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Article

Bifurcation analysis and multiobjective nonlinear model predictive control of sustainable ecosystems

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Abstract: Problem: All optimal control work involving ecological models involves single objective optimization. In this work, we perform multiobjective nonlinear model predictive control (MNLMPC) in conjunction with bifurcation analysis on an ecosystem model. **Methods:** Bifurcation analysis was performed using the MATLAB software MATCONT MATLAB CONTINUITION, while multiobjective nonlinear model predictive control was performed by using the optimization language PYOMO (PYTHON OPTIMIZATION). **Results:** Rigorous proof showing the existence of bifurcation (branch) points is presented along with computational validation. It is also demonstrated (both numerically and analytically) that the presence of the branch points was instrumental in obtaining the Utopia solution when the multi-objective nonlinear model prediction calculations were performed. **Conclusions:** The main conclusions of this work are that one can attain the utopia point in MNLMPC calculations because of the branch points that occur in the ecosystem model, and the presence of the branch point can be proved analytically.

Keywords: ecosystem; bifurcation; optimal control

1. Introduction

Sustainability is a significant factor to be considered in almost all physical and chemical phenomena. Beneficial activities and situations must be sustained over a considerable amount of time. This is especially true in ecosystem management, where the conservation of natural species is essential for ensuring a healthy environment for the long-term well-being of the human population. The issue of sustainability should be implemented in optimization and control studies of ecosystems. In this work, MNLMPC calculations are performed in conjunction with bifurcation analysis maximizing sustainability.

2. Literature review

Cabezas and co-workers [1–9] have applied the fisher index [10] as a sustainability criterion for ecosystems. Specifically, the sustainability concept has been applied in the management of ecosystems by controlling the population of various species.

Shastri and DIwekar [11] and Sorayya et al. [12] performed single objective optimal control calculations on ecological models maximizing the fisher index to ensure maximum sustainability.

In this article, bifurcation analysis and multiobjective nonlinear model predictive control tasks on the ecological model described in Shastri and Diwekar [11]. The bifurcation analysis reveals the existence of branch points. A rigorous mathematical analysis (which is also computationally validated) demonstrating the existence of branch points is presented. The branch point causes the multiobjective nonlinear model predictive control calculations to converge to the utopia solution. This demonstrates that one can maximize the conservation of the natural habitat and maintain maximum sustainability.

3. Equations in ecological model

The equations are the following:

$$\phi_{12} = \frac{a_2 x_2 x_1}{b_2 + x_1}; \phi_{23} = \frac{a_3 x_3 x_2}{b_3 + x_2} \quad (1)$$

$$\begin{aligned} \frac{dx_1}{dt} &= f_1 = x_1(r(1 - \frac{x_1}{K}) - \frac{a_2 x_2}{b_2 + x_1}) = x_1(r(1 - \frac{x_1}{K})) - \phi_{12} = r x_1 - \frac{r x_1^2}{k} - \phi_{12} \\ \frac{dx_2}{dt} &= f_2 = x_2(e_2 \frac{a_2 x_1}{b_2 + x_1} - \frac{a_3 x_3}{b_3 + x_2} - d_2) = (e_2 \phi_{12}) - \phi_{23} - d_2 x_2 \\ \frac{dx_3}{dt} &= f_3 = x_3(e_3 \frac{a_3 x_2}{b_3 + x_2} - d_3) = e_3 \phi_{23} - d_3 x_3 \end{aligned} \quad (2)$$

The base parameter values are $a_2 = 2.0$; $a_3 = 0.1$; $b_2 = 235.50$; $b_3 = 250$; $e_2 = 1.35$; $e_3 = 1.29$; $d_2 = 1.0$; $d3 = 0.04$; $k = 710$; $r = 1.2$.

4. Computational procedures used

4.1. Bifurcation analysis

Bifurcations that lead to multiple steady-state solutions can be classified as a) branch points and b) limit points. At these bifurcation points, the Jacobian matrix of the set of steady-state equations has a determinant of 0. There are 2 tangents at a branch point. At a limit point, there is only one tangent software to locate these bifurcations: CL_MATCONT [13,14] (a MATLAB software) is commonly used to locate limit points, branch points, and Hopf bifurcation points. Hopf bifurcation points do not cause multiple steady states.

