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Article

On the analytical mechanics methods in mathematical modeling the dynamics systems with geometric constraints

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Abstract: A necessary condition for the most effective application of the mathematical control theory results to the modern automatic devices dynamics consideration is the presence of an adequate nonlinear mathematical model obtained by strict general methods. Methods for reducing the dimensions of dynamic models of systems with geometric constraints by analytical mechanics methods for non-free systems are considered due to the transition to equations in redundant coordinates free of constraint multipliers. A detailed algorithm for this procedure and its justification is given. Using the theory of critical cases, a complete solution is given to the stabilizing problem of a given configuration of systems with geometric constraints.

Keywords: mathematical dynamics model; geometric constraints; stabilization

1. Introduction

The trouble-free operation of modern automatic technical devices is determined by the reliability of their control systems. The mathematical control theory with incomplete state information [1] has developed methods for increasing this reliability by not only reducing the control vector dimension (the number of actuators) [2,3], but also by reducing the volume of current measurement information (the number of measuring sensors) that ensure the formation and implementation of the necessary control actions in real time.

A necessary condition for the these results effective application is the adequate nonlinear mathematical model of the dynamics device under study, obtained by strict abstract-theoretical methods. Despite intensive research, the methods development for the strict nonlinear controlled dynamics modeling of modern automatic devices requires further development. All of the above fully applies to such a widespread and rapidly developing class of automatic systems as manipulators with parallel kinematics [4].

At the initial stage, completely incorrect attempts were made to describe the such systems dynamics using classical Lagrange equations of the second kind. The such equations use is based on the generalized coordinate introduction, i.e. independent parameters in the smallest number, uniquely determining the system state. This procedure is not feasible in the presence of relationships describing geometric connections.

Later, traditionally, studies began on modeling the dynamics of manipulators with parallel kinematics using the Lagrange equations with constraint multipliers. The need for a general solution to the inverse kinematics problem that arises with this approach makes it impossible to obtain an analytical form of a nonlinear model. Since in the absence of such a model, the effective application for the mathematical control theory results is impossible, published studies are limited to considering specific technical devices.

2. Methods

Until now, to construct mathematical models of these devices as systems with geometric constraints, a method [5–9] has been used that is far from being the most effective, based on the use of equations with constraint multipliers [10–14]. As is known, such an approach increases the dimensionality of the model: the variables of the model are, in addition to all coordinates, all (including dependent) velocities, and additionally constraint multipliers. Attempts are being made to simplify the study (reduce the dimensionality of the model) of specific devices by excluding constraint multipliers using an extremely labor-intensive method associated with double differentiation with respect to time of the geometric constraint equations and substitution into the obtained acceleration ratios of all coordinates expressed from equations with constraint multipliers.

The algorithm of such a study method was proposed and substantiated by Lyapunov [15] in 1885: the unique nontrivial solvability of the obtained ratios as linear inhomogeneous algebraic equations with respect to constraint multipliers was proven. The dimensionality of the mathematical model is reduced by this method by excluding the constraint multipliers from consideration. The applicability of this cumbersome method is obvious only to the study of the specific device dynamics; it cannot have general theoretical significance.

But even with the use of modern software for processing symbolic information, it is not possible to obtain the dynamics mathematical model for a specific technical device in analytical form due to the already noted extreme labor intensity by the practical implementation of the Lyapunov algorithm. Consideration using this method, in particular, of the Delta robot dynamics was limited [6,8,9] only to the device behavior computer simulation with an arbitrarily specified vector of control actions.

Analytical mechanics of non-free systems with geometric constraints (which include manipulators with parallel kinematics) has effective general theoretical methods for reducing the dimensions of mathematical models [10,11,13,14] of their dynamics by excluding from consideration not only the constraint multipliers, but also the velocities of coordinates dependent on the constraints [12].

However, as the analysis of publications shows, despite the numerous articles [16–19], these results have not yet found application in technical practice. In stabilizing problems for the given configuration manipulators with parallel kinematics, a further reduction of the mathematical model dimension for determining the stabilizing controls can be achieved by applying the theory of critical cases of nonlinear stability theory [20–22]. But for a justified conclusion, due to this theory, not only a strict selection of the first approximation is necessary, but also a complete analysis of the nonlinear terms of the equations of perturbed motion in analytical form. In such situations, no computer modeling can solve the problem. A complete nonlinear model obtained by strict abstract-theoretical methods with the necessary reduction of the equations to a special form using the replacement [23] is required. The use of general methods makes it possible to further reduce the control problems dimensions by moving, in particular, to Routh variables [24,25].

3. Traditional method for modeling the non-free systems dynamics

Let us consider the algorithm for compiling Lagrange equations with constraint multipliers for a system with coordinates $q_1, \dots, q_{n+m}$ and geometrical constraints,

$$F(q) = 0;\ F'(q) = \left(F_1(q), \dots, F_m(q)\right) \tag{1}$$

$$\det \left\| \frac{\partial\left(F_1(q), \dots, F_m(q)\right)}{\partial(q_{n+1}, \dots, q_{n+m})} \right\| \neq 0; \tag{2}$$

with kinetic energy of the most general form (here and below, summation is performed over repeating indices)

$$T(q, \dot{q}) = {}^1\!/_2\, a_{ik}(q)\dot{q}_i\dot{q}_k + a_i(q)\dot{q}_i + T_0;\ i,k = \overline{1, n+m}; \tag{3}$$

Due to the presence of relations (1), the following conditions are imposed on the variations of the coordinates:

$$b_{\sigma i}(q)\delta q_i = 0;\ b_{\sigma i} = \frac{\partial F_\sigma}{\partial q_i}\ ;\ \sigma = \overline{1, m};\ i = \overline{1, n+m}; \tag{4}$$

imposed on the variations of coordinates due to the presence of constraints (1) on which the forces Q_i, related to the coordinates (potential and non-potential) act.

