Sound & Vibration 2025, 59(2), 2358.
https://doi.org/10.59400/sv2358

Article

Scrutinizing highly nonlinear oscillators using He's frequency formula

Gamal M. Ismail[1,*], Galal M. Moatimid[2], Ibrahim Alraddadi[1], Stylianos V. Kontomaris[3,4,5]

[1] Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah 42351, Saudi Arabia
[2] Department of Mathematics, Faculty of Education, Ain Shams University, Roxy, Cairo 11566, Egypt
[3] Department of Engineering and Construction, Metropolitan College, 15125 Athens, Greece
[4] School of Sciences, European University Cyprus, Nicosia 2043, Cyprus
[5] BioNanoTec Ltd., Nicosia 2043, Cyprus
*** Corresponding author:** Gamal M. Ismail, gismail@iu.edu.sa

CITATION

Ismail GM, Moatimid GM, Alraddadi I, Kontomaris SV. Scrutinizing highly nonlinear oscillators using He's frequency formula. Sound & Vibration. 2025; 59(2): 2358. https://doi.org/10.59400/sv2358

ARTICLE INFO

Received: 23 December 2024
Accepted: 6 February 2025
Available online: 5 March 2025

Abstract: Highly nonlinear oscillators are examined in their capacity to simulate intricate systems in engineering, physics, biology, and finance, as well as their diverse behavior, rendering them essential in the development of resilient systems and technological advancement. Therefore, the fundamental purpose of the current work is to analyze He's frequency formula (HFF) to get theoretical explanations of many types of very nonlinear oscillators. We investigate, in both analytical and computational, the relationship between elastic forces and the solution of a specific oscillator. This oscillator exhibits significant nonlinear damping. It is assumed that the required quantity of trigonometric functions matches the solution of a strong nonlinear ordinary differential equation (ODE) that explains the motion. The novel approach definitely takes less processing time and is less complex than the traditional perturbation methods that were widely used in this field. This novel method, which is essentially giving a linearization of the nonlinear ODE, is known as the non-perturbative approach (NPA). This procedure produces a new frequency that is similar to a linear ODE, much as in a fundamental harmonic scenario. Readers will benefit from an in-depth account of the NPA. The theoretical findings are validated by numerical examination using Mathematical Software (MS). The theoretical and numerical solution (NS) tests yielded fairly similar findings. It is a well-established principle that classical perturbation methods trust on Taylor expansions to approximate restoring forces, therefore simplifying the current situation. When the NPA is used, this vulnerability does not present. Furthermore, the NPA enables a thorough assessment of the problems' stability analysis, which was a not possible using prior conventional methodology. Consequently, the NPA is a more appropriate responsibility tool for examining approximations in extremely nonlinear oscillators in MS.

Keywords: non-perturbative approach; analytical solutions; nonlinear oscillators; numerical solutions

1. Introduction

Rapid expansion of nonlinear science appears to be motivating interest among scientists and engineers in analytical asymptotic techniques for nonlinear problems. While it has been easier recently to solve linear systems through numerical simulations, addressing nonlinear issues analytically is still very difficult. The Duffing oscillator (DO) is one of the most well-known instances of a Hamiltonian system. However, there has not been much research done on simple generalizations of these oscillators, such as cubic-quintic DO. In numerous fields, both linear and nonlinear ODEs are employed to describe issues relevant to mathematics, physics, biology, chemistry, and engineering. Unlike nonlinear ODEs, which are traditionally thought to have approximate solutions via many perturbation techniques, the solutions of a linear ODE may be easily obtained using recognized approaches. Furthermore, nonlinear oscillations have attracted the attention of a growing number of scientists. Because scientific and technical phenomena regularly took the form of nonlinear kinds, the nonlinear ODE is therefore highly helpful in explaining them. The nonlinear ODEs were therefore essential in mathematics, applied physics, and engineering [1]. Emphasizing the significance of mathematical calculations in various research works and publications pertaining to nonlinear ODEs that emerge in various scientific and engineering domains was imperative [2]. Few nonlinear ODEs have direct solutions, even if many have numerical approximations. The literature has used a variety of approximate analytical techniques to determine the relationship between the nonlinear oscillators' frequency-amplitude. The most multipurpose method in evaluating non-linear engineering problems was the perturbation technique, which was widely utilized to derive approximate analytical solutions to nonlinear ODEs [3–5]. The past twenty years have seen an unbelievable growth in the nonlinear sciences, which has sparked an increasing interest in analytical methodologies in nonlinear difficulties among scientists and engineers. It was developed to examine the behavior of these nonlinear ODEs using both numerical and non-numerical approximation techniques [6–12]. A number of novel techniques have recently been developed in analytically solving the nonlinear ODEs. Consequently, several researchers developed a few special methods. In order to realise analytical responses that closely approximate the exact solutions, many researchers have explored numerous innovative and unique methods. It was done using the Lindstedt-Poincaré process [13]. The homotopy perturbation method (HPM) is one of these techniques [14,15]. A multitasking grapheme electromagnetic detecting device utilizing second harmonic generation was presented [16]. The generalized forms of these non-equilibrium work theorems, applicable to dissipative transformations involving simultaneous mechanical work and pressure-temperature or volume-temperature variations, were presented [17]. A multifunctional device was developed, capable of passive multiplication and division, along with high-performance sensing of multiple physical quantities. This was achieved through a design integrating optical Tamm states, the inherent absorption properties of liquid crystals, and nonlinear optical effects [18]. Precise soliton solutions of certain novel changing nonlinear coefficients were derived [19]. The profile characteristics of the evolution wave functions, contingent upon the composite functions, were obtained. These solutions possess potential applicability in molecular physics. Gaussian solitary wave solutions were derived for a specific class of logarithmic non-linear Schrödinger equations incorporating a conventional harmonic oscillator potential [20]. The behavior of the Gaussians was depicted and the relationships of the pertinent parameters were analyzed.