For a dynamic system,

$$\dot{x} = f(x, \beta) \quad x \in R^n \quad (3)$$

Let the tangent plane at any point *x* be $[v_1, v_2, v_3, v_4, \ldots . v_{n+1}]$. Defining matrix, A as:

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \cdots\cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial \beta} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \cdots\cdots & \frac{\partial f_2}{\partial x_n} & \frac{\partial f_2}{\partial \beta} \\ \cdots\cdots & & & & & \\ \cdots\cdots & & & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \frac{\partial f_n}{\partial x_3} & \frac{\partial f_n}{\partial x_4} \cdots\cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial \beta} \end{bmatrix} \quad (4)$$

A can be written as:

$$A = [B|\frac{\partial f}{\partial \beta}] \quad (5)$$

The tangent plane, being orthogonal to the gradient vector, will satisfy the equation.

$$Av = 0 \quad (6)$$

For both limit and branch points, the matrix B must be singular. For a limit point (LP), the n+1th component of v; v_{n+1} = 0 the branch point (BP) condition is that the matrix $\begin{bmatrix} A \\ v^T \end{bmatrix}$ must be singular [15–17]. MATCONT detects all the singularities.

4.2. Multiobjective nonlinear model predictive (MNLMPC)M algorithm

In this article, the MNLMPC strategy [18,19] does not involve the use of weighting functions or impose additional constraints [20]. For an optimization problem:

$$\begin{aligned} &min J\,(x,u) = (x_1, x_2 \ldots . x_k) \\ &subject\ to\ \ \frac{dx}{dt} = F(x,u) \end{aligned} \quad (7)$$

First, the dynamic optimization problems are independently minimizing/maximizing each variable z_i individually. The minimization/maximization of z_i will provide the values z_i^*. Then the optimization problem:

$$\begin{aligned} &min\{ z_i - z_i^*\}^2 \\ &subject\ to\ \ \frac{dx}{dt} = F(x,u) \end{aligned} \quad (8)$$

Will be solved.

This will provide the control values for various times. The first obtained control value is implemented, and the remaining are discarded. This procedure is repeated until the implemented and the first obtained control value are the same.

The optimization package, Pyomo [21], was used for the calculations, Pyomo automatically differential equations to a Nonlinear Program (NLP) using the orthogonal collocation method [22]. 10 finite elements are chosen, and the Lagrange-

Radau quadrature with three collocation points is used to solve the optimal control problems. The resulting nonlinear optimization problem was solved using the IPOPT [23], and the globality of the solutions is confirmed with BARON [24]. The algorithm is as follows:

1) Optimize z_i with Pyomo to obtain the z_i^*.
2) Minimize $\{z_i - z_i^*\}^2$.
3) Implement only the first obtained control values.
4) Repeat until there is no difference between the implemented and the first obtained control values.

The utopia point is when $z_i = z_i^*$ for all i. Sridhar [25] has shown that the presence of branch points will cause the MNLMPC algorithm to converge to the uptopia point.

5. Results and discussion

5.1. Bifurcation analysis of ecological model

The software CL_MATCONT was used to perform the bifurcation analysis. Two cases were considered. In the first case, d3 was the bifurcation parameter while k was the bifurcation parameter in the second case. **Figures 1** and **2** show the bifurcation diagrams that were obtained. In both instances, branch points from which two different branches originated are shown.

The derivatives of f_1, f_2, f_3 with respect to the variables x_1, x_2, x_3 are:

$$\begin{aligned} &\frac{\partial f_1}{\partial x_1} = r - \frac{2rx_1}{k} - \frac{\partial \phi_{12}}{\partial x_1};\ \frac{\partial f_1}{\partial x_2} = -\frac{\partial \phi_{12}}{\partial x_2};\ \frac{\partial f_1}{\partial x_3} = 0 \\ &\frac{\partial f_2}{\partial x_1} = e_2\frac{\partial \phi_{12}}{\partial x_1};\ \frac{\partial f_2}{\partial x_2} = e_2\frac{\partial \phi_{12}}{\partial x_2} - \frac{\partial \phi_{23}}{\partial x_2} - d_2;\ \frac{\partial f_2}{\partial x_3} = -\frac{\partial \phi_{23}}{\partial x_3} \\ &\frac{\partial f_3}{\partial x_1} = 0;\ \frac{\partial f_3}{\partial x_2} = e_3\frac{\partial \phi_{23}}{\partial x_2};\ \frac{\partial f_3}{\partial x_3} = e_3\frac{\partial \phi_{23}}{\partial x_3} - d_3 \end{aligned} \tag{9}$$

