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Article

Design of optimum spacecraft reorientation control with a combined criteria of quality based on the quaternions

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Abstract: Specifical original problem of attitude controlling for spacecraft was proposed in this paper. Problem of optimal rotation from a known initial state in a prescribed spatial orientation was studied in detail (turnaround time is not fixed). Design of optimal program of reorientation is based on new indicator of quality that combines energy costs including the contribution of controlling torques and integral of rotary energy (in a known proportion) and reorientation time; presence of duration factor bounds time of rotation finish. To construct an optimal control of angular momentum changing, quaternionic method and the maximum principle were applied. Differential equation that relates spacecraft angular momentum and quaternion of spacecraft orientation is a base to obtain analytic solution to a problem. We reveal the properties of optimum control program analytically, and study key features of optimum motion in details. Also, we write the formalized equations, mathematical formulas to design optimal law for change of spacecraft's angular momentum. Analytic relations and equations are given for finding the optimal solution. Control law (in as explicit dependence between phase variables and con-trolling variables has been formulated. Main relations determining optimum values of parameters for rotation control algorithm were given. The closed-form law for rotation was obtained for dynamically symmetric solids. Numerical example as well as results of mathematical modeling of spacecraft motion that formed using optimum control are presented. This data as addition to the made theoretic descriptions shows reorientation process (in virtual form) and demonstrates practical feasibility of the developed control method. A designed algorithm for optimal control of rotation improves an efficiency of attitude system, and originates more economical performing of space vehicle during its flight along orbit.

Keywords: quaternion; spatial attitude; optimal control; angular momentum; maximum principle; combined criterion of quality; boundary value problem

1. Introduction

Spacecraft transfer problem from its initial spatial position into a given angular position has been solved. Spatial reorientation consists in moving of spacecraft axis from the known attitude in another given position within finite time *T*. Spacecraft attitude is specified with respect to the chosen frame of reference. A case when inertial frame was reference basis has been considered. Use of new quality indicator (without any constraint for the control variables) is principial difference in solution proposed.

Problem of controlling the angular position of a solid is known very well [1–28]. It was considered in various formulations and studied by different methods in wide range, including analytic design of optimum controllers [1], or basing on inverse dynamics problem [2] and others, as well as issues of optimum control [1,3–

26]. Methods for optimizations are very different, for example, optimization based on the maximum principle [9–26], where classical criteria for estimate control process (a speed-optimal [4–12], energy costs [11,13,14,19], minimum fuel consumption [14], etc.) have been used, or kinematic problems on turning [15–18]. Optimum control problem in dynamic statements is of special interest; in particular case (when process end time is fixed) method of variables separation was applied [14]. Analytic solution of optimum turn controlling in a closed form remain practically important because this solution allow a finished law of program control to be applied onboard (as well as variation of optimal rotation trajectory of spacecraft). But extremely difficult to solve problem and find optimum law for solids when moments of inertia are arbitrary. Consequently, approximate numeric methods can be used only for solving the problem of a turn. Some special cases of control were known in case of spherically symmetric [12,19] or solids with dynamical symmetry [8–11,16,20–22]. Specifical features of control for the spacecrafts equipped with inertial actuators (for example, the gyrodins) were investigated earlier [29–31]. Attitude system based on control using the gyrodins (or other control moment gyroscopes) can use the patented method [31].

Personal contribution of authors of papers published earlier is: Sinitsin and Kramlikh [1] advanced the method of analytical construction of optimal regulators for problems of spacecraft attitude. Velishchanskii and Krishchenko [2] successfully adapted the method of inverse problems of dynamics well-known in the theory of dynamic systems to the problems of angular motion control. Branets and Shmyglevskii [12] have developed theory of quaternion application to is-sues of description of solid rotations kinematics and control of a spacecraft (in other domains of sciences, quaternionic approach was used also). Zelepukina et al. [16] investigated in detail the construction of optimal laws of variation in the angular momentum vector during spatial turn (including for axisymmetric rigid space-craft). Huge range of problems of optimal control with application of maximum principle basing on the quaternions were solved by Molodenkov and Sapunkov [10] in different years (approximate solutions or numerical solutions are given); analytical solutions were obtained for axisymmetric spacecraft and spherically symmetric spacecrafts, moreover, different criteria of quality was used. Lastly, Aipanov and Zhakypov [14] proposed the method of separation of variables for solving the problems of optimal turn control.

