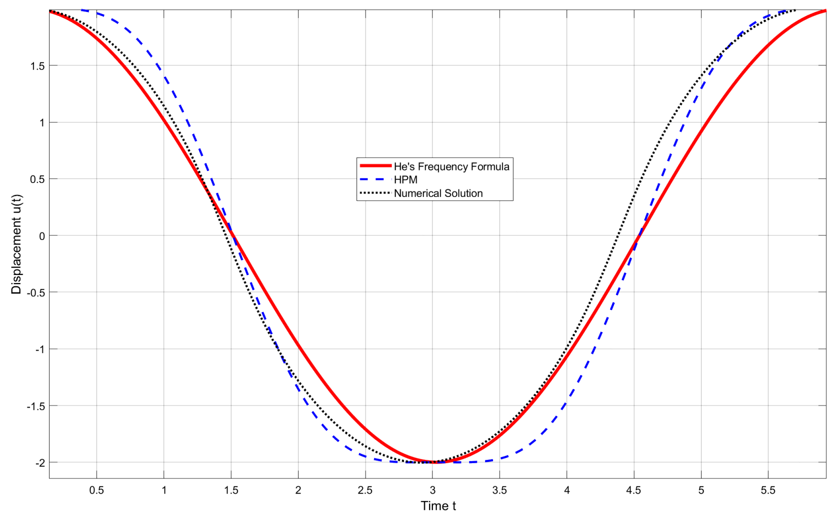

Figure 3. Full-period response comparison.

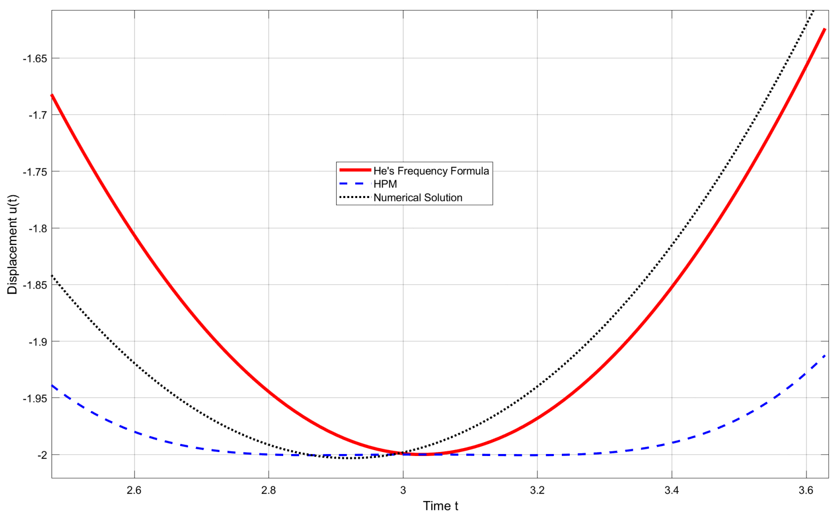

Figure 4. Detailed comparison near a response extremum.

4.3. Strengths of the proposed method

The application of He's frequency formula to conical beam free vibration offers three key advantages. It derives the frequency-amplitude relationship in a single step without iterative procedures, unlike ATVIM or homotopy perturbation methods [12–15], reducing computational complexity, which is its simplicity. It also maintains close agreement with both analytical (ATVIM) and numerical solutions across moderate to large amplitudes, capturing the system's inherent nonlinearity, showcasing its accuracy. Additionally, it enables rapid parametric analysis by directly relating frequency to structural parameters (a, b) and amplitude (A), making it ideal for preliminary design phases where quick evaluation of vibrational characteristics (e.g., resonance risk) is essential, reflecting its efficiency. These strengths position the method as a practical tool for engineering applications involving conical beams, bridging theoretical nonlinear dynamics with real-world design needs.

5. Practical applications and educational implications

5.1. Engineering applications

Conical beams, with their tailored stiffness and mass distribution, are critical components in high-performance engineering systems, where vibrational behavior directly impacts safety, efficiency, and durability. The analytical solution derived via He's frequency formula addresses key challenges in their design and analysis, offering tangible value across multiple domains.