The equations of motion with constraint multipliers

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i + \lambda_\sigma b_{\sigma i};\ \sigma = \overline{1, m};\ i = \overline{1, n+m}; \tag{5}$$

can be obtained [10–14] from the d'Alembert-Lagrange principle. The variables of the model are

$$q_i,\ \dot{q}_i,\ \lambda_\sigma i = \overline{1, n+m};\ \sigma = \overline{1, m};$$

In general, to obtain a complete nonlinear dynamics mathematical model of a system with geometric constraints, m nonlinear algebraic geometric constraints in Equation (1) should be added to $n + m$ second-order differential Equation (5). The model includes all coordinates and all velocities (including those dependent on constraints). To determine the Lagrange multipliers, it is necessary to take into account relations (1) for determining the constraint multipliers. When passing to the normal form, we obtain $2(n + m)$ first-order differential equations and m algebraic constraint equations. The variables of such a high-dimensional model are all $n + m$ coordinates, all $n + m$ velocities and m constraint multipliers. The total dimension of the model is $2(n + m) + m = 2n + 3m$.

This approach leads to an unjustified increase in the dimensions of the models, which greatly exceeds the number of degrees of freedom of the system. Therefore, for controlled systems with geometric constraints, there is still practically no analytical form of mathematical dynamics models, including the actuator models.

The situation persists [5–9], despite the fact that in the analytical mechanics of non-free systems, methods have been developed [12,16–19] for reducing the dimensions of models by switching to equations in redundant coordinates by excluding dependent velocities from consideration using differentiated equations of geometric constraints and the expression for the Lagrange multipliers found in the general case.

Remark 1. *It should be noted that attempts have been made to model the system's dynamics with geometric constraints using Lagrange equations of the second kind. In particular, such an approach was used to study the dynamics of the well-known Ball and Beam (**Figure 1**).*

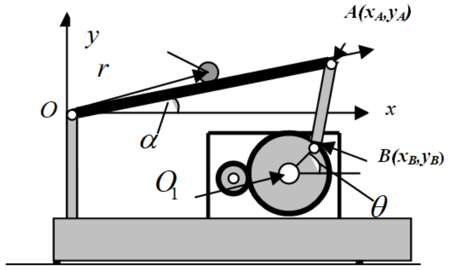

Figure 1. Scheme of the "Ball and Beam" stand.

In this system, the electric drive, due to the inclination of the chute OA, can roll the ball C to any predetermined position on the chute and stabilize this equilibrium. The chute is connected on one side at point O to a fixed supporting post, and on the other to a movable lever AB. The motion of the lever is controlled by a DC motor. A nonlinear geometric relationship is imposed on the system: the distance between points A(x_A, x_B) and B(x_B, y_B) is constant:

$$\left(L(cos\alpha - 1) + d(1 - cos\theta)\right)^2 + (Lsin\alpha + l - dsin\theta)^2 = l^2; \tag{6}$$

(L = OA − length of the trough, l = AB − length of the lever, d − radius of the output drive wheel).

Starting with [26], in all studies of the dynamics of the Ball and Beam stand with a geometric constraint [26–31], when constructing a mathematical model, immediately, starting with [26] and up to now [30,31], they completely unjustifiably move from a nonlinear Equation (6) to a linear dependence $\alpha \approx \frac{d}{L}\theta$ and exclude the dependent coordinate from consideration. Despite the fact that the incorrectness of such an approach has already been discussed in detail earlier [16–18], such a model is still used [30,31], and even referring to the article [17] (compare with [32]). Therefore, the author considered it appropriate to once again present in detail an alternative algorithm for obtaining, using strict methods of analytical mechanics, non-free systems of complete nonlinear models, the dimensions of which are significantly smaller than those traditionally used.

4. Method of excluding constraint multipliers in dynamics models for specific systems with geometric constraints.

The labor intensity by the dynamics research of specific technical devices with geometric constraints is determined by the dimensionality of the model used. Most often, information on constraint reactions is not required in technical practice. Therefore, it is possible to simplify the study by defining an explicit form of constraint multipliers to exclude them from consideration.

The general procedure for this operation was developed by Lyapunov [15]. With this modeling method, the dynamics of a specific system, it is necessary to consistently perform extremely labor-intensive and cumbersome operations:

1) Express all accelerations $\ddot{q}_i$ from the equations of motion with constraint multipliers (6).

2) Obtain a system of linear algebraic relations with respect to accelerations by differentiating the expressions of geometric constraints twice with respect to time.

3) By substituting the expressions for accelerations from the motion equations into these relations, obtain a linear inhomogeneous algebraic equations system with respect to the constraint multipliers λ_σ.

4) Express the constraint multipliers from these equations as $\lambda_{\sigma(}(q, \dot{q})$.

5) Substitute the expressions $\lambda_{\sigma(}(q, \dot{q})$ into Equation (6) to obtain a mathematical model of the system dynamics without constraint multipliers.

Despite the extreme labor intensity of this method of obtaining a model, proposed back in the 19th century, it is this method that is still used in the study of the Delta robot [5]. The complexity of the resulting mathematical model (only the mechanical component of the robot without taking transient processes into account for the actuators is considered) leads to the possibility of considering only a computer simulation.

It should be noted that (using modern information technologies) the method [15] was used [5] without references. It is obvious that it is fundamentally impossible to obtain an explicit analytical form of a nonlinear mathematical model of the device under study dynamics in this way. In our opinion, that leads to the fact that in modern dynamics studies for robotic devices with parallel kinematics, the mechanical robot component behavior analysis from the analytical mechanics standpoint is not at all carried out, and immediately a transition to computer modeling is carried out.