The approximations of the analytical responses are valid throughout the solution. To address the limitations of traditional perturbation methods, alternative approaches are employed, often combined with mathematical tools like variational theory, the homotopy perturbation method (HPM), and iterative techniques. It is necessary to expeditiously estimate a nonlinear oscillator's periodic characteristic for engineering applications. Moatimid et al. [21–23] employed HPM to examine various problems in Fluid Mechanics and Dynamical Systems. A nonlinear ODE can be made almost linear by using the frequency approach. A review was conducted on some recent advancement in asymptotic techniques in powerfully and weakly nonlinear systems [24]. Some of the more straightforward techniques for nonlinear oscillators that were covered were the HPM, the max-min approach, and the HFF [25]. The HFF was explained mathematically, and the frequency of projected accuracy was increased by adding the weighted average. A severely nonlinear oscillator was studied using an available and straightforward technique [26]. The simplest calculation can be used to rapidly determine its frequency quality. The outcome demonstrated that the approach provided a rationally accurate response. The nonlinearity of a vibration system was reflected in the link between frequency and amplitude. For nonlinear oscillators with arbitrary initial conditions (ICs), our work offered a simple frequency prediction method [27]. The results from the HPM and the ones from the study agreed fairly well. To quickly and precisely understand the nonlinear vibration of the system parameters, a very helpful technique is developed. A discussion of periodic qualities, unstable features, and a spinning pendulum ensued regarding the use of the HFF. Studying the characteristics dynamic of a pendulum attached to a solid rotational frame with an unchanged angular velocity, the vertical axis that goes through the pendulum’s pivot point was considered. The controlling nonlinear ODE of an analytical solution was created using the HFF since linear problems frequently have perfect solutions. The linearized equation, also referred to as a quasi-exact solution, depicts an almost accurate solution to the nonlinear ODE. No matter what, solving a linear problem was easier than solving a nonlinear. In order to address the damping nonlinear oscillator, a developed HFF was constructed. By linearizing a nonlinear oscillator using its conservative restoring force, a frequency component arising from the odd nonlinear damping was revealed. The amplitude-frequency formulation for nonlinear oscillators elucidates the crucial mechanism of pseudo-periodic motion and identifies that the quadratic nonlinear force contributes to the pull-down phenomenon in each cycle of periodic motion. When the force attains a threshold value, pull-down instability ensues [28]. The periodic motion of the micro-electro-mechanical system, influenced by a singularity that complicates the determination of an accurate solution and the comprehension of its dynamic features, was analysed [29]. It was discovered that when the amplitude attains a threshold value, the periodic motion transitions to pull-in instability. A survey of the periodic properties of micro-electro-mechanical systems was conducted using the HPM, variational iteration method, variational theory, HFF, and Taylor series method [30]. The HPM is a common technique for nonlinear oscillators; however, the results are only relevant in cases of weak nonlinearity. Alternative analytical techniques, such the variational iteration approach and the HPM, can produce an acceptable approximate solution; however, each technique requires the execution of numerous calculations. A one-step frequency formulation for nonlinear oscillators was recommended, with this part 1 concentrating on odd nonlinearity [31]. An early Babylonian procedure for computing the square root of 2 was revealed, and the possible connection between this rudimentary technology and an ancient Chinese method was examined [32]. Subsequently, the approach was innovatively expanded to address algebraic equations. The fundamental HFF for non-linear oscillators was presented and validated, and a variation was proposed [33]. A fractal vibration within a porous medium was investigated, and its low-frequency behavior was characterized using a frequency-based formulation. Nonlinear oscillation is a progressively significant and highly intriguing subject in engineering. A straightforward method introduced by Prof. He and effectively formulates a fractal un-damped DO utilizing the two-scale fractal derivative in a fractal space was reported [34]. The numerical outcome indicates that HFF is an exceptional instrument for fractal equations. The variational principle and frequency formula of the fractal non-linear equation are derived using straightforward methods presented by Professor He. A fractal adaptation of the non-linear oscillator in a porous basis vibration was established [35].

The practical applications of each of the following problems are:

1) Micro-electromechanical systems (MEMS) are extensively utilized in sensors, actuators, and microelectronics. They are utilized in accelerometers for airbag activation, gyroscopes for navigational systems, pressure sensors in medical apparatus, and microfluidics for lab-on-chip technologies. MEMS technology facilitates the miniaturization and integration of mechanical and electrical systems, resulting in high-precision, cost-effective devices.
2) Nonlinear systems derived from the natural oscillation of a conservative oscillator: These systems are examined to mimic and analyze real-world events, including pendulums, vibrating molecules, and energy transmission in mechanical systems. Useful applications encompass the creation of energy harvesters, the analysis of chaos in mechanical systems, and the optimization of oscillatory systems in engineering.
3) A rigid rod oscillating on a circular surface without slipping serves as a model for comprehending contact mechanics and stability in rocking structures. Applications encompass the earthquake-resistant design of monuments, the analysis of toys such as rocking horses, and mechanical systems where stability under periodic stresses is essential.
4) The motion of a particle on a rotating parabola has both experimental and theoretical implications for comprehending stability and equilibrium in rotating systems. It simulates phenomena in astrophysics (e.g., motion within gravity wells), particle dynamics in electromagnetic traps, and rotational dynamics in mechanical and robotic systems.

It is clear that for the situations stated previously, the current approach produces findings that are more accurate than comparable approximations. As seen, the NPA has an allowance of promise and can be used to handle more significantly nonlinear scenarios. The NPA addresses multiple real-world scenarios when combined. Previously, various popular analytical techniques that had been previously included in the literature were used to resolve these real-world problems. However, employing our current methodology yields better outcomes more quickly. Though, compared to other analytical techniques, the calculations with MS aid are significantly simpler when utilizing the NPA, and the methods for figuring out the analytical solutions are well demonstrated. Some computation techniques were time-consuming to employ or required an allocation of work to analyze the result. In light of the unique approach used or notable findings, the following facts should be highlighted:

1) An alternative comparable linear ODE generated by the approach is identical to the existing non-linear one.
2) When employing this method, these two ODEs are completely matching one another.
3) When restoring forces are present, all traditional techniques employ Taylor expansion to simplify the situation. This weakness has been addressed in the current plan.
4) The NPA offers an advantage over conventional methods by enabling us to analyze the stability of the problem.
5) The new approach seems like an intriguing, useful, and simple-to-use tool. It can be applied to the analysis of various nonlinear oscillators.

To crystalize the presentation of the paper, its remainder will be organized as follows: The current work is divided into five sections that aid in making its presentation more understandable. In **§ 2**, we illustrate and briefly recapitulate the NPA description. In **§ 3**, some real-world nonlinear ODEs are analyzed with the NPA. An overview of the present study's dissections is provided in **§ 4**. Finally, **§ 5** offers a synopsis of the closing thoughts.

2. A brief explanation of NPA

Let us consider a highly nonlinear ODE in the following form:

$$\eta'' + F(\eta,\eta',\eta'') + G(\eta,\eta',\eta'') = H(\eta,\eta',\eta'') \tag{1}$$

where $F(\eta,\eta',\eta'')$ and $G(\eta,\eta',\eta'')$ are third-order functions. In addition, $H(\eta,\eta',\eta'')$ is a quadratic function.

The above functions can be expressed as follows:

$$\left.\begin{aligned} F(\eta,\eta',\eta'') &= a_1\eta' + b_1\eta\eta''\eta' + c_1\eta^2\eta' + d_1\eta'^3 + e_1\eta''\eta'^2,\\ G(\eta,\eta',\eta'') &= \omega^2\eta + b_2\eta'\eta^2 + c_2\eta\eta'^2 + d_2\eta^3 + e_2\eta''\eta^2,\\ H(\eta,\eta',\eta'') &= a_2\eta\eta' + b_2\eta'^2 + c_2\eta^2 + d_2\eta'\eta'' + e_3\eta\eta'', \end{aligned}\right\} \tag{2}$$

where $a_j, b_j, c_j, d_j, e_j (j = 1,2,3)$ are constant, and ω denotes the natural frequency of the structure.