The Jacobian matrix is:

$$J = \begin{pmatrix} (r - \frac{2rx_1}{k} - \frac{\partial \phi_{12}}{\partial x_1}) & (-\frac{\partial \phi_{12}}{\partial x_2}) & 0 \\ (e_2\frac{\partial \phi_{12}}{\partial x_1}) & (e_2\frac{\partial \phi_{12}}{\partial x_2} - \frac{\partial \phi_{23}}{\partial x_2} - d_2) & (-\frac{\partial \phi_{23}}{\partial x_3}) \\ 0 & e_3(\frac{\partial \phi_{23}}{\partial x_2}) & (e_3\frac{\partial \phi_{23}}{\partial x_3} - d_3) \end{pmatrix} \tag{10}$$

The determinant is given by:

$$
\begin{aligned}
det(J) &= (r - \frac{2rx_1}{k} - \frac{\partial \phi_{12}}{\partial x_1})(e_2 \frac{\partial \phi_{12}}{\partial x_2} - \frac{\partial \phi_{23}}{\partial x_2} - d_2)(e_3 \frac{\partial \phi_{23}}{\partial x_3} - d_3) \\
&+ e_3(\frac{\partial \phi_{23}}{\partial x_2})(\frac{\partial \phi_{23}}{\partial x_3}) + (\frac{\partial \phi_{12}}{\partial x_2})(e_2 \frac{\partial \phi_{12}}{\partial x_1}) e_3 (\frac{\partial \phi_{23}}{\partial x_2}) \\
&= (r - \frac{2rx_1}{k} - \frac{\partial \phi_{12}}{\partial x_1})(e_2 \frac{\partial \phi_{12}}{\partial x_2} - \frac{\partial \phi_{23}}{\partial x_2} - d_2)(e_3 \frac{\partial \phi_{23}}{\partial x_3} - d_3) \\
&+ \{e_3((\frac{\partial \phi_{23}}{\partial x_3}) + (\frac{\partial \phi_{12}}{\partial x_2})(e_2 \frac{\partial \phi_{12}}{\partial x_1}) e_3\}(\frac{\partial \phi_{23}}{\partial x_2}) \\
&= (r - \frac{2rx_1}{k} - \frac{\partial \phi_{12}}{\partial x_1})(e_2 \frac{\partial \phi_{12}}{\partial x_2} - \frac{\partial \phi_{23}}{\partial x_2} - d_2)(\frac{e_3 a_3 x_2}{b_3 + x_2} - d_3) \\
&+ \{e_3((\frac{\partial \phi_{23}}{\partial x_3}) + (\frac{\partial \phi_{12}}{\partial x_2})(e_2 \frac{\partial \phi_{12}}{\partial x_1}) e_3\}(\frac{a_3}{b_3 + x_2} - \frac{a_3 x_2}{b_3 + x_2}) x_3
\end{aligned} \tag{11}
$$

For steady-state to be attained $\frac{dx_3}{dt} = f_3 = 0$ This implies that $(\frac{e_3 a_3 x_2}{b_3+x_2} - d_3) = 0$; and/or $x_3 = 0$.

If both these terms are 0 det(J)=0 and the Jacobian matrix is singular. This is the only singular point because:

a) $(\frac{e_3 a_3 x_2}{b_3+x_2} - d_3) = 0$; and det (J)=0 t will imply that $x_3 = 0$; and

b) det (J)=0 and $x_3 = 0$; will imply that $(\frac{e_3 a_3 x_2}{b_3+x_2} - d_3) = 0$;

This singular point will be a branch point with 2 branches that $(\frac{e_3 a_3 x_2}{b_3+x_2} - d_3) = 0$; and$x_3 = 0$.

5.2. Computational validation

Case 1 d_3 bifurcation parameter

At the branch point (singular point) $x_1 = 138.529412; x_2 = 180.631105; x_3 = 0; d_3 = 0.054110$.

$b_3 = 250;\ e_3 = 1.29; a_3 = 0.1$ b3 = 250; the value of $(\frac{e_3 a_3 x_2}{b_3+x_2} - d_3) = 0$.

Case 2 *K* is a bifurcation parameter

At the branch point (singular point) $x_1 = 138.529412$; $x_2 = 112.359551$; $x_3 = 0$; k=277.431490.

d_3 = 0.04; $b_3 = 250$; $e_3 = 1.29; a_3 = 0.1; x_2 = 112.359551$ b3 = 250; the value of $(\frac{e_3 a_3 x_2}{b_3+x_2} - d_3) = 0$.