Below, we will solve a problem of optimum spacecraft rotation with new indicator of quality that combines (in an assigned proportion) the contribution of control for performing a maneuver (in energy consumption sense) and an integral of rotary energy; a presence of such integral leads to limitation of energy of rotation during a turn). Phase variables are orientation quaternion and spacecraft angular momentum. This problem statement differs significantly from earlier research with the complex functional by type of optimality index which includes both control variables and phase variables, as well as time of maneuver [21–26]. In our paper, the adopted functional characterizes complex combination of the energy costs and time, necessary for a turning. Minimization of such functional is important issue for spaceflight practice. Currently, cost-efficiency control remains very relevant for space engineering. Defining and studying an optimum rotation control from one attitude to other attitude (with minimum of a selected optimality index) is topic and subject of research.

2. Statement of the optimal control problem

Let us describe spatial motion about center mass by mathematic apparatus of quaternions (Rodrigues-Hamilton parameters). A motion of body frame E relative to reference frame is given by normalized quaternion Λ (body frame E, or spacecraft coordinate system, is formed by the principal central axis of inertia), $||\Lambda||=1$ [12]. We assume that reference frame is inertial. Initial and final spacecraft attitude in inertial frame we specify by quaternions Λ_{in} and Λ_f. Kinematic equation has the form [12]:

$$2\dot{\Lambda} = \Lambda \circ (I^{-1}L) \quad (1)$$

where symbol $\circ$ is mark for quaternion multiplication. Spacecraft angular momentum (as rigid body) changes according to the equation [24,26]:

$$\dot{L} + (I^{-1}L) \times L = M \quad (2)$$

where L is spacecraft angular momentum, I is inertia tensor, M is control torque (symbol $\times$ means a vector product of vectors). Disturbance torques (atmosphere and perturbation caused by external field) are negligibly small, it is very important for spacecraft control during spaceflight. The spacecraft is controlled around center of mass by vector M. The boundary conditions for controlled system Equations (1) and (2) are:

$$\Lambda(0) = \Lambda_{\text{in}}, L(0) = 0 \quad (3)$$

$$\Lambda(T) = \Lambda_f, L(T) = 0 \quad (4)$$

where T is rotation end time; Λ_{in}, and Λ_f specify attitude of a spacecraft at initial and final instant (they must satisfy a requirement $||\Lambda_{in}|| = ||\Lambda_f|| = 1$). It is set that rotary motion is regulated using control system which creates torques about three main central axis of inertia. For optimization of control program, its efficiency is specified by indicator:

$$G = \int_0^T (M_1^2/J_1 + M_2^2/J_2 + M_3^2/J_3)dt + k_1 \int_0^T (L_1^2/J_1 + L_2^2/J_2 + L_3^2/J_3)dt + k_2 \mathrm{T} \quad (5)$$

where M_i are projections of control torque M on principal central axis of spacecraft's inertia ($i = 1, 2, 3$); L_i are projections of angular momentum L on axis of spacecraft frame; J_i are principal central moments of inertia; $k_1 > 0$, $k_2 > 0$ are the positive constant coefficients ($k_1 \neq 0$, $k_2 \neq 0$).

We formulate optimal control problem as following: spacecraft needs to be turned from Equation (3) into Equation (4) according to Equations (1) and (2) and minimum sum Equation (5), where $\Lambda_f \neq \pm\Lambda_{in}$. Duration of turn T is unfixed. Solution-function M(t) is found within piecewise-continuous functions of time.

Accepted criteria of optimality Equation (5) distinguishes our problem of optimization from other considered tasks by functional type that must be minimized. Index Equation (5) combines energy costs (including the contributions of control) and time (in its proportion). An adopted criteria allows to estimate an energetically advantageous trajectory from initial orientation Λ_{in} to a required termination attitude Λ_f with minimum costs of the controlling resources and energy, and for determination of control mode corresponding to optimum turn. Energy integral in (2.5) limits kinetic energy during rotation. The required turn may be executed for any values Λ_{in}, Λ_{f} and J_1, J_2, J_3, k_1 и k_2 because the time T is not fixed. Since combination of quadratic performance criterion and time is optimized, optimal duration of a turn T_{opt} exists to be minimum of Equation (5). The optimal rotation problem, when control process quality is estimated by the index Equation (5), is relevant.