As previously proven in the traditional perturbation approaches [3–5], the third quadratic function does not yield any secular terms, but the first two terms do. Currently, the NPA's primary purpose is to generate a replacement linear ODE. Three constants will be determined in order to form the required linear ODE. To do this, in accordance with Ismail et al. [9], a guessing (trial) solution of the specified nonlinear ODE has the form:

$$\tilde{x} = A \cos \Omega\, t \tag{3}$$

The ICs are as follows: $\tilde{x}(0) = A$ and $\tilde{x}'(0) = 0,$

where Ω refers to the total frequency, which will be calculated later.

One possible design of the essential linear ODE is as follows:

$$x'' + \sigma_{eqv}x' + \omega_{eqv}^2 x = \Lambda \tag{4}$$

As well-known, the HFF used a combination of the HPM and an averaging technique to estimate the frequency of nonlinear oscillators. It provides an analytical framework that specifies the relationship between oscillation frequency and amplitude, especially beneficial in systems demonstrating moderate nonlinearity. The NPA employs the system's periodic properties by integrating across a complete period. The oscillatory motion completes one cycle in certain duration, and integrating over this period ensures that the frequency estimation corresponds with the system's inherent dynamics. Integrating over a complete period offers the following benefits:

1) The nonlinear effects are thoroughly documented.
2) The total energy or action is conserved.
3) The response function includes higher-order nonlinearities.

Physical interpretation

Nonlinear Frequency Alteration

The frequency of a nonlinear system deviates from the linear case due to amplitude dependence. This method enables the estimation of this frequency without requiring series expansions, making it advantageous even amongst considerable nonlinearities.

1) Viewpoint on Energy Conservation
2) External Perturbation Theory
3) The HFF in integral form accurately balances kinetic and potential energy, ensuring that the frequency estimation aligns with the complete energy distribution over a full cycle.

Outstanding Perturbation Theory

Unlike conventional approaches that assume modest modifications within a linearized framework, the NPA inherently incorporates substantial nonlinear elements, making it suitable for systems displaying high amplitudes or hard/soft stiffness characteristics. Therefore, the NPA, based on HFF and thorough evaluation over A, provides a robust method for analyzing highly nonlinear oscillators, avoiding the limitations of perturbation theory.

As previously demonstrated [31–35], the aforementioned three parameters can be assessed as follows:

$$\sigma_{eqv} = \int_0^{2\pi/\Omega} \tilde{x}' F(\tilde{x}, \tilde{x}', \tilde{x}'')\, dt \Big/ \int_0^{2\pi/\Omega} \tilde{x}'^2\, dt = \sigma_{eqv}(\Omega) \tag{5}$$

Take into consideration the equivalent frequency that can be ascertained as follows in terms of a function of the total frequency:

$$\omega_{eqv}^2 = \int_0^{2\pi/\Omega} \tilde{x} G(\tilde{x}, \tilde{x}', \tilde{x}'')\, dt \Big/ \int_0^{2\pi/\Omega} \tilde{x}^2\, dt = \omega_{eqv}^2(\Omega) \tag{6}$$

The non-secular component can be solved using the quadratic formula. Therefore, the even non-secular function will be replaced by the following to compute the inhomogeneity: $\eta \to kA, \eta' \to kA\Omega,$ and $\eta'' \to kA\Omega^2$. It was demonstrated that the parameter k can be described as follows: $k = 1/2\sqrt{n - r}$: where n = 2 denotes the system’s degree of freedom and r = 1. Therefore, in this instance, after which the value of becomes k = 1/2. The value of the quadratic (a non-secular term) follows. As a result, the inhomogeneity part Λ will be calculated by substituting: $\eta \to \frac{A}{2}, \eta' \to \frac{A\Omega}{2}$, and $\eta'' \to \frac{A\Omega^2}{2}$.

For additional convenience, Equation (4) may be stated in an appropriate normal form by applying the substitution:

$$x(t) = \tilde{x}(t)Exp(-\sigma_{eqv}t/2) \tag{7}$$

By putting Equation (7) into Equation (4), one obtains

$$\tilde{x}'' + \left(\omega_{eqv}^2 - \frac{1}{4}\sigma_{eqv}^2\right)\tilde{x} = \Lambda Exp(\sigma_{eqv}t/2) \tag{8}$$

Therefore, the overall frequency is provided by $\Omega^2 = \omega_{eqv}^2 - \frac{1}{4}\sigma_{eqv}^2$.

3. Applications

This section examines various highly nonlinear problems utilizing the previously NPA.

3.1. Example 1

This example focuses on an adaptable vibrating structure applicable to both nano and micro electromechanical systems (N/MEMS). It aims to illustrate the previously established theoretical framework by applying it to a general model of N/MEMS oscillators. Subsequently, the example will demonstrate how this general model can be adopted to represent three specific, well-known N/MEMS devices commonly used in Nano science and nanotechnology, as documented in source [36]. We will analyze the movement of these microstructures, which are governed by a nonlinear ODE that captures the behavior of a class of oscillators found in N/MEMS.

The main equation of motion may be expressed as follows.

$$\begin{gathered}(1 + d_1 y + d_2 y^2 + d_3 y^3 + d_4 y^4)\ddot{y} + d_5 + d_6 y + d_7 y^2 \\ + d_8 y^3 + d_9 y^4 + d_{10} y^5 + d_{11} y^6 + d_{12} y^7 = 0\end{gathered} \tag{9}$$

where $d_j = \alpha_j/\alpha_0, j = 1,2,3, \ldots .12$. Additionally, the constants $\alpha_0, \alpha_1, \ldots, \alpha_{12}$ are derived from the transformation of a multivariable differential equation into an ordinary differential equation (ODE) using the Galerkin approach [37].

In accordance with the NPA, Equation (9) may be written as follows:

$$\ddot{y} + f_1(y, \ddot{y}) + f_2(y, \ddot{y}) = 0 \tag{10}$$

where

$$\left.\begin{aligned} f_1(y, \ddot{y}) &= d_2 y^2 \ddot{y} + d_4 y^4 \ddot{y} + d_6 y + d_8 y^3 + d_{10} y^5 + d_{12} y^7, \\ f_2(y, \ddot{y}) &= d_2 + d_1 y \ddot{y} + d_3 y^3 \ddot{y} + d_7 y^2 + d_9 y^4 + d_{11} y^6. \end{aligned}\right\} \tag{11}$$

As seen from Equation (11), the function $f_1(y, \ddot{y})$ represents an odd function, which produces the secular terms. Meanwhile, the function $f_2(y, \ddot{y})$ signifies an even function, which does not yield any secular terms. Following NPA as augmented previously [9], the process in finding the equivalent linear equation may be introduced as follows:

Assuming that the guessing (trial) solution is given by

$$h = A\cos\Omega\, t, I.C.\, h(0) = A, \text{ and } \dot{h}(0) = 0 \tag{12}$$

It is referred to be a "trial" solution since it is an informed hypothesis that may not yield a straightforward resolution to the problem. The term implies the temporary nature of the assumption, acknowledging that the preliminary estimate may require adjustments or improvements. In iterative methods, the initial trial solution serves as a basis, and the objective is to systematically enhance this answer to converge on the accurate solution. In analytical methods, the trial solution often facilitates the development of a general solution, which is then adjusted by integrating the unique conditions or constraints of the problem.