In both cases, at the singular point, $x_3 = 0$ and $(\frac{e_3 a_3 x_2}{b_3+x_2} - d_3) = 0$.

Figures 1 and **2** show the bifurcation diagrams when *d*3 and *K* are the bifurcation parameters.

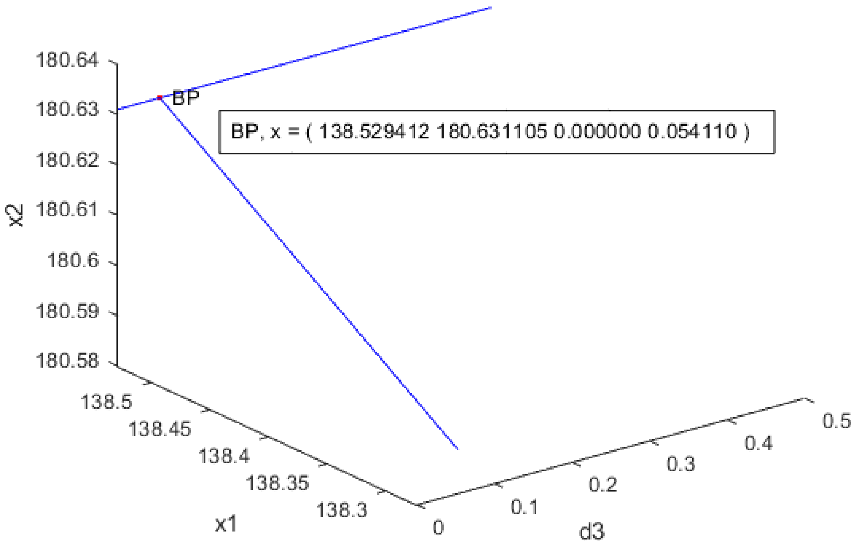

Figure 1. Bifurcation diagram with d3 as bifurcation parameter.

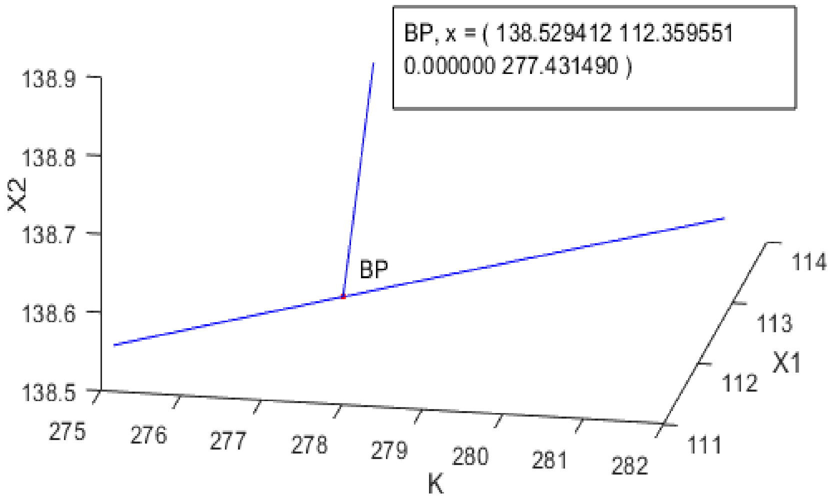

Figure 2. Bifurcation diagram with K as bifurcation parameter.

5.3. Multiobjective nonlinear model predictive control of the ecological model

The averaged fisher index (FI) is given by:

$$FI = \frac{1}{t_f}\int_0^{t_f} \frac{(a(t))^2}{(v(t))^4} dt$$

$$v(t) = \sqrt{\sum_{i=1}^{3} (\frac{dx_i}{dt})^2} = \sqrt{\sum_{i=1}^{3} (f_i)^2} \tag{12}$$

$$a(t) = \frac{1}{v(t)} \sum_{i=1}^{3} (\frac{dx_i}{dt})(\frac{d^2 x_i}{dt^2})$$

$$(\frac{d^2 x_i}{dt}) = \frac{df_i}{dt} = \sum_{j=1}^{3} (\frac{df_i}{dx_j})(f_j) \ldots i = 1,2,3$$

The expressions of the functions f_i and the derivatives $\frac{df_i}{dx_j}$ are provided in equation sets 2 and 3. Both d3 and k were used as control variables. Both $\sum_0^{t_f} x_3$ and the Fisher index (FI) were maximized individually. The maximization of $\sum_0^{t_f} x_3$ resulted in a value of 716.534, while the maximization of FI resulted in a value of 3.965×10^{-5}. For the multiobjective nonlinear model predictive calculations, the function minimized was $(\sum_0^{t_f} x_3 - 716.534)^2 + (FI - 3.965e - 05)^2$ subject to the equation set 2. The resulting objective function value obtained was the utopia point 0. The multiobjective nonlinear model control variables obtained were $d3 = 0.0274$ and $k = 680.00$.

