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Article

# Frequency formulation for nonlinear oscillators (part 1)

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**Abstract:** The perturbation method is a prevalent approach for nonlinear oscillators; however, the outcomes are only applicable to situations with weak nonlinearity. Other analytical methods, such as the variational iteration method and the homotopy perturbation method, can yield a satisfactory approximate solution; however, each method necessitates the completion of multiple calculations. Hereby is recommended a one-step frequency formulation for nonlinear oscillators, and this part 1 focuses itself on odd nonlinearity.

**Keywords:** nonlinear vibration; frequency-amplitude relationship; non-perturbative approach; mathematical pendulum; Cubic-Quintic Duffing Oscillator

## 1. Introduction

Nonlinear oscillators have a variety of applications in modern technologies, with the frequency-amplitude relationship representing a particularly crucial factor. In numerous fields, including electronic engineering, physics, and communication technology, nonlinear oscillators are of paramount importance. For example, in radio frequency circuits, nonlinear oscillators can be employed to generate signals at specific frequencies. In the field of quantum mechanics, certain quantum systems have been observed to exhibit nonlinear oscillatory behavior, which is of paramount importance for the comprehension and regulation of quantum states.

The importance of the frequency-amplitude relationship lies in its role in determining the output characteristics of the oscillator. The specific combinations of frequency and amplitude required for a given application scenario must be determined. By studying and understanding this relationship, we can better design and control nonlinear oscillators to meet the needs of different technologies. For instance, in communication systems, it is essential to exercise precise control over the oscillator's frequency in order to guarantee accurate signal transmission. In certain sensor applications, alterations in amplitude may be associated with the physical quantity being measured. Consequently, an understanding of the frequency-amplitude relationship facilitates the enhancement of sensor performance, the frequency-amplitude relationship can make in-depth research and optimized design of a nonlinear oscillator.

Previous research has primarily concentrated on the approximate solution of a specific oscillator, such as large-amplitude oscillations [1], nonlinear advanced noncanonical dynamic systems [2], neutral nonlinear differential equations [3,4], second-order quasilinear functional dynamic equations [5], nonlinear quartic oscillators [6], delayed oscillators [7], Duffing oscillators [8,9], and fractional oscillators [10]. The present study focuses on the frequency-amplitude relationship of a nonlinear oscillator in the following form:

$$u'' + f(u) = 0 \tag{1}$$

with initial conditions:

$$u(0) = \alpha, u'(0) = \beta \tag{2}$$

For conservation oscillator, it requires $f(u)/u > 0$, and $f(-u) = -f(u)$, and we assume that the approximate solution can be expressed in the following form:

$$u(t) = A\cos(\omega t + \sigma) \tag{3}$$

and its frequency can be calculated by the following simple equation [11,12]

$$\omega = \sqrt{\frac{f(\bar{u})}{\bar{u}}} \tag{4}$$

where $A$ is the amplitude, $\omega$ is the frequency, $\sigma$ is the initial phase angle, $\bar{u}$ is a location point, generally it is recommended as $\bar{u} = 0.8660A$ [11].

By the initial conditions of Equation (2), we have

$$A\cos(\sigma) = \alpha \tag{5}$$

$$-\omega A \sin\sigma = \beta \tag{6}$$

Or

$$\alpha^2 + \frac{\beta^2}{\omega^2} = \mathrm{A}^2 \tag{7}$$

$$\tan\sigma = -\frac{\beta}{\alpha\omega} \tag{8}$$

From Equations (4), (7) and (8), the frequency-amplitude relationship can be obtained.