Moreover, a completely unfounded conclusion about the model adequacy constructed using modern information technologies with selected specific numerical values of the device parameters is made on the basis of the model's behavior proximity (obtained as a numerical experiment result) to the real object dynamics. Analysis of the particular solution behavior for a nonlinear differential equations system, especially one obtained using one or another approximate calculation method, in principle cannot serve as the basis for any conclusions about the nature of this system's general solution.

5. Some general methods for reducing the dimension of the mathematical model in modeling the non-free systems dynamics

General methods for such an exclusion, associated with a single or double differentiation with respect to time of the equation of geometric constraints (1) Lurye [10], Suslov [11] developed.

Separating in Equation (4) the variations of independent and dependent coordinates we obtain

$$b_{\sigma\rho}(q)\delta q_\rho + b_{\sigma\mu}(q)\delta q_\mu = 0;\ \mu = \overline{n-m+1, n+m};\ \sigma = \overline{1,m};\ \rho = \overline{1,n}; \quad (7)$$

According to Equation (2) we can express from Equation (7) the dependent variations.

$$\delta q_\mu = B_{\mu\rho}(q)\delta q_\rho;\ \|B_{\sigma\rho}(q)\| = -\left\|\frac{\partial(F_1,\dots,F_m)}{\partial(q_{n+1},\dots,q_{n+m})}\right\|^{-1}\left\|\frac{\partial(F_1,\dots,F_m)}{\partial(q_1,\dots,q_n)}\right\|; \quad (8)$$

The conditions imposed by the constraints Equation (1) on the variations take the form

$$\delta q_\mu - B_{\mu\rho}(q)\delta q_\rho = 0 \quad (9)$$

Using Equation (9) we can separate Equation (5) for dependent and independent coordinates.

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q_\rho} - \frac{\partial T}{\partial q_\rho} = Q_\rho - B_{\sigma\rho}\lambda_\sigma;\ \frac{d}{dt}\frac{\partial T}{\partial \dot q_\sigma} - \frac{\partial T}{\partial q_\sigma} = Q_\sigma + \lambda_\sigma;\ \sigma = \overline{1,m};\ \rho = \overline{1,n}; \quad (10)$$

Expressing the multipliers from the second Equation (10)

$$\lambda_\sigma = \frac{d}{dt}\frac{\partial T}{\partial \dot q_\sigma} - \frac{\partial T}{\partial q_\sigma} - Q_\sigma; \sigma = \overline{n+1, n+m}; \quad (11)$$

After substituting Equation (11) for the first equation of system Equation (10), we obtain

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q_\rho} - \frac{\partial T}{\partial q_\rho} = Q_\rho - B_{\sigma\rho}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot q_\sigma} - \frac{\partial T}{\partial q_\sigma} - Q_\sigma\right);$$

From the equations of the constraints (1), differentiated once with respect to time, it is possible to express the dependent velocities.

$$\dot q_\sigma = B_\mu(q)\dot q_j;\ \sigma = \overline{1,m}; j = \overline{1,n}; \quad (12)$$

In our opinion, a much more effective way to reduce the dimensionality of the model is the alternative method of Shulgin [12] of transition to decreases in redundant coordinates by excluding from consideration the multipliers and dependent velocities.

Denoting the result of eliminating dependent velocities using Equation (12) from the kinetic energy Equation (3) through $T^*(q_1, \dots, q_{n+m}, \dot q_1, \dots, \dot q_n)$, and from the acting forces through Q_i^*.

$$T^* = {}^1\!/_2\, a^*_{\gamma\rho}(q)\dot q_\rho \dot q_\gamma + a^*_\rho(q)\dot q_\rho + T_0;\ \rho, \gamma = \overline{1,n}; \quad (13)$$

$$a^*_{\gamma\rho}(q) = a_{\gamma\rho} + a_{\mu\gamma}B_{\mu\gamma}B_{\mu\rho} + a_{\mu\rho}B_{\mu\gamma} + a_{\mu\sigma}B_{\mu\rho}B_{\sigma\gamma};\ a^*_\rho(q) = a_\rho + a_\mu B_{\mu\rho}.$$

Comparing the corresponding derivatives $T^*(q_1, \dots, q_{n+m}, \dot q_1, \dots, \dot q_n)$ and $T(q_1, \dots, q_{n+m}, \dot q_1, \dots, \dot q_{n+m})$ and taking into account the integrability of the constraints (12), we will have as a result, in the general case, a nonlinear mathematical model of the system dynamics is obtained in the form of M.F. Shulgin’s equations free of multipliers [12] in redundant coordinates (cf. the equations of Voronets [33–35] into account the integrability of the kinematic constraints (12):

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_\rho} - \frac{\partial T^*}{\partial q_\rho} = Q_\rho^* + B_{\sigma\rho}(q)\left(\frac{\partial T^*}{\partial q_\sigma} + Q_\sigma^*\right);\ \dot{q}_\sigma = B_{\sigma\rho}(q)\dot{q}_\rho; \tag{14}$$

$$\sigma = \overline{n+1, n+m};\ \ \rho = \overline{1, n};$$

It should be noted that, unlike the approach of Lyapunov [15], this is a general theoretical method. In the general case, the dimension of the mathematical model of the dynamics of systems with geometric constraints in the form (14) is reduced in comparison with Equation (5) by a double number of constraints (1). In addition to the multipliers, the model does not contain the velocity coordinates that depend on the presence constraints (1). Moreover, it should be especially noted that the application of this modeling method to the study of the dynamics of specific technical devices generally eliminates the need to solve the inverse kinematics problem (cf. [5]). The strict nonlinear model is obtained in analytical form and includes in normal form only $2n + m$ differential equations of the first order, i.e., it is reduced by $2m$ in comparison with the dimension $2n + 3m$ of model (1), (5).