Note that angular momentum control for spatial turn minimizing an indicator Equation (5) is useful for spacecraft's attitude system using electric-jet engines (EJE): when controlled using EJE (in particular, with use of ion-engines), in Equation (5) first integral is proportional to electric energy consumed (for EJE, thrust and electric current are directly-proportional [32], and torque of EJE is proportional to an arm of engine installation). We know about a necessity of all-round decrease of electrical consumption for control of motion, a select of functional is clear.

3. Materials and methods

Optimal control problem proposed is dynamical problem of optimum turn [12]; consequently, M_i are the functions that should be found (control by optimal way of spatial motion), but L and Λ are the phase variables. A norm of quaternion Λ is constant, $\|\Lambda\|$ = const [12]. We solve optimum control issue and find optimum law of spacecraft's angular momentum changing during spatial rotation basing on the method of quaternions and Pontryagin's maximum principle [12,33]. They were used by many scientists and researchers, but for other index form [1,12,13–17,21–28]. Quaternions is mathematical tool that were effective applied in another domains of sciences [34–36], also in research process on controled motion of solids. Also, we apply numerical simulation (mathematical modeling) (for proof feasibility in engineering practice). To solve differential equations systems, we made method of successive approximations (shooting algorithm). Method of iteration was applied for solving two-point boundary value problem; other numerical methods have been used also.

The procedure of solving the optimization problem proposed and formulated in section 2 is as follows. In accordance with the Pontryagin's maximum principle [33], conjugate variables ϕ_i corresponding to angular momentum projections on spacecraft axis L_i are introduced. The Hamiltonian H for the problem Equations (1)–(5) is:

$$\begin{aligned} H = -k_2 - k_1(L_1^2/J_1 + L_2^2/J_2 + L_3^2/J_3) - M_1^2/J_1 - M_2^2/J_2 - M_3^2/J_3 + \phi_1(M_1 \\ + (1/J_3 - 1/J_2)L_2L_3) + \phi_2(M_2 + (1/J_1 - 1/J_3)L_1L_3) + \phi_3(M_3 \\ + (1/J_2 - 1/J_1)L_1L_2) + L_1r_1/J_1 + L_2r_2/J_2 + L_3r_3/J_3 \end{aligned}$$

where r_i are [19,24]:

$$r_1 = (\lambda_0\psi_1 + \lambda_3\psi_2 - \lambda_1\psi_0 - \lambda_2\psi_3)/2,$$
$$r_2 = (\lambda_0\psi_2 + \lambda_1\psi_3 - \lambda_2\psi_0 - \lambda_3\psi_1)/2,$$
$$r_3 = (\lambda_0\psi_3 + \lambda_2\psi_1 - \lambda_3\psi_0 - \lambda_1\psi_2)/2$$

ψ_j are conjugate variables, corresponding to components of quaternion λ_j (j = 0, 1, 2, 3).

Writing and structure of H does not take into account phase constraint $\|\Lambda\|=1$ due to $\|\Lambda(0)\| = 1$. We use universal variables r_i proposed earlier [12,19] because the minimized index (2.5) does not depend on quaternion Λ. The equations for vector **r** formed by variables r_i (optimal functions r_i also) are [24–26]:

$$\dot{r}_1 = L_3 r_2/J_3 - L_2 r_3/J_2,$$
$$\dot{r}_2 = L_1 r_3/J_1 - L_3 r_1/J_3,$$
$$\dot{r}_3 = L_2 r_1/J_2 - L_1 r_2/J_1,$$

$$\dot{r} = r \times (I^{-1}L) \qquad (6)$$

It is known that r turns out to be motionless in inertial frame, $|r|$ = const ≠ 0 [19].