The corresponding linear ODE may be formulated as follows:

$$h = A\cos\Omega\, t, IC.\, h(0) = A, \text{ and } \dot{h}(0) = 0 \tag{13}$$

$$\ddot{h} + \Omega^2 h = -\Lambda, \tag{14}$$

where the parameters Ω and Λ may be evaluated as follows:

The total frequency may be obtained using the following integration:

$$\Omega^2 = \int_0^{2\pi/\Omega} h f_1(h, \ddot{h})dt \Big/ \int_0^{2\pi/\Omega} h^2 dt \tag{15}$$

By means of the MS with some simplifications, Equation (15) yields

$$\Omega^2 = \frac{64d_6 + 48A^2 d_8 + 40A^4 d_{10} + 35A^6 d_{12}}{64 + 48A^2 d_2 + 40A^4 d_4} \tag{16}$$

The constant Λ can be evaluated from the following substitution:

$$\Lambda = f_2(y, \ddot{y})\Big|_{y \to \frac{A}{2}, \ddot{y} \to \frac{A\Omega^2}{2}} \tag{17}$$

The direct substitution between the second Equation in (11) and (16) produces

$$\Lambda = \frac{1}{64}(d_{11}A^6 + 64d_5 + 16(d_7 + d_1\Omega^2)A^2 + 4(d_9 + d_3\Omega^2)A^4) \tag{18}$$

Now, Equation (14) is well-defined. For enhanced convenience, the numerical solutions of Equations (9) and (14) can be obtained with the command *NDSolve* in the MS.

Therefore, consider a sample of data choices as follows:

$$d_1 = 2.0, d_2 = 3.0, d_3 = 2.0, d_4 = 5.0, d_5 = 3.0, d_6 = 4.0, d_7 = 2.0,$$

$$d_8 = 5.0, d_9 = 3.0, d_{10} = 7.0, d_{11} = 4.0, d_{12} = 3.0, \text{ and } A = 0.02.$$

The validation of the NPA is shown in **Figure 1**.

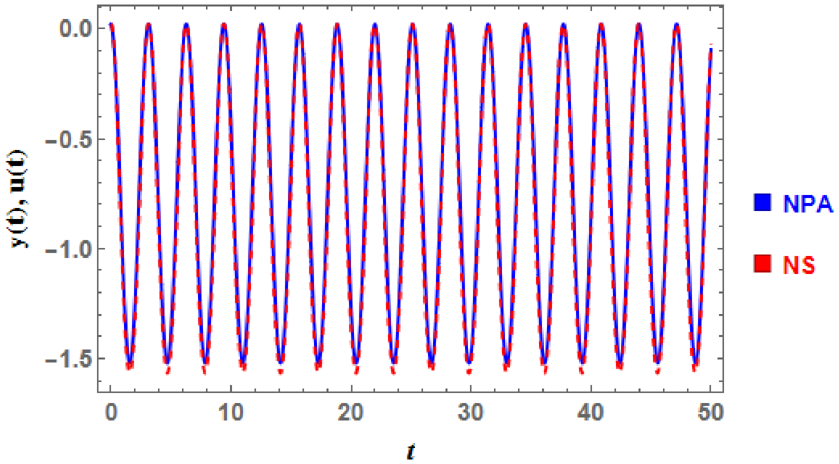

Figure 1. Displays the comparison between the NS of Equations (9) and (14).

The MS showed that the absolute error between the two solutions is 0.140187. The convergence of a highly nonlinear ODE and a linear one transpires at particular places, where their solutions cross. This occurrence indicates that, despite the fundamentally distinct behavior of the two ODEs-one defined by simple, proportional relationships and the other by intricate, dynamic changes-they exhibit comparable values under specific conditions. At these places of intersection, the complex curvature of the nonlinear ODE temporarily coincides with the constant trajectory of the linear one, forming an intersection. This indicates a temporary equilibrium in which the systems represented by these equations produce identical outputs, despite their differing governing principles.

From Equation (16), it should be noted that the stability condition is given by

$$\frac{64d_6 + 48A^2d_8 + 40A^4d_{10} + 35A^6d_{12}}{64 + 48A^2d_2 + 40A^4d_4} > 0 \tag{19}$$

3.2. Example 2

The nonlinear system arises from the natural oscillation of a conservative oscillator [6,38]. It is suitable for simulating the movement of a mass attached to both linear and nonlinear springs in a series arrangement on a smooth contact surface, as displayed in **Figure 2**. Here k_1 is the stiffness of linear spring, m is the mass, k_2 and β represent the coefficients of the linear and nonlinear components of the nonlinear spring, respectively.

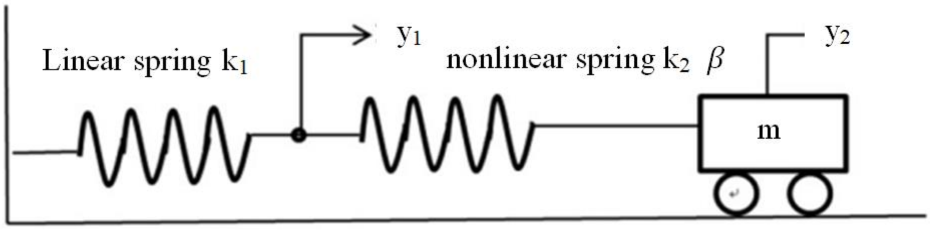

Figure 2. Geometry of the problem in example 2.

where

$$\varepsilon = \frac{\beta}{k_2}, \nu = \frac{k_2}{k_1}, z = \frac{\nu}{1+\nu}, \text{ and } \omega = \sqrt{\frac{k_2}{m(1+\nu)}} \quad (20)$$

The equation of motion may be formulated as [38]:

$$(1 + 3\varepsilon z x^2)\ddot{x} + 6\varepsilon z \dot{x}^2 \omega^2 x + \varepsilon \omega^2 x^3 = 0 \quad (21)$$

With the following ICs:

$$x(0) = A, \text{ and } \dot{x}(0) = 0 \quad (22)$$

where the dots over letters show the time derivatives.

Equation (21) may be written as follows:

$$\ddot{x} + f(x, \dot{x}, \ddot{x}) = 0 \quad (23)$$

where

$$f(x, \dot{x}, \ddot{x}) = 3\varepsilon z x^2 \ddot{x} + 6\varepsilon z \dot{x}^2 + \omega^2 x + \varepsilon \omega^2 x^3 \quad (24)$$

As seen from Equation (24), the function $f(x, \dot{x}, \ddot{x})$ represents an odd function, which produces the secular terms. Following the NPA as previously augmented [9], the process in finding the equivalent linear ODE may be introduced as follows:

Assuming that the solution obtained through guessing is represented by

$$u = B cos\tilde{\Omega}t, ICs\ u(0) = B, \text{ and } \dot{u}(0) = 0 \quad (25)$$

The corresponding linear ODE may be formulated as follows:

$$\ddot{u} + \tilde{\Omega}^2 u = 0 \quad (26)$$

where $\tilde{\Omega}$ may be evaluated as follows:

The following integration may be used to find the total frequency:

$$\tilde{\Omega}^2 = \int_0^{2\pi/\tilde{\Omega}} u f(u, \dot{u}, \ddot{u}) dt \Big/ \int_0^{2\pi/\ddot{\Omega}} u^2 dt \quad (27)$$

Using the MS with simplifications, Equation (27) provides

$$\tilde{\Omega}^2 = \frac{(4 + 3\varepsilon B^2)}{4 + 3\varepsilon z B^2} \omega^2 \quad (28)$$

Equation (28) is clearly defined. For enhanced convenience, the NS of Equations (23) and (26) can be obtained using the *NDSolve* command in the MS. Consequently, examine a selection of data options as follows:

$$\varepsilon = 3.0, z = 4.0, \omega = 2.0, \text{ and } B = 0.1.$$

The validation of the NPA is illustrated in **Figure 3**.