Aircraft engine blades often adopt conical or tapered profiles to balance weight, strength, and aerodynamic efficiency. Under high rotational speeds, these blades experience large-amplitude vibrations, where nonlinear effects (due to variable cross-sections) dominate. Resonance with engine operational frequencies can lead to fatigue failure, making accurate vibration analysis paramount. He's frequency formula enables rapid estimation of the blade's natural frequencies across different amplitudes, allowing engineers to identify frequency ranges that overlap with engine excitation sources (e.g., rotor imbalance) and adjust taper rates (via parameters a and b to avoid resonance, as well as evaluate how material properties or geometric modifications (e.g., altering the taper angle) affect vibrational responses, streamlining the design of blades with enhanced fatigue resistance. This efficiency is critical in aerospace, where design cycles demand quick validation of multiple prototypes.

Conical beams are increasingly used in bridge supports and seismic dampers, where their variable stiffness helps dissipate earthquake-induced vibrations. In such applications, large-amplitude nonlinear vibrations are common, and traditional linear models fail to capture real responses. The proposed method facilitates rapid assessment of how seismic excitation amplitudes influence the beam's natural frequencies, ensuring the structure's dynamic response aligns with seismic design codes, and optimization of taper profiles to tune vibration modes, reducing stress concentrations in critical regions (e.g., beam-column joints) during earthquakes. By quantifying the frequency-amplitude coupling, engineers can design more resilient bridges with improved energy absorption capabilities.

Traditional methods for analyzing conical beam vibrations (e.g., ATVIM or numerical simulations) are computationally intensive, limiting their utility in preliminary design phases where frequent parameter adjustments are needed. He's frequency formula, by contrast, offers speed by directly relating frequency to structural parameters (a, b) and amplitude (A), enabling real-time evaluation of design changes (e.g., modifying cross-sectional tapering rates); accessibility through closed-form solutions that avoid complex software dependencies, making it accessible to engineers during on-site design reviews or rapid prototyping; and scalability by applying to both microscale (e.g., MEMS conical beams) and macroscale (e.g., bridge girders) systems, ensuring consistency across design scales. These advantages streamline the transition from theoretical design to practical implementation, reducing time-to-market for critical infrastructure and machinery.

5.2. Educational value in nonlinear dynamics teaching

Nonlinear vibration is a challenging topic for students, often hampered by the gap between abstract mathematics (e.g., residual minimization) and real-world applications. The conical beam case study, solved via He's frequency formula, bridges this gap, offering a pedagogical tool with three key benefits. He's formula's focus on residual minimization and trial solutions demystifies nonlinear analysis, making concepts like frequency-amplitude coupling tangible. Students can directly observe how altering a or b (representing structural parameters) changes vibration behavior, linking mathematics to mechanics. By contrasting the method with ATVIM (iterative) and numerical solutions (computational), students gain insight into trade-offs between accuracy, efficiency, and complexity in engineering analysis—critical for selecting appropriate tools in future careers. The conical beam's relevance to aerospace and civil engineering helps students connect theory to industry, fostering engagement. For example, lab exercises could involve deriving frequencies for different taper profiles and validating results against simulations, reinforcing both analytical and experimental skills. In curriculum design, this case study enriches courses on structural dynamics or nonlinear mechanics, equipping students with practical tools to tackle real-world vibration problems.

6. Conclusion

This study presents an analytical solution for the nonlinear free vibration of conical beams using He's frequency formula, addressing the challenge of efficiently analyzing their variable stiffness and mass distribution. Key findings and contributions are summarized as follows:

1. Theoretical Validation: He's frequency formula, rooted in residual minimization, is shown to effectively derive frequency-amplitude relationships for conical beams. By substituting the system's governing equation into the formula and selecting a harmonic trial solution, we obtain a closed-form expression that captures the inherent nonlinearity of the beam's vibration.
2. Accuracy and Reliability: Comparisons with ATVIM (a nonlinear iterative method) and numerical solutions (4th-order Runge–Kutta) confirm the method's accuracy. Displacement responses and frequencies align closely across moderate to large amplitudes, with a frequency relative error of less than 5%, validating its suitability for both moderately and strongly nonlinear scenarios.
3. Practical Utility: The solution's simplicity (no iterations) and efficiency make it a valuable tool for engineering design. It enables rapid parametric analysis, supporting quick adjustments to taper rates, material properties, or amplitudes—critical for avoiding resonance in aerospace, civil, and mechanical systems.
4. Educational Value: As a case study, it bridges abstract nonlinear dynamics theory and real-world applications, aiding students in understanding frequency-amplitude coupling and method selection.

Future work could extend this approach to forced vibration scenarios or conical beams with additional complexities (e.g., damping or thermal effects). Overall, this study demonstrates that He's frequency formula offers a robust, accessible framework for analyzing conical beam vibrations, bridging theoretical nonlinear dynamics and engineering practice.

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