In accordance with the maximum principle, for functions ϕ_i, the equations are [33]:

$$\dot{\phi}_i = -\frac{\partial H}{\partial L_i} \ (i = 1, 2, 3)$$

Conjugate equations system (after differentiation of H) has the form

$$\dot{\phi}_1 = 2k_1 L_1/J_1 + L_3\phi_2(1/J_3 - 1/J_1) + L_2\phi_3(1/J_1 - 1/J_2) - r_1/J_1$$
$$\dot{\phi}_2 = 2k_1 L_2/J_2 + L_1\phi_3(1/J_1 - 1/J_2) + L_3\phi_1(1/J_2 - 1/J_3) - r_2/J_2 \qquad (7)$$
$$\dot{\phi}_3 = 2k_1 L_3/J_3 + L_2\phi_1(1/J_2 - 1/J_3) + L_1\phi_2(1/J_3 - 1/J_1) - r_3/J_3$$

Searching an optimum mode is in writing and solving the equations of motion Equations (1) and (2) and the Equations (6) and (7) under condition that a found control itself is selected by maximization of Hamiltonian. Equation (6) that defines behavior of r in body frame E, will be used, replacing the conjugate equations for ψ_j. Optimum function $r(t)$ and quaternion $\Lambda(t)$ are related [19,24–28]

$$r = \tilde{\Lambda} \circ c_E \circ \Lambda,$$

where $\boldsymbol{c}_E$= const=$\Lambda_{\text{in}} \circ$ r(0) $\circ \tilde{\Lambda}_{\text{in}}$
where $\tilde{\Lambda}$ is quaternion conjugate to Λ [12, p. 10–20]; and $r(0) \neq 0$ (in another case $r_1 = r_2 = r_3 = 0$ and solution of problem loses sense).

Vector direction c_E is determined by initial and terminal spacecraft attitude: c_E must be defined by solution of Equation (1) and $\Lambda(T) = \Lambda_f$. Differential Equations (6) and (7), together with requirement of maximum for Hamilton function H, are necessary optimality conditions. Conditions of maximum for H determine control function $M(t)$; also, $\Lambda(0)$, $L(0)$, $\Lambda(T)$, $L(T)$ determine solutions $\Lambda(t)$ and $r(t)$, $L(t)$ [24].

A boundary-value problem of maximum principle is in calculating such $r(0)$ for which solving of differential Equations (1) and (2), Equations (6) and (7) together with Hamiltonian maximization, at any time, satisfies the conditions Equations (3) and (4). The restriction $\|\Lambda(t)\|=1$ for phase variable Λ not taken into account because it holds for Equation (1).

4. Principle scheme and algorithm of solving control problem

To specify controlling function M(t) and vector r, maximality conditions for H should be formalized. If $u_i = M_i/\sqrt{J_i}$ and $n_i = \phi_i\sqrt{J_i}$, writing of H gives, using variables u_i and n_i

$$H = u \cdot n - |u|^2 + H_{inv} = |n||u|\cos\delta - |u|^2 + H_{inv}$$

where H_{inv} not explicitly depends on the functions M_i; n and u are two vectors of u_i and n_i; angle between u and n is δ (sign "·" means scalar product of vectors). The Hamiltonian H is quadratic function of functions M_i, H is maximum if only $\delta = 0$. Necessary conditions of extremum are $\partial H/\partial M_i = 0$. After applying them:

$$M_i = J_i\varphi_i/2 \tag{8}$$

Problem of determination of optimum control is in a solving of Equations (1) and (2), and Equations (6) and (7) under requirement that a sought control M is generated by Equation (8). The problem of optimum control Equations (1)–(5) formulated above is completely solved. Since $|r| = \text{const} = |r(0)| \neq 0$, transfer to a normalized vector p: $p = r/|r|$, and $|r(0)| = r_0$ ($|p| = 1$). For p, or elements p_i:

$$\begin{aligned} \dot{p} &= p \times (I^{-1}L) \\ \dot{p}_1 &= L_3p_2/J_3 - L_2p_3/J_2 \\ \dot{p}_2 &= L_1p_3/J_1 - L_3p_1/J_3 \\ \dot{p}_3 &= L_2p_1/J_2 - L_1p_2/J_1 \end{aligned} \tag{9}$$

Note: $r_i = |r(0)|p_i$. Since $L(0) = L(T) = 0$, we see that our issue Equations (1)–(5) has single and unique solving:

$$\phi_i = a(t)p_i/J_i \tag{10}$$

$$L_i = b(t)p_i \tag{11}$$

where $p_i = r_i/r_0$; $a(t)$ and $b(t)$ are scalar time functions (but $b(t) \geq 0$). Respectively, $M_i = a(t)p_i/2$.