The frequency calculated by Equation (4) is relatively accurate and has been widely used to gain an in-depth understanding of the frequency-amplitude relationship in a nonlinear vibration system. In a remarkable contribution to the field, He and Liu [13] offered a mathematical perspective on the frequency formulation, demonstrating that the frequency formulation presented in Equation (4) is not only mathematically solid but also provides valuable insights into the underlying physical phenomena. In the literature, Equation (4) is referred to as He's frequency formulation, He's non-perturbative approach, or He's frequency-amplitude formulation. The frequency formulation has been successfully applied to a number of different systems, including the Toda oscillator [14], fractal vibration systems [15], Duffing oscillators [16], and vibration-proof designs [17]. It has been utilized to gain rapid and reliable insights into the frequency-amplitude relationship of nonlinear vibration systems. Tsaltas demonstrated that the one-step formulation is a simple and effective approach for complex vibration systems [18], while Ismail and colleagues showed that the formulation is applicable to strongly nonlinear oscillators [19]. Chun-Hui He and colleagues have employed the formulation to analyze the low-frequency property of a concrete beam's vibration [20]. El-Dib and colleagues have extended the formulation to cover more complex cases [21,22]. The formulation was successfully used to analyze the vibration system of a jet engine with great success [23]. Hashemi compared the formulation with the energy balance method and found that the former is simpler yet effective [24]. Alyousef et al. applied the formulation to pendulum oscillators [25]. Moatimid and Mohamed applied the formulation to study the nonlinear stability of fluid motion [26,27]. Big-Alabo compared the frequency obtained by the formulation with the exact ones and demonstrated high accuracy [28]. Furthermore, the frequency formulation can be applied to MEMS systems [29–31] and nonlinear vibration systems with arbitrary initial conditions [32].

The location point is indeed a topic that has sparked considerable debate, and a number of modifications have been put forward. Lyu and colleagues proposed an alternative location point [33]. On the other hand, He and others recommended the utilization of multiple location points, followed by the calculation of an average value [34]. Shen suggested the employment of Lagrange interpolation for the location points [35], while Mohammadian introduced a novel approach for determining the location point [36].

## 2. A new frequency formulation

In this section, the following nonlinear oscillator with odd nonlinearities is considered [37–39].

$$u'' + \sum_{n=0}^{N} a_{2n+1} u^{2n+1} = 0 \tag{9}$$

where $a_{2n+1}$ ($n$=0~$N$) are constants. In light of Equation (4), we obtain the following frequency-amplitude relationship.

$$\omega^2 = \sum_{n=0}^{N} a_{2n+1} \bar{u}^{2n} \tag{10}$$

where $\bar{u} = bA$, $b$ is a constant, it was recommended $b$ = 0.8660[11]. Practical applications show that the proposed methodology is effective in cases where $A$ is small. Nevertheless, Equation (10) generates a significant error when $A$ is large or even when $A$ approaches infinity. To tackle this problem, a proposed modification is presented as follows.

$$\omega^2 = a_1 + \sum_{n=1}^{N} a_{2n+1} b_{2n+1} A^{2n} \tag{11}$$

where $b_{2n+1}$ is recommended as:

$$b_{2n+1} = \frac{3}{2n+2} \tag{12}$$

We take the cubic-quintic Duffing oscillator [40,41] as an example to illustrate its extreme effectiveness and high reliability.

$$u'' + a_1 u + a_3 u^3 + a_5 u^5 = 0 \tag{13}$$

with initial conditions:

$$u(0) = A, u'(0) = 0 \tag{14}$$

By Equations (11) and (12), we have:

$$\omega^2 = a_1 + \frac{3}{4} a_3 A^2 + \frac{1}{2} a_5 A^4 \tag{15}$$

To verify the validity of Equation (15), we consider a special case where $a_5 = 0$ for the Duffing oscillator. From Equation (15), we obtain.

$$\omega = \sqrt{a_1 + \frac{3}{4} a_3 A^2} \tag{16}$$

This result is exactly the same as that obtained by the homotopy perturbation method [42,43], its approximate period is:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{a_1 + \frac{3}{4} a_3 A^2}} \tag{17}$$

To demonstrate the accuracy of Equation (17) for the Duffing oscillator, we consider an extreme case where $A$ tends to infinity. In this case, a simplified equation is obtained as follows

$$u'' + a_3 u^3 = 0 \tag{18}$$

Its exact period is [40,41],

$$\lim_{A \to \infty} T_{exact} = 7.4164 (a_3)^{-1/2} A^{-1} \tag{19}$$

while the approximate one is

$$\lim_{A \to \infty} T = \frac{2\pi}{\sqrt{\frac{3}{4} a_3 A^2}} = 7.2552 (a_3)^{-1/2} A^{-1} \tag{20}$$

The relative error is 2.17%. Given its simplicity, this is good enough.