Let us once again present a step-by-step algorithm for obtaining a generally nonlinear mathematical model (14) of the dynamics of a system with geometric constraints (1):

1) Differentiate the equations of geometric relationships (1) once with respect to time.

2) Express the velocities of dependent coordinates (12) from this linear algebraic system.

3) After substituting (12) into the acting forces, kinetic energy (3) is obtained as kinetic energy (13) with the dependent velocities excluded.

4) Obtain a mathematical model (14) of the dynamics of systems in redundant coordinates.

It is extremely important to note that, for the practical application of this mathematical modeling method for the systems dynamics with geometric constraints, mathematical training in the amount of standard engineering education is quite sufficient. No additional competencies in the field of analytical mechanics of non-free systems are required.

The obtained general analytical form of the mathematical model of nonlinear dynamics allows, in the general case, to present the mathematical dynamics model for any system with geometric constraints (under the action of arbitrary potential and non-potential forces that do not violate the conditions for the existence and uniqueness of the differential equations solutions) in the form of the Shulgin Equation (14) in explicit scalar form [16,17,36] without resolving the inverse problem of kinematics (cf. [5,7,8,37,38]):

$$\begin{aligned} a_{lr}^*\ddot{q}_r + \frac{\partial a_{lr}^*}{\partial q_j}\dot{q}_r\dot{q}_j + \frac{\partial a_{lr}^*}{\partial q_\mu}B_{\mu j}\dot{q}_l\dot{q}_j - \frac{1}{2}\frac{\partial a_{lj}^*}{\partial q_l}\dot{q}_r\dot{q}_j - \frac{1}{2}\frac{\partial a_{lj}^*}{\partial q_\mu}B_{\mu l}\dot{q}_r\dot{q}_j + \left(\frac{\partial a_l^*}{\partial q_r} - \frac{\partial a_r^*}{\partial q_l} + B_{\mu l}\frac{\partial a_l^*}{\partial q_r} - B_{\mu l}\frac{\partial a_r^*}{\partial q_\mu}\right)\dot{q}_r \\ + \frac{\partial W}{\partial q_l} + B_{\mu l}\frac{\partial W}{\partial q_\mu} = Q_l^* + B_{\mu l}Q_\mu^*; \dot{q}_\sigma = B_{\sigma j}(q)\dot{q}_j; \end{aligned} \tag{15}$$

$\mu, \sigma = \overline{n+1, n+m}; \; l, j, r = \overline{1, n};$

here $W(q) = \Pi(q) - T_0(q)$ is the changed (reduced) potential energy, $\Pi(q)$ - potential energy, and Q_l^*, Q_μ^* now denotes the non-potential forces corresponding to the coordinates $q_l, \; q_\mu$ when they are introduced in excess.

One of the most important problems in technical practice of the technical objects control is ensuring stable implementation of the object specified behavior. The analytical form of the obtained mathematical model created the possibility for the Krasovskii's method using by determining the stabilizing control that solves this problem. The developed complex application of analytical mechanics strict methods for non-free systems [10–14] and nonlinear stability theory [20–22] to real modern technical devices due to the reduction of the model dimension allows including the actuators dynamics in the consideration. In particular, in detailed studies of the Ball and Beam stand, additional voltage on the armature winding of the actuator drive commutator motor was considered as a stabilizing control. The control coefficients were determined [16–19] from the linear-quadratic stabilization problem solution by the Krasovskii method [39]. To conclude about the asymptotic (despite the presence of the zero root of the characteristic equation) stability [20–24] in the complete nonlinear system closed by the found control, a proof of the general theorem [16,17] was required.

The results obtained in the complete study of the Ball and Beam stand dynamics created [16–18,34] the basis for the development of such a general modeling method for the controlled dynamics of non-free mechanical systems with geometrical constraints in general [19], which turned out to be an effective tool for the nonlinear dynamics modeling of manipulators with parallel kinematics [36]. The rigorous nonlinear mathematical models obtained with its use made it possible to apply general methods of nonlinear stability theory [20–24] for a complete solution, in particular, in stabilization problems of given parallel manipulator configurations.

6. Application of the developed mathematical model to the general stabilization problem for non-free systems

Let the system admit an equilibrium position.

$$q_i = q_{i0}, \quad q_\mu = q_{\mu 0} \tag{16}$$

Equilibrium equations can be obtained in the general case from the Equation (15).

$$\frac{\partial W}{\partial q_i} + B_{\mu i}\frac{\partial W}{\partial q_\mu} = Q_{i(\dot{q}=0)} + \left(B_{\mu i} Q_\mu\right)_{(\dot{q}=0)} \tag{17}$$

When $T_0(q) = 0$ and there are no non-potential forces, we obtain

$$\frac{\partial \Pi}{\partial q_i} + B_{\mu i}\frac{\partial \Pi}{\partial q_\mu} = 0.$$

Remark 2. *It follows that the equilibrium position when using redundant coordinates may not be a stationary point of potential energy. However, if the* *potential energy can be expressed through independent generalized coordinates $\Pi(q_1, \dots, q_n)$, then for it, in the absence of non-potential positional forces, all*

$$\frac{\partial \Pi}{\partial q_i} = 0, i = \overline{1, n}.$$

Remark 3. *From the equilibrium Equation (17) it follows that potential and non-potential positional forces may contain constant non-zero terms*

$$\left(\frac{\partial W}{\partial q_i}\right)_0 \neq 0, \left(\frac{\partial W}{\partial q_\mu}\right)_0 \neq 0,\ Q_i(q_0) \neq 0, Q_\mu(q_0) \neq 0: \tag{18}$$

In order to formulate the equations of disturbed motion necessary for solving the stabilization problem, we introduce disturbances.

$$q_{i0} = q_{i0} + x_i, q_\mu = q_{\mu o} + y_\mu.$$

To solve the stabilization problem, equations of perturbed motion should be drawn up. Let us introduce disturbances, compose equations of perturbed motion and select the first approximation in them. Let us formulate the equations of the perturbed motion and select the first approximation in them.