Figures 3–6 show the profiles for the MNLMPC calculations.

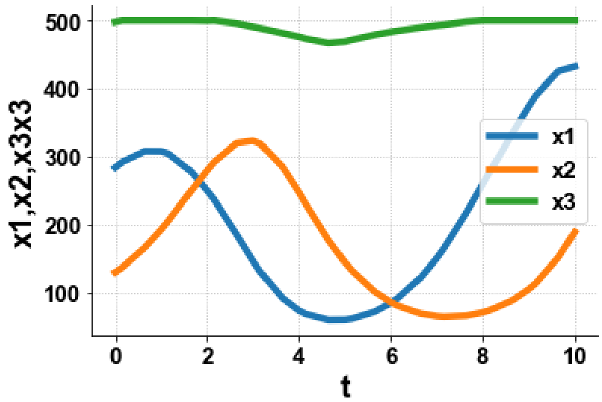

Figure 3. X1, X2, X3 profiles for MNLMPC calculations.

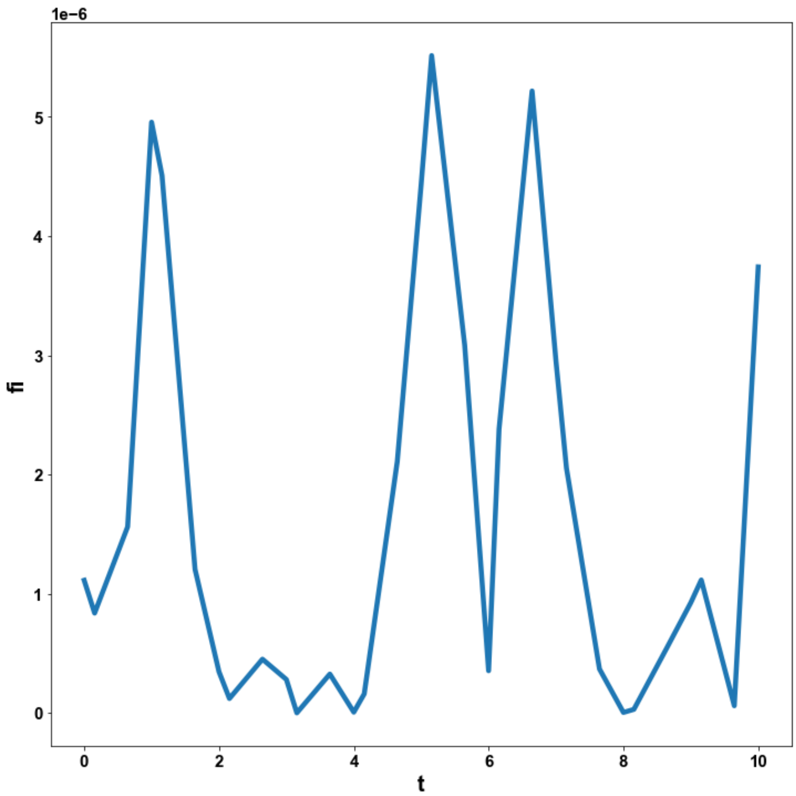

Figure 4. FI versus *t*.

The bifurcation points are instrumental in maximizing the sustainability and the amount of prey.