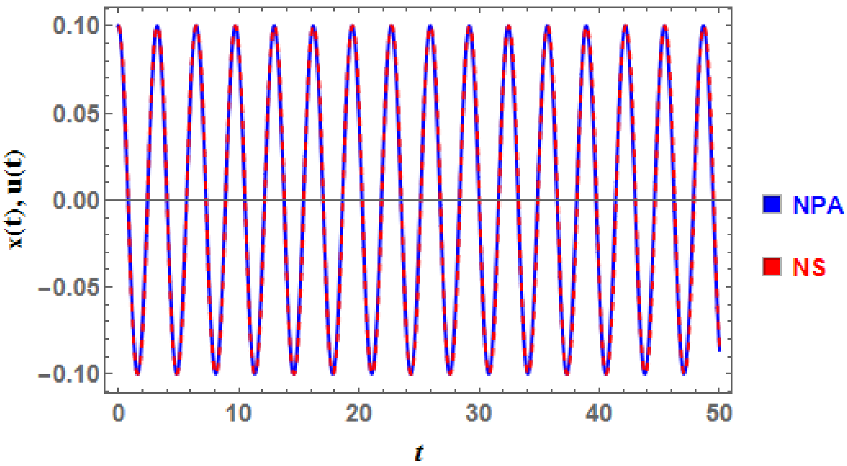

Figure 3. Displays the comparison between the NS of Equations (21) and (26).

The MS showed that, the absolute error between the two solutions is 0.0109541. The convergence of a highly nonlinear ODE and a linear one occurs at specific points, where their solutions intersect. This phenomenon suggests that, despite the fundamentally different behavior of the two ODEs-one characterized by simple, proportional relationships and the other by complex, dynamic changes-they display similar values under certain conditions. At these points of intersection, the intricate curvature of the nonlinear ODE briefly aligns with the constant trajectory of the linear one, creating an intersection. This signifies a transient equilibrium, where the systems described by these equations yield identical outputs, notwithstanding their divergent controlling principles.

From Equation (28), it should be noted that the stability condition is given by

$$\frac{(4+3\varepsilon A^2)}{4+3\varepsilon z A^2}\omega^2 > 0 \tag{29}$$

3.3. Example 3

The equation of motion for a rigid rod rocking on a circular surface without slipping is stated as follows [39]:

$$\left(\frac{1}{12}+\frac{1}{16}u^2\right)\frac{d^2u}{dt^2}+\frac{1}{16}u\left(\frac{du}{dt}\right)^2+\frac{g}{4l}u\cos u = 0 \tag{30}$$

With the ICs $u(0) = A,$ and $u^{'}(0) = 0$ and A is the initial amplitude, g is the gravitational acceleration, a is the length of the rod and t is time. The problem is illustrated in **Figure 4** corresponding to the solution $u(t)$ in Equation (30).

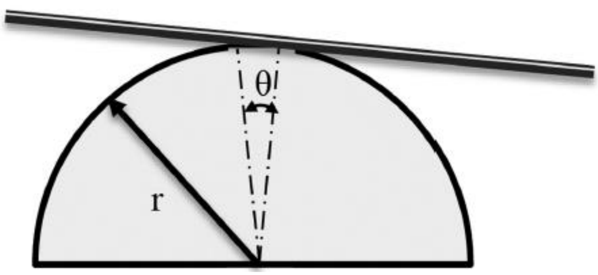

Figure 4. Geometry of the problem in example 3.

This problem was extensively analyzed by El-Dib and Moatimid [40] by means of the modified HPM, then by Moatimid and Amer [12] in case of the time delay. In the previous two cases, the authors were forced to adopt Taylor expansion in or to expand the forcing force. In contrast, the usage of the NPA does not use Taylor expansion.

Equation (30) can be expressed as:

$$\ddot{u}+\frac{3}{4}u^2\ddot{u}+\frac{3}{4}u\dot{u}^2+\frac{3g}{l}u\cos u=0 \tag{31}$$

where the dots over letters show the time derivatives.

Equation (31) may be written as follows:

$$\ddot{u}+f(u,\dot{u},\ddot{u})=0 \tag{32}$$

where

$$f(u,\dot{u},\ddot{u})=\frac{3}{4}u^2\ddot{u}+\frac{3}{4}u\dot{u}^2+\frac{3g}{l}u\cos u \tag{33}$$

As seen from Equation (33), the function $f(u,\dot{u},\ddot{u})$ represents an odd function, which produces the secular terms. Following the NPA as previously augmented [9], the process in finding the equivalent ODE may be introduced as follows:

Assuming that the solution obtained through guessing is represented by

$$x=C\cos\hat{\Omega}t, I.C.\, x(0)=C, \text{and } \dot{x}(0)=0 \tag{34}$$

The corresponding linear ODE may be formulated as follows:

$$\ddot{x}+\hat{\Omega}^2x=0 \tag{35}$$

where the parameters $\hat{\Omega}$ may be assessed as follows:

The following integration may be used to find the total frequency:

$$\hat{\Omega}^2=\int_0^{2\pi/\hat{\Omega}}xf(x,\dot{x},\ddot{x})\,dt\Big/\int_0^{2\pi/\hat{\Omega}}x^2dt \tag{36}$$

Using the MS and certain simplifications, Equation (36) provides

$$\hat{\Omega}^2=\frac{48g}{lA}\left(\frac{J_1(A)-AJ_2(A)}{8+3A^2}\right) \tag{37}$$

where $J_1(A)$ and $J_2(A)$ are the Bessel functions of order one and two, respectively.

Equation (36) is clearly specified. For heightened convenience, the numerical solutions of Equations (31) and (37) can be obtained through the MS using the *NDSolve* command. Consequently, examine a selection of data options as follows:

$$\text{g}=12, l=10, \text{and } C=0.5,$$

Figure 5 displays the justification of the NPA.