Here, for ease of understanding the taking-into-account problem of the required order in the constraint equations with a strict selection of the first approximation, the equations are written without taking into account non-potential forces (among which there may be control ones):

$$\begin{aligned} a_{is}^*(0)\ddot{x}_s &+ \left[\left(\frac{\partial a_i^*}{\partial q_s}\right)_0 - \left(\frac{\partial a_s^*}{\partial q_i}\right)_0 + B_{\mu s}(0)\left(\frac{\partial a_i^*}{\partial q_\mu}\right)_0 - \left(\frac{\partial a_s^*}{\partial q_\mu}\right)_0 B_{\mu i}(0)\right]\dot{x}_s \\ &+ \left[\left(\frac{\partial^2 W}{\partial q_i \partial q_s}\right)_0 + B_{\mu i}(0)\left(\frac{\partial^2 W}{\partial q_\mu \partial q_s}\right)_0 + \left(\frac{\partial W}{\partial q_\mu}\right)_0 \left(\frac{\partial B_{\mu i}}{\partial q_s}\right)_0\right] x_s \\ &+ \left[\left(\frac{\partial^2 W}{\partial q_i \partial q_k}\right)_0 + B_{\mu i}(0)\left(\frac{\partial^2 W}{\partial q_k \partial q_\mu}\right)_0 + \left(\frac{\partial B_{\mu i}}{\partial q_k}\right)_0 \left(\frac{\partial W}{\partial q_\mu}\right)_0\right] y_k = X_i^{(2)}(\dot{x}, x, y); \end{aligned} \tag{19}$$

here $(\cdots)_0$ means the value of the expression in brackets in equilibrium (16), and the number in the brackets in the upper index is the order of the lowest terms in the corresponding expression expansion.

Let us add to these equations the equations of differential constraints with the selected first approximation

$$\dot{y}_k = B_{jk}(0)\dot{x}_k + B_{jk}^{(1)}(x, y)\dot{x}_k \tag{20}$$

$$B_{jk}^{(1)}(x, y) = B_{jk}(q_{i0} + x, q_{\mu 0} + y) - B_{jk}(q_{i0}, q_{\mu 0});\ B_{jk}(q_{i0}, q_{\mu 0}) = B_{jk}(0).$$

If a linear replacement is carried out in system (18), (19) [23]

$$z_k = y_k - B_{kj}(0)x_j \tag{21}$$

then the constraint equations will take the form

$$\dot{z}_k = B_{kj}^{(1)}(x, z + B(0)x)\dot{x}_j; \tag{22}$$

After replacement (21), there are no linear terms on the right in Equation (22). Obviously, the variables z_k correspond to the zero roots of the characteristic equation of the system.

$$\begin{aligned} a_{is}^*(0)\ddot{x}_s &+ \left[\left(\frac{\partial a_i^*}{\partial q_s}\right)_0 - \left(\frac{\partial a_s^*}{\partial q_i}\right)_0 + B_{\mu s}(0)\left(\frac{\partial a_i^*}{\partial q_\mu}\right)_0 - \left(\frac{\partial a_s^*}{\partial q_\mu}\right)_0 B_{\mu i}(0)\right]\dot{x}_s \\ &+ \left[\left(\frac{\partial^2 W}{\partial q_i \partial q_s}\right)_0 + B_{\mu i}(0)\left(\frac{\partial^2 W}{\partial q_\mu \partial q_s}\right)_0 + \left(\frac{\partial W}{\partial q_\mu}\right)_0\left(\frac{\partial B_{\mu i}}{\partial q_s}\right)_0\right]x_s \\ &+ \left[\left(\frac{\partial^2 W}{\partial q_i \partial q_k}\right)_0 + B_{\mu i}(0)\left(\frac{\partial^2 W}{\partial q_k \partial q_\mu}\right)_0 + \left(\frac{\partial B_{\mu i}}{\partial q_k}\right)_0\left(\frac{\partial W}{\partial q_\mu}\right)_0\right]\left(z_k + B_{kj}(0)x_j\right) \\ &= X_i^{(2)}(\dot{x}, x, z + B(0)x) \end{aligned} \tag{23}$$

$$\dot{z}_k = B_{kj}^{(1)}(x, z + B(0)x)\dot{x}_j.$$

The system of Equation (23) represents a mathematical model of the dynamics of an arbitrary system with geometric constraints with a strictly selected first approximation. It should be noted that in the equations of the first approximation, in the general case, due to conditions (18), there may be terms

$$\left(\frac{\partial B_{\mu i}}{\partial q_k}\right)_0, \left(\frac{\partial B_{\mu i}}{\partial q_s}\right)_0,$$

determined by the coefficients of quadratic terms in the geometric constraint expansion in the neighborhood equilibrium (16), since the matrix

$$\left\|B_{\sigma\rho}(q)\right\| = -\left\|\frac{\partial(F_1,\ldots,F_m)}{\partial(q_{n+1},\ldots,q_{n+m})}\right\|^{-1}\left\|\frac{\partial(F_1,\ldots,F_m)}{\partial(q_1,\ldots,q_n)}\right\|,$$

through the first-order terms in this expansion. The need to take into account quadratic terms in the constraint equations expansion when studying the stability of systems with geometric constraints was pointed out by Rouse [13].