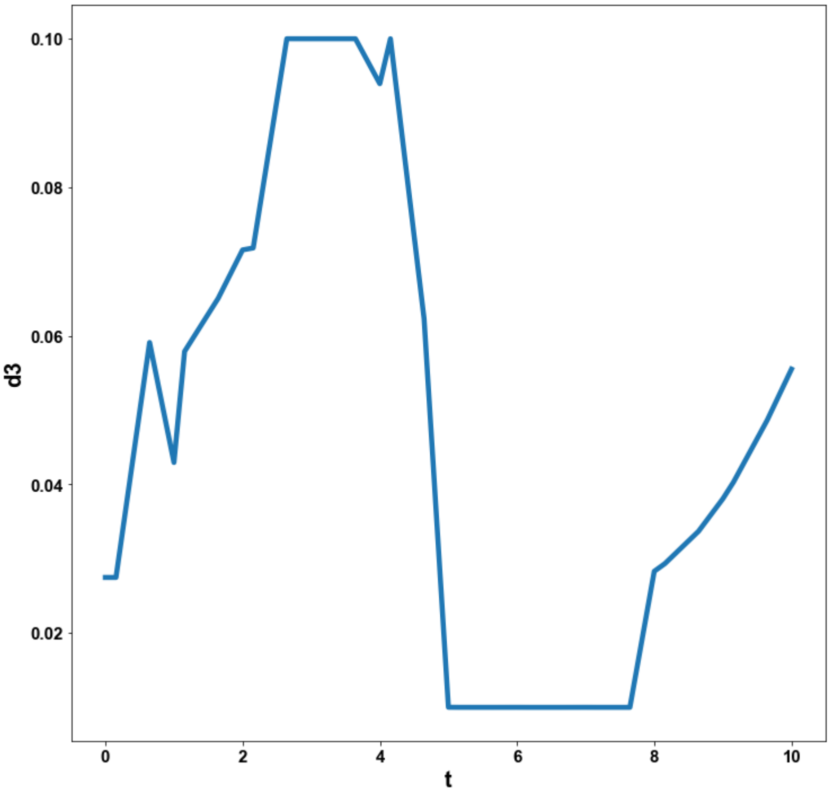

Figure 5. *d*3 versus *t*.

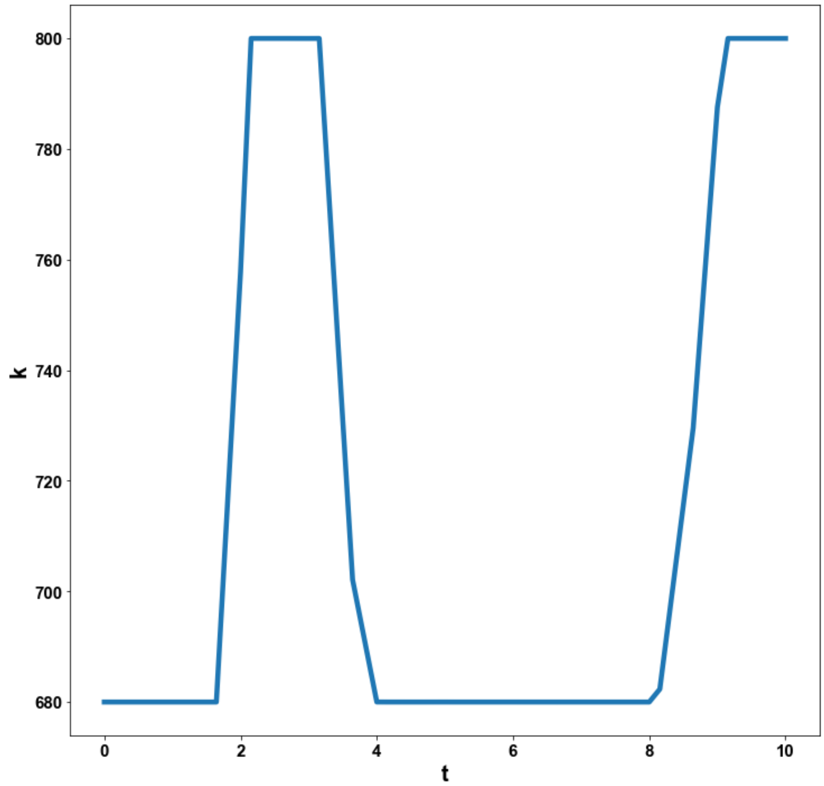

Figure 6. *K* versus *t*.

6. Conclusions

The main conclusions of this work are that one can attain the utopia point in MNLMPC calculations because of the branch points that occur in the ecosystem model, and the presence of the branch point can be proved analytically. The use of rigorous mathematics to enhance sustainability will be a significant step in encouraging sustainable development and a significant addition to the work of Manioudis and Meramveliotakis [26] and Meramveliotakis and Manioudis [27]. The main practical implication of this work is that the strategies developed here can be used by all researchers involved in maximizing sustainability. The future work will involve applying these mathematical strategies to other ecosystem models and food chain models, which will be a huge step in developing strategies to address problems involving nutrition. The broader impact of this work is that the MNLMPC calculations can be performed for other problems in conjunction with the bifurcation analysis.

Conflict of interest: The author declares no conflict of interest.

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