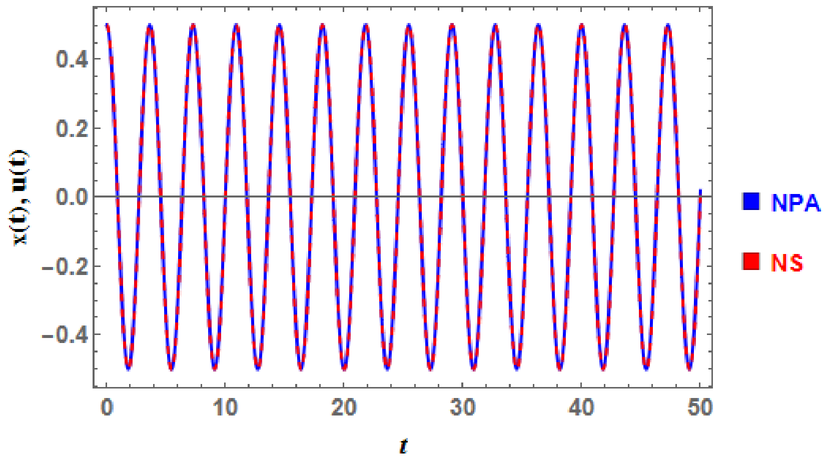

Figure 5. Displays the comparison between the NS of Equations (31) and (35).

The MS showed that, the absolute error between the two solutions is 0.0151677.

The convergence of a highly nonlinear ODE and a linear one transpires at distinct locations when their solutions cross. This occurrence indicates that, despite the fundamentally distinct behavior of the two ODEs-one defined by simple, proportional relationships and the other by intricate, dynamic changes-they exhibit analogous values under specific conditions. At these junction sites, the complex curvature of the nonlinear ODE momentarily coincides with the constant path of the linear ODE, resulting in an intersection. This indicates a temporary equilibrium in which the systems represented by these equations produce the same outputs, despite their differing governing principles.

From Equation (37), it should be noted that the stability condition is given by

$$\frac{48g}{lA}\left(\frac{J_1(A) - AJ_2(A)}{8 + 3A^2}\right) > 0 \tag{38}$$

3.4. Example 4

Motion of a ring on a rotating parabola

Let us explore a single degree-of-freedom conservative system represented by a complex ODE. Imagine a particle of mass sliding without friction along a wire shaped like a parabola $z = px^2$. This parabola rotates at a uniform angular velocity σ around the z-axis, as described in [4]. The problem is illustrated in **Figure 6**. This specific problem has been previously studied using the HPM and the expanded frequency concept [41]. To derive the main equation of motion for the particle, we can utilize the Euler-Lagrange formulation. Since the system is conservative and holonomic (meaning its constraints can be integrated), its equation of motion can be represented as follows:

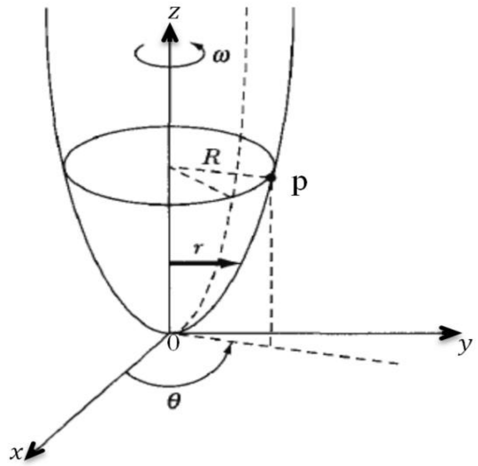

Figure 6. Geometry of the problem in example 4.

$$(1+4p^2x^2)\ddot{x}+\Lambda x+4p^2\dot{x}^2x=0, \Lambda=2gp-\sigma^2 \tag{39}$$

The following is the way to express Equation (39):

$$\ddot{x}+f(x,\dot{x},\ddot{x})=0 \tag{40}$$

where, $f(x,\dot{x},\ddot{x})=\Lambda x+4p^2\dot{x}^2x+4p^2x^2\ddot{x}$.

Assuming that the proposed (trial) solution is provided by

$$v=D\cos\bar{\Omega}\,t, I.C.\, v(0)=D, \text{and } \dot{v}(0)=0 \tag{41}$$

The corresponding frequency may be calculated using the following integration:

$$\omega_{eqv}^2=\int_0^{2\pi/\bar{\Omega}} vf(v,\dot{v},\ddot{v})dt \,/\int_0^{2\pi/\bar{\Omega}} v^2dt \tag{42}$$

Using the non-perturbative technique, Equation (42) provides.

$$\omega_{eqv}^2=\int_0^{2\pi/\bar{\Omega}} vf(v,\dot{v},\ddot{v})dt \,/\int_0^{2\pi/\bar{\Omega}} v^2dt \tag{43}$$

$$\omega_{eqv}^2=\Lambda-3B^2p^2\bar{\Omega}^2 \tag{44}$$

As there is similar damping in the investigated case, it follows that the total frequency.

$$\bar{\Omega}^2=\frac{\Lambda}{1+3B^2p^2} \tag{45}$$

The analogous linear ODE is presented as follows:

$$\ddot{v}+\bar{\Omega}^2v=0 \tag{46}$$

The stability condition necessitates that $\Omega^2>0$. This means that $2gp-\sigma^2>0$, or $\sigma^2<2gp$.

To make things easier, the MS may be used to match the original nonlinear differential equation provided in Equation (39) to the corresponding linear ODE given in Equation (45) for the sample system:

$$\sigma=2.0, g=4.0, p=1.0, \text{and } D=0.05,$$

Figure 7 depicts the justification of the NPA.

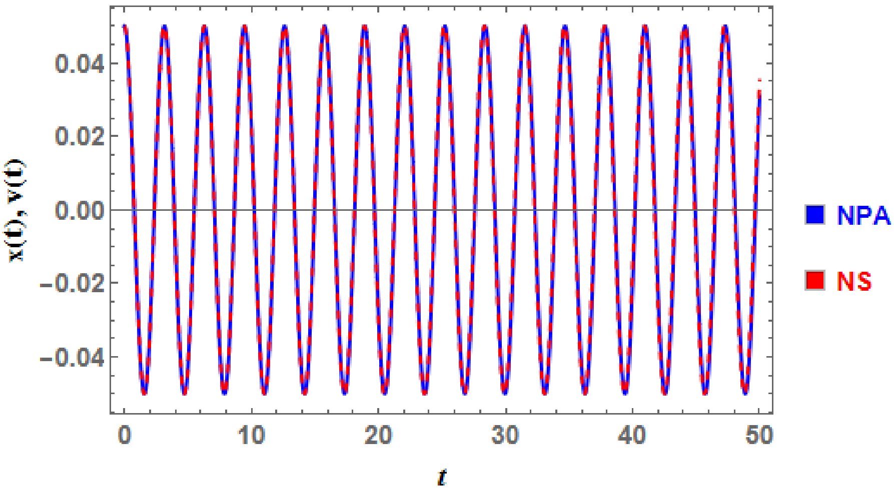

Figure 7. A matching between the two solutions of Equations (39) and (45).