Now let us assume that in addition to potential forces with energy *Π(q)*, non-potential positional forces and forces depending on velocities act on the system. Having singled out the first approximation in the motion equations for such systems and using the vector-matrix form of recording, we will have

$$Q_i = f_{ij}\dot{q}_j + f_{i\mu}\dot{q}_\mu + p_{ij}q_j + iq_\mu + \cdots.$$

After eliminating dependent velocities and replacing (21) for dependent coordinates, we obtain the characteristic equation of the first approximation system in the general case by the action of potential and non-potential forces.

$$\lambda^m \, det[A\lambda^2 + (G + D)\lambda + C_1 + C_2 B + B'C_3 + B'C_4 B + C^B + P] = 0 \tag{24}$$

$$\mathrm{D} = \left\|f_{ij} + f_{i\mu}B_{\mu i} + f_{\mu i}b_{\mu j} + f_{\mu i}b_{\mu k}B_{kj}\right\|;\ G = \{g_{is}(0)\};$$

$$\mathrm{P} = \left\|p_{ij} + p_{i\mu}B_{\mu i} + B_{\mu j}p_{\mu j} + B_{\mu i}p_{\mu k}B_{kj}\right\|;$$

$$\mathrm{A} = \|a^*(0)\|;\ G = \|g_{is}(0)\|; C_1 = \left\|\left(\frac{\partial^2 W}{\partial q_i \partial q_s}\right)_0\right\|; C_2 = \left\|\left(\frac{\partial^2 W}{\partial q_i \partial q_\mu}\right)_0\right\|;$$

$$C_4 = \left\| \left(\frac{\partial^2 W}{\partial q_\mu \partial q_\sigma} \right)_0 \right\|;\ C^B = \left\| \left(\frac{\partial W}{\partial q_\mu} \right)_0 \left(\frac{\partial B_{\mu i}}{\partial q_s} \right)_0 + \left(\frac{\partial W}{\partial q_\mu} \right)_0 \left(\frac{\partial B_{\mu i}}{\partial q_k} \right)_0 B_{ks}(0) \right\|;$$

$$\mathrm{B} = \|B_{\mu i}(0)\|;\ g_{is}(0) = \left[\left(\frac{\partial a_i^*}{\partial q_s} \right)_0 - \left(\frac{\partial a_s^*}{\partial q_i} \right)_0 + B_{\mu s}(0) \left(\frac{\partial a_i^*}{\partial q_\mu} \right)_0 - \left(\frac{\partial a_s^*}{\partial q_\mu} \right)_0 B_{\mu i}(0) \right].$$

Let us note once again that in the equations of the perturbed motion after the replacement (21), in the general case, linear terms appear with the matrix C^B, which depends on the linear terms of the expansion of the coefficients of the kinematic constraints $\left(\frac{\partial B_{\mu i}}{\partial q_s}\right)_0, \left(\frac{\partial B_{\mu i}}{\partial q_k}\right)_0$. Hence, the conclusion follows: in the general case, to obtain a linear approximation of the equations of perturbed motion, one cannot limit oneself to considering only the linear term of the expansion of the kinematic constraint Equation (1) (cf. [26–28,30–32]).

Obviously, the characteristic equation (24) in the general case always, under the action of any forces, including control forces, has m zero roots. Consequently, the stability of the equilibrium states of systems with geometric connections is possible only in critical cases [20–22]. Using Kamenkov's [22] theorem, the following is proved:

Theorem [16,17]: If the real parts of all other, except m zero, roots of the characteristic equation are negative, then the equilibrium position [16] is asymptotically stable.

Remark 4. *The developed method basis is the reduction principle results of the nonlinear variables stability theory, when in addition to roots with zero real parts, there necessarily exist roots with negative real parts. The rigorous results of the general reduction principle are formulated after two variable changes. The first of them, linear, should reduce the equations to the so-called special form. With the indicated arrangement of roots, this change always exists. The second nonlinear change existence over all noncritical variables is a difficult problem to solve, since these functions are defined as the series in the variable powers, which convergence conditions in the theorem in the general case cannot be formally verified.*

However, in the mechanical systems special case with differential constraints, due to the nonlinear equations structure, it turned out to be possible to carry out the change only over the part of noncritical variables. And for such systems, the nonlinear change solves the problem of stability (non-asymptotic) in a special case. When passing from geometric constraints in the form of finite equations (1) to the differential constraints in the form (12), taking into account the structure of the arising perturbed motion equations, in contrast to the general case, it is the asymptotic stability of the unperturbed motion that is proven.