Now we consider another case

$$u'' + a_5 u^5 = 0 \tag{21}$$

The frequency formulation leads to

$$\omega = \sqrt{\frac{1}{2} a_5 A^4} = 0.7071 (a_5)^{1/2} A^2 \tag{22}$$

The period is

$$\lim_{A\to\infty} T = \frac{2\pi}{\omega} = 8.8858(a_5)^{-1/2} A^{-2} \tag{23}$$

while the exact one is [40,41].

$$\lim_{A\to\infty} T_{exact} = 8.41309(a_5)^{-1/2} A^{-2} \tag{24}$$

The relative error is 5.61%. For the large amplitude oscillator, this is also good enough considering its simplicity.

Now we consider a case when $a_1 = a_3 = a_5 = 1$ and $A$=1, the frequency formulation leads to the result

$$\omega = \sqrt{1 + \frac{3}{4} + \frac{1}{2}} = 1.5 \tag{25}$$

while the exact one is [44]

$$\omega_{exact} = 1.523586029 \tag{26}$$

The relative error is 1.54%. Thus, the accuracy is acceptable for all values of $A$ greater than 0.

We take another example of a mathematical pendulum [45].

$$\begin{cases} u'' + \sin u = 0 \\ u(0) = A, u'(0) = 0 \end{cases} \tag{27}$$

The mathematical pendulum acts as a fundamental example for understanding concepts such as periodic motion, energy conservation, and simple harmonic motion. Equation (27) can be represented in a Taylor series in terms of variable $u$. This allows for a more detailed analysis and approximation of the behavior described by the frequency formulation.

$$u'' + u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 - \frac{1}{7!}u^7 = 0 \tag{28}$$

The frequency-amplitude relation is

$$\omega^2 = 1 - \frac{1}{3!} \times \frac{3}{4} A^2 + \frac{1}{5!} \times \frac{3}{6} A^4 - \frac{1}{7!} \times \frac{3}{8} A^6 \tag{29}$$

For $A = \pi / 2$, we obtain $\omega = 0.8473$. With a relative error of 0.85%, this is good enough for practical applications. The relatively low error percentage indicates that the result is reliable and can be effectively utilized in real-world scenarios where such approximations are acceptable.

It should be noted that Ma proposed a modified frequency-amplitude formulation based on the Hamilton principle [46]. Coupling the Hamilton principle with the present method may also offer a new perspective for future research.

## 3. Discussion and conclusion

The perturbation method is truly a common technique employed for nonlinear oscillators. Nevertheless, its limitation of being applicable only to weak nonlinearity situations can be a disadvantage. On the contrary, methods such as the variational iteration method and the homotopy perturbation method can offer satisfactory approximate solutions, but they involve multiple calculations which can be time-consuming and complex.

The one-step frequency formulation for nonlinear oscillators, with an emphasis on odd nonlinearity as recommended in this part 1, has the potential to provide a more efficient alternative. This approach could simplify the analysis and offer a quicker means to obtain approximate solutions without the need for extensive iterative calculations.

Although the frequency formulation has liberated engineers from the burden of complex computations, the mathematical proof and counterexample still remain a subject for future investigation. A complex vibration system can be reduced to a simplified form by applying the Taylor series method, thereby enabling the use of the frequency formulation of Equation (11). Future papers will address damped and forced vibration systems, nonlinear oscillators with quadratic nonlinearity, and fractal-fractional vibration equations.

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