The absolute error between the two answers is 0.0059. It should be highlighted that, unlike classic perturbation approaches, the current NPA allows us to discuss the stability requirement. The convergence of a highly non-linear ordinary differential equation and a linear one occurs at different points when their solutions intersect. This phenomenon demonstrates that, despite the fundamentally different behavior of the two ODEs-one characterized by simple, proportional relationships and the other by complex, dynamic changes-they display similar values under certain conditions. At these junctions, the intricate curvature of the nonlinear ordinary differential equation temporarily aligns with the constant trajectory of the linear equation, leading to an intersection. This signifies a transient equilibrium wherein the systems denoted by these equations yield identical outputs, notwithstanding their divergent controlling principles.

From Equation (44), the stability criteria becomes

$$\frac{\Lambda}{1+3B^2p^2} > 0 \tag{47}$$

4. Results and discussions

It is common to match analytical responses to numerical ones to determine the correctness of approximate solutions derived using a certain NPA. It will use the given solution (4) to generate numerical simulations in solving Equation (2). **Figures 1**, **3**, **5**, and **7** compare the numerical and analytical solutions, demonstrating the great accuracy of the current technique. **Figures 1**, **3**, **5**, and **7** demonstrate great agreement between the numerical and analytical solutions. Finally, we may conclude that the NPA is suitable for providing an accurate solution to the strong non-linear oscillator.

5. Concluding remarks

Because nonlinear oscillators have become increasingly more community, the primary aim of the present study was to investigate the HFF to attempt to demonstrate the theoretical justifications for various types of highly nonlinear oscillators. We studied mathematically and computationally the connection between elastic forces and the solution of a certain kind of oscillators with substantial nonlinear damping. The motion was thought to be explained by a strong nonlinear ODE, whose solution was matched by the proper value of the trigonometric functions. We gave several instances drawn from various scientific and technological domains. It was evident that the new approach was easier to use and takes less time to process than the traditional perturbation methodologies that were being used extensively in this field. The NPA refers to this novel approach, which is essentially a linearization of the nonlinear ODE. Using this approach, a new frequency is produced that is similar to a linear ODE in fundamentally harmonic situations. This straightforward process yields findings that not only agree well with numerical results when computed for physiologically descriptive specialist instances, but also turn out to be more accurate than the results from several widespread approximation approaches. For the reader's understanding, a thorough explanation of the NPA was presented. A numerical analysis, validated by MS, corroborated the theoretical findings. There was a strong correlation between the results obtained from both the theoretical analysis and the NS test. It was a well-established fact that all traditional perturbation methods rely on Taylor expansion to approximate restoring forces, which simplifies the analysis but introduces certain limitations. These limitations are overcome with the NPA. Furthermore, the NPA allows for a proper investigation of the stability analysis of the problems, an area that was previously inaccessible with standard methods. Therefore, the NPA represents a more appropriate analytical tool when dealing with approximations of highly nonlinear oscillators within MS. Its adaptability makes the NPA a valuable asset across scientific, technological, and applied research domains, allowing it to tackle a wide variety of nonlinear problems.

The ensuing outcomes should focus on the original methodology or notable results:

1) The specified technique produced a supplementary linear ODE that was equivalent to a non-linear one.
2) There existed a robust link between these two equations.
3) In the face of restoring forces, all traditional methods employed Taylor expansion to simplify the situation at hand. This weakness has been eradicated in the current strategy.
4) The current methodology, unlike earlier traditional methodologies, allowed us to conduct a stability study of the problem.

In the coming study, we intend to analyze the state spaces of multi-degrees of freedom in accordance with the following characteristics:

1) The concept of multiple degrees of freedom in basic pendulums extends their conventional single-degree-of-freedom motion to complex systems where many interconnected pendulums function either independently or interactively.

2) This significantly improves their physical behavior, enabling the description of intricate dynamical systems observed in both natural and artificial environments.
3) Multi-degree-of-freedom pendulums are crucial for analyzing coupled oscillations, wave propagation, and energy transfer, with applications in mechanical and civil engineering, robotics, and seismology.
4) Understanding their dynamics may improve vibration control in structures, optimize the design of coupled oscillatory systems in equipment, or promote the development of advanced robotics with flexible joints.
5) Moreover, they serve as simplified representations of more complex phenomena, such as molecular vibrations in chemistry or chaotic systems in physics.
6) By analyzing their degrees of freedom, researchers can gain insights into resonance, stability, and energy distribution, making multi-degree-of-freedom pendulums crucial for exploring nonlinear dynamics and developing innovative solutions.

Author contributions: Conceptualization, GMM and GMI; methodology, GMM and GMI; software, GMM and GMI; validation, GMM, GMI, IA and SVK; formal analysis, GMM, GMI, IA and SVK; investigation, GMM; resources, GMM; data curation, GMM, GMI, IA and SVK; writing-original draft preparation, GMM; writing-review and editing, GMM, GMI, IA and SVK; visualization, GMM, GMI, IA and SVK; supervision, GMM. All authors have read and agreed to the published version of the manuscript.

Acknowledgments: The researchers wish to extend their sincere gratitude to the Deanship of Scientific Research at the Islamic University of Madinah for the support provided to the Post Publishing Program.

Conflict of interest: The authors declare no conflict of interest.

References

1. Mickens RE. Oscillations in Planar Dynamic Systems. Series on Advances in Mathematics for Applied Sciences; 1996. doi: 10.1142/2778
2. Kumar D, Singh J, Baleanu D. A hybrid computational approach for Klein–Gordon equations on Cantor sets. Nonlinear Dynamics. 2016; 87(1): 511-517. doi: 10.1007/s11071-016-3057-x
3. Nayfeh AH. Perturbation Methods. John Wiley & Sons, New York; 1973.
4. Nayfeh AH, Mook DT. Nonlinear Oscillations. John Wiley & Sons, New York; 1979.
5. Nayfeh AH. Introduction to Perturbation Technique. John Wiley & Sons, New York; 1981.
6. Telli S, Kopmaz O. Free vibrations of a mass grounded by linear and nonlinear springs in series. Journal of Sound and Vibration. 2006; 289(4-5): 689-710. doi: 10.1016/j.jsv.2005.02.018
7. He JH. The simplest approach to nonlinear oscillators. Results in Physics. 2019; 15: 102546. doi: 10.1016/j.rinp.2019.102546
8. Moatimid GM, Amer TS. Analytical Approximate Solutions of a Magnetic Spherical Pendulum: Stability Analysis. Journal of Vibration Engineering & Technologies. 2022; 11(5): 2155-2165. doi: 10.1007/s42417-022-00693-8
9. Ismail GM, Moatimid GM, Yamani MI. Periodic Solutions of Strongly Nonlinear Oscillators Using He's Frequency Formulation. European Journal of Pure and Applied Mathematics. 2024; 17(3): 2155-2172. doi: 10.29020/nybg.ejpam.v17i3.5339
10. Moatimid GM, Amer TS. Nonlinear suppression using time-delayed controller to excited Van der Pol–Duffing oscillator: analytical solution techniques. Archive of Applied Mechanics. 2022; 92(12): 3515-3531. doi: 10.1007/s00419-022-02246-7