Therefore, to solve the problem of stabilizing equilibrium (16) it is necessary to ensure the negativity of the real roots of the equation.

$$det[A\lambda^2 + (G + D)\lambda + C_1 + C_2 B + B' C_3 + B' C_4 B + C^B + P] = 0 \quad (25)$$

The location of the roots of Equation (25) is determined by the linear subsystem.

$$
\begin{aligned}
a_{is}^{*}(0)\ddot{x}_s &+ \left[\left(\frac{\partial a_i^*}{\partial q_s}\right)_0 - \left(\frac{\partial a_s^*}{\partial q_i}\right)_0 + B_{\mu s}(0)\left(\frac{\partial a_i^*}{\partial q_\mu}\right)_0 - \left(\frac{\partial a_s^*}{\partial q_\mu}\right)_0 B_{\mu i}(0)\right]\dot{x}_s \\
&+ \left[\left(\frac{\partial^2 W}{\partial q_i \partial q_s}\right)_0 + B_{\mu i}(0)\left(\frac{\partial^2 W}{\partial q_\mu \partial q_s}\right)_0 + \left(\frac{\partial W}{\partial q_\mu}\right)_0 \left(\frac{\partial B_{\mu i}}{\partial q_s}\right)_0\right] x_s \\
&+ \left[\left(\frac{\partial^2 W}{\partial q_i \partial q_k}\right)_0 + B_{\mu i}(0)\left(\frac{\partial^2 W}{\partial q_k \partial q_\mu}\right)_0 + \left(\frac{\partial B_{\mu i}}{\partial q_k}\right)_0 \left(\frac{\partial W}{\partial q_\mu}\right)_0\right] B_{kj}(0)x_j = 0
\end{aligned}
\tag{26}
$$

The dimension of this subsystem in normal form is $2n$, i.e., it is further reduced by the number of constraints. Thus, using the analytical mechanics of non-free systems, strict methods, and nonlinear stability theory, a mathematical model is obtained for the general stabilizing problem of given configurations for systems with geometric constraints (in particular, manipulators with parallel kinematics), which dimension is equal to the number of freedom degrees of the system.

The mathematical model (26) includes any possible (not violating the theorem's conditions about the existence and uniqueness of solutions for the corresponding differential equations) potential and non-potential forces. Among these forces, there may also be control mechanical actions (forces and moments) that ensure the specified behavior of the system.

The achieved significant reduction in the model dimensionality and the methods rigor used to obtain it make it possible to include in the controlled dynamics model for a controlled subsystem a mathematical actuators model that creates these control actions. By virtue of the proven theorem on asymptotic stability (taking into account the nonlinear terms) for the model (26) in additional equations modeling the dynamics of actuators, it is sufficient to limit ourselves to only linear terms: with respect to these variables, under the controllability condition, it is always possible to achieve asymptotic stability by the first approximation.

To simplify the further presentation, in accordance with the different nature of the dependence of kinetic and potential energies, geometric constraints and non-potential forces, we introduce vectors.

$$
q = \begin{pmatrix} q_1 \\ \vdots \\ q_{n+m} \end{pmatrix}; r = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}; s = \begin{pmatrix} q_{n+1} \\ \vdots \\ q_{n+m} \end{pmatrix}; F(q) = \begin{pmatrix} F_1 \\ \vdots \\ F_m \end{pmatrix}; Q(q,\dot{q}) = \begin{pmatrix} Q_1 \\ \vdots \\ Q_{n+m} \end{pmatrix}
$$

$$
Q_r(q,\dot{q}) = \begin{pmatrix} Q_1 \\ \vdots \\ Q_n \end{pmatrix}; Q_s(q,\dot{q}) = \begin{pmatrix} Q_{n+1} \\ \vdots \\ Q_{n+m} \end{pmatrix},
$$

and we will move on to the vector form of the dynamics model

$$
\frac{d}{dt}\frac{\partial T^*}{\partial \dot{r}} - \frac{\partial T^*}{\partial r} = Q_r^* - B'(q)\left(\frac{\partial T^*}{\partial s} - Q_s^*\right); \dot{s} = B(q)\dot{r}; \tag{27}
$$

Equilibrium position (16) in vector form

$$
r = r_0; s = s_0: \tag{28}
$$

Let us introduce disturbances and write down the equations of the disturbed motion with the selected first approximation.

$$
\mathrm{r} = r_0 + x; \dot{r} = x_1; s = s_0 + y:
$$

$$\dot{x} = x_1;$$

$$\dot{x}_1 = ax + bx_1 + hy + X_1^{(2)}(x, x_1, y); \tag{29}$$

$$\dot{y} = B(x, y)x_1.$$

The matrices in system (29) are expressed in a known [16–19,32,36] manner through the parameters of the system.

After replacement (21) in vector form

$$z = y - B(0)x; \tag{30}$$

the system (29) will go into

$$\dot{x} = x_1.$$

$$\dot{x}_1 = (a + hB(0))x + bx_1 + hz + X_1^{(2)}(x, x_1, z + B(0)x); \tag{31}$$

$$\dot{z} = B^{(1)}(x, z + B(0)x)x_1.$$

The characteristic equation of this system is

$$\begin{vmatrix} E_n\lambda & -E_n & 0 \\ -a - hB(0) & E_n\lambda - b & 0 \\ 0 & 0 & E_m\lambda \end{vmatrix} = 0: \tag{32}$$

According to the Theorem [16,17] asymptotic stability sufficient condition for the zero solution in the complete nonlinear system (29), the real parts of the roots equation must be negative.

$$\begin{vmatrix} E_n\lambda & E_n\lambda \\ -a - hB(0) & E_n\lambda - b \end{vmatrix} = 0 \tag{33}$$

If this condition is not met, a stabilization problem arises: it is necessary to introduce control that ensures the desired location of the characteristic equation roots for the closed system. If we do not consider the actuators dynamics, to determine the stabilizing mechanical action we obtain the following model for linear system of dimension 2*n.*

$$\dot{x} = x_1;$$

$$\dot{x}_1 = \big(a + hB(0)\big)x + bx_1 + Vu; \tag{34}$$

Here the matrix *V* models the way the control *u* is applied. To formulate sufficient conditions for the solvability of the stabilization problem, we will reduce system (34) to the standard form of mathematical control theory.

$$\dot{\eta} = P\eta + Qu;\ \eta' = (x', x'_1);\ P = \begin{pmatrix} 0 & E_n \\ a + hB(0) & b \end{pmatrix};\ Q = \begin{pmatrix} 0 \\ V; \end{pmatrix} \tag{35}$$

When the controllability condition for the system is met Equation (35)

$$rank[Q\ PQ\ P^2\ Q\ \cdots\ P^{2n-1}\ Q] = 2n; \tag{36}$$

control

$$u = K_1 x + K_2 x_1; \tag{37}$$

can be determined by solving the linear-quadratic problem for system Equation (35) using Krasovskii's method [39]. The minimum dimension r of the control vector sufficient to satisfy condition Equation (36) is determined [2] by the number of the non-trivial invariant polynomials matrix *P*.