11. Moatimid GM, Amer TS, Amer WS. Dynamical analysis of a damped harmonic forced duffing oscillator with time delay. Scientific Reports. 2023; 13(1). doi: 10.1038/s41598-023-33461-z
12. Moatimid GM, Amer TS. Dynamical system of a time-delayed of rigid rocking rod: analytical approximate solution. Scientific Reports. 2023; 13(1). doi: 10.1038/s41598-023-32743-w
13. Liu CS, Chen YW. A Simplified Lindstedt-Poincaré Method for Saving Computational Cost to Determine Higher Order Nonlinear Free Vibrations. Mathematics. 2021; 9(23): 3070. doi: 10.3390/math9233070
14. He JH, Amer TS, Elnaggar S, et al. Periodic Property and Instability of a Rotating Pendulum System. Axioms. 2021; 10(3): 191. doi: 10.3390/axioms10030191
15. Bayat M, Head M, Cveticanin L, et al. Nonlinear analysis of two-degree of freedom system with nonlinear springs. Mechanical Systems and Signal Processing. 2022; 171: 108891. doi: 10.1016/j.ymssp.2022.108891
16. Yang C, Guo CM, Xu J, et al. Device Design for Multitask Graphene Electromagnetic Detection Based on Second Harmonic Generation. IEEE Transactions on Microwave Theory and Techniques. 2024; 72(7): 4174-4182. doi: 10.1109/tmtt.2023.3347528
17. Chelli R, Marsili S, Barducci A, et al. Generalization of the Jarzynski and Crooks nonequilibrium work theorems in molecular dynamics simulations. Physical Review E. 2007; 75(5). doi: 10.1103/physreve.75.050101
18. Xu J, Zhang MZ, Tang Z, et al. Layered heterogeneous structures integrated device for multiplication, division arithmetic unit and multiple-physical sensing. Physics of Fluids. 2024; 36(9). doi: 10.1063/5.0228552
19. López RC, Sun GH, Camacho-Nieto O, et al. Analytical traveling-wave solutions to a generalized Gross–Pitaevskii equation with some new time and space varying nonlinearity coefficients and external fields. Physics Letters A. 2017; 381(35): 2978-2985. doi: 10.1016/j.physleta.2017.07.012
20. Guo YS, Li W, Dong SH. Gaussian solitary solution for a class of logarithmic nonlinear Schrödinger equation in (1 + n) dimensions. Results in Physics. 2023; 44: 106187. doi: 10.1016/j.rinp.2022.106187
21. Ghaleb A.F., Abou-Dina M.S., Moatimid G.M., and Zekry M.H., Analytic approximate solutions of the cubic-quintic Duffing Van der Pol equation with two-external periodic forces terms: Stability analysis, Mathematics and Computers in Simulation, 2021, 180, 129-151. https://doi.org/10.1016/j.matcom.2020.08.001
22. Moatimid G.M., El-Dib Y.O., and Zekry M.H., Stability analysis using multiple scales homotopy approach of coupled cylindrical interfaces under the influence of periodic electrostatic fields, Chinese Journal of Physics, 2018, 56(5), 2507-2522. https://doi.org/10.1016/j.cjph.2018.06.008
23. Moatimid G.M., and Amer T.S., Analytical solution for the motion of a pendulum with rolling wheel: Stability Analysis, Scientific Reports, 2022, 12(1), 12628. https://doi.org/10.1038/s41598-022-15121-w
24. He JH. Some asymptotic methods for strongly nonlinear equations. International Journal of Modern Physics B. 2006; 20(10): 1141-1199. doi: 10.1142/s0217979206033796
25. He JH. The simpler, the better: Analytical methods for nonlinear oscillators and fractional oscillators. Journal of Low Frequency Noise, Vibration and Active Control. 2019; 38(3-4): 1252-1260. doi: 10.1177/1461348419844145
26. Qie N, Houa WF, He JH. The fastest insight into the large amplitude vibration of a string. Reports in Mechanical Engineering. 2021; 2(1): 1-5. doi: 10.31181/rme200102001q
27. He JH, Yang Q, He CH, et al. A Simple Frequency Formulation for the Tangent Oscillator. Axioms. 2021; 10(4): 320. doi: 10.3390/axioms10040320
28. He JH, Yang Q, He CH, et al. Pull-down instability of the quadratic nonlinear oscillators. Facta Universitatis, Series: Mechanical Engineering. 2023; 21(2): 191. doi: 10.22190/fume230114007h
29. He JH. Periodic solution of a micro-electromechanical system. Facta Universitatis, Series: Mechanical Engineering. 2024: 187. doi: 10.22190/fume240603034h
30. Tian Y, Shao Y. Mini-review on periodic properties of MEMS oscillators. Frontiers in Physics. 2024; 12. doi: 10.3389/fphy.2024.1498185
31. He JH. Frequency formulation for nonlinear oscillators (part 1). Sound & Vibration. 2024; 59(1): 1687. doi: 10.59400/sv1687
32. He JH. An Old Babylonian Algorithm and Its Modern Applications. Symmetry. 2024; 16(11): 1467. doi: 10.3390/sym16111467
33. He CH, Liu C. A modified frequency–amplitude formulation for fractal vibration systems. Fractals. 2022; 30(03). doi: 10.1142/s0218348x22500463

34. Feng GQ. He’s frequency formula to fractal undamped Duffing equation. Journal of Low Frequency Noise, Vibration and Active Control. 2021; 40(4): 1671-1676. doi: 10.1177/1461348421992608
35. Feng GQ, Niu JY. He’s frequency formulation for nonlinear vibration of a porous foundation with fractal derivative. GEM - International Journal on Geomathematics. 2021; 12(1). doi: 10.1007/s13137-021-00181-3
36. Anjum N, Ul Rahman J, He JH, et al. An Efficient Analytical Approach for the Periodicity of Nano/Micro electromechanical Systems’ Oscillators. Mathematical Problems in Engineering. 2022; 2022: 1-12. doi: 10.1155/2022/9712199
37. Qian Y, Pan J, Qiang Y, et al. The spreading residue harmonic balance method for studying the doubly clamped beam-type N/MEMS subjected to the van der Waals attraction. Journal of Low Frequency Noise, Vibration and Active Control. 2018; 38(3-4): 1261-1271. doi: 10.1177/1461348418813014
38. Lu J, Liang Y. Analytical approach to the nonlinear free vibration of a conservative oscillator. Journal of Low Frequency Noise, Vibration and Active Control. 2019; 40(1): 84-90. doi: 10.1177/1461348419881831
39. Coskun SB, Şenturk E, Atay MT, Analysis of the motion of a rigid rod on a circular surface using interpolated variational iteration method, Sigma Journal of Engineering and Natural Sciences, 2022; 40(3): 577-584. doi: 10.14744/sigma.2022.00062
40. El-Dib YO, Moatimid GM. Stability Configuration of a Rocking Rigid Rod over a Circular Surface Using the Homotopy Perturbation Method and Laplace Transform. Arabian Journal for Science and Engineering. 2019; 44(7): 6581-6591. doi: 10.1007/s13369-018-03705-6
41. Moatimid GM. Sliding bead on a smooth vertical rotated parabola: stability configuration. Kuwait Journal of Science. 2020; 47(2), 6-21.