The simplification study due to the model dimension reduction creates the possibility for including the actuators dynamics in the consideration. For example, if the implementation of mechanical control actions is ensured by electric drives with collector motors with independent excitation, a linear approximation of transient processes in the motors anchor windings should be added to the model [40].

$$I_\rho \frac{di_\rho}{dt} + R_\rho i_\rho + k_{1(\rho)} \dot{x}_{1(\rho)} = e_\rho;\ \rho = \overline{1, r}; \tag{38}$$

The meaning of the designations in Equation (38): I_ρ is the inductance of the armature winding, $R_\rho$is its ohmic resistance, $k_{1(\rho)}$ is the coefficient of back-EMF, and e_ρ is voltage on the armature winding, additional to the voltage $e_{\rho 0}$, that ensures the flow $i_{\rho 0}$ of current in the armature winding, creating a moment that realizes equilibrium (28). The most detailed presentation of the complete solution to the stabilization problems by moments created by DC electric drives with collector motors is given in [16–18,36].

7. Result

The abstract theoretical results on analytical mechanics of non-free systems obtained in the Soviet Union and Russia apparently remain unknown to specialists in technical practice. Despite the fact that the main works have been translated and the author has published numerous publications on their application, they remain unclaimed. This article once again describes in detail how effective the application of analytical mechanics methods of non-free systems is to modeling the behavior of manipulators with parallel kinematics.

The abstract theoretical results of the application of the non-free systems analytical mechanics to modeling the behavior of manipulators with parallel kinematics as systems with geometric constraints in the general case frees one from considering the inverse problem of kinematics.

Due to the exclusion of constraint multipliers and velocities of coordinates dependent on constraints from consideration, the dimensionality of the model, compared to traditional models in the form of equations with constraint multipliers (expressions of which in the coordinates and velocities functions form require additional definition), is reduced by the number of constraints. The developed transition to constraint multipliers-free equations in redundant coordinates allows, in contrast to all known results, in the general case to include the dynamics of actuators in consideration.

8. Discussion

Numerous articles on modeling the dynamics of systems with geometric constraints develop a method whose mandatory stage is the consideration of the inverse kinematics problem. It is obvious that it is impossible to obtain a general solution to the inverse kinematics problem in an analytical form. Therefore, all publications model the dynamics of only a specific device. Moreover, the inverse problem is solved using software environments for processing symbolic information.

Our approach is based on the transition to equations in redundant coordinates free of constraint multipliers. Such a procedure not only reduces the model dimensionality by excluding dependent velocities from consideration, but, most importantly, does not require solving the inverse kinematics problem in principle. Equations (14) and (15) generally represent the analytical form for a complete nonlinear dynamics model of an arbitrary system with geometric constraints (1).

The explicit analytical form of the model (cf. the simulation methods [41–46] created the possibility for a complete solution to the general problem of stabilizing a given configuration of a non-free system with constraints (1) by applying the mathematical control theory results with the involvement of the critical cases theory. The proposed method's effectiveness is demonstrated by a complete solution to the stabilization problem at the specified grip position of the Delta robot. Unfortunately, our highly effective, strictly substantiated approach is not only not used by technical practitioners, but is not reflected in any way even in such review articles as [41,46] with a list of references of many dozens of articles.

The analytical model form creates the possibility not only for the rigorous selection of the first approximation in the perturbed motion equations, but also for conducting an analysis of the nonlinear term structure from the point of the critical cases theory view. On this basis, in the general stabilizing problem, a given manipulator configuration, the control problem dimension is additionally reduced by the number of dependent coordinates. At the same time, it is strictly proven that the control that solves the problem of stabilization to asymptotic stability for the unperturbed configuration in the complete nonlinear system is a linear function of only independent velocities and perturbations of the corresponding coordinates.

In the presence of cyclic coordinates, it is possible to further simplify the study by switching to Routh variables. The reduction in the dimensionality of the problem for determining the stabilizing control is achieved by excluding uncontrolled impulses from consideration [24,25].

9. Conclusion

The developed simplification of the mathematical model makes it possible to include a description of the dynamics of the actuators. If we take an electric drive with collector motors with independent excitation, mechanical moments proportional to the currents in the anchor windings of the motors should be added to the model [16–18,36]. In this case, additional voltages on the anchor windings of the drive motors should be considered as stabilizing controls.

This approach has shown high efficiency in solving current stabilization problems for complex modern automatic devices with parallel kinematics. A full description of its application is presented in detail in several studies. The very first object with one geometric constraint was the stand Ball and Beam [16–18,32].

The solution algorithm developed there was then applied to the complete solution of the stabilization problem for the given position of the Delta robot gripper. This device was considered as a system with three geometric constraints [36] (without the need to consider the inverse problem of kinematics, cf. [5–9,37,38]). The rigor of the applied abstract-theoretical methods created not only the possibility of taking into account the dynamics of actuators in the nonlinear model, but also the possibility of obtaining a complete solution to the stabilization problem with the determination of the stabilizing control coefficients controlled by the method of Krasovskii [39].

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