Successive substitution of dependences Equation (10) into Equation (7), by taking into account relations Equation (11) with $r_i = r_0\,p_i$, proofs a validity of solution which we specified. i.e., Equations (10) and (11) for the differential equations system Equations (2) and (7), Equations (8) and (9) is truly (relation Equation (11) follows directly from the system Equations (2), (8) and (10). From Equations (7), (10) and (11), we see that optimal functions satisfy the relation.

$$\dot{a}(t) = 2k_1b - r_0 \tag{12}$$

The Equations (7), (8) and (11) give dependence $\dot{b} = a/2$ for $b(t)$, or

$$b(t) = 0.5\int_0^t a(t)dt$$

Taking into account last dependence and condition Equation (12), we obtain $\ddot{a} = k_1 a$ for $a(t)$, which has analytical form:

$$a(t) = C_1 exp(-t\sqrt{k_1}) + C_2 exp(t\sqrt{k_1}) \quad (13)$$

where C_1 and C_2 are some constants that will be determined below.

It is seen, $b(0) = b(T) = 0$ and $\dot{a}(0) = \dot{a}(T) = -r_0$, since $L(0)=0$, $L(T)=0$. As consequence, $r_0 = \sqrt{k_1}(C_1 - C_2)$. Therefore, for optimal rotation, $b(t)$ is:

$$b(t) = (\dot{a}(t) + r_0)/(2k_1) = [C_2 exp(t\sqrt{k_1}) - C_1 exp(-t\sqrt{k_1}) + C_1 - C_2]/(2\sqrt{k_1}) \quad (14)$$

Hamilton function H not explicitly depends on the time explicitly, and time of rotation finish T is not fixed. Whence, for optimal control H = const = 0 [37]. At ends of trajectory L = L = 0, hence:

$$H(0) = H(T) = -K_2 - \frac{J_1\phi_1^2 + J_2\phi_2^2 + J_3\phi_3^2}{4} + \frac{J_1\phi_1^2 + J_2\phi_2^2 + J_3\phi_3^2}{2} = (J_1\phi_1^2 + J_2\phi_2^2 + J_3\phi_3^2)/4 - k_2$$

or $H(0) = H(T) = M_1^2/J_1 + M_2^2/J_2 + M_3^2/J_3 - k_2 = 0$. Whence $a2(0) = a2(T) = 4k_2/(p_1^2/J_1 + p_2^2/J_2 + p_3^2/J_3)$.

Optimal rotation has specific property: $p_1^2/J_1 + p_2^2/J_2 + p_3^2/J_3 = \text{const}$ for p_i (or ort p). To confirm this feature, left side was differentiated (with accounting for Equations (9) and (11)). After substitution $\dot{p}_i$ by Equation (9) and then L_i by the formulas Equation (11), we show that the resulting derivative is zero. For optimum function $a(t)$, must satisfy the requirements:

$$a(0) = 2\sqrt{k_2}/C \; a(T) = -2\sqrt{k_2}/C \; a(T/2) = 0, \dot{a}(0) = \dot{a}(T) = -r_0 (C = \sqrt{p_{10}^2/J_1 + p_{20}^2/J_2 + p_{30}^2/J_3})$$

Last equality follows from Equation (12), since $b(0) = b(T) = 0$; conditions $L(0) = 0$ and $L(T) = 0$ for optimal turn. The property $a(T) = -a(0)$ follows from the necessary condition of optimality $H(0) = H(T) = 0$ (and Equations (8) and (10)). We can write (on the base Equation (13)):

$$\dot{a}(t) = \sqrt{k_1}(C_2 exp(t\sqrt{k_1}) - C_1 exp(-t\sqrt{k_1}))$$

$$\dot{a}(0) = \sqrt{k_1}(C_2 - C_1) = \dot{a}(T) = \sqrt{k_1}(C_2 exp(T\sqrt{k_1}) - C_1 exp(-T\sqrt{k_1}))$$

Comparation of $\dot{a}(0)$ and $\dot{a}(T)$ gives the equations $C_2(exp(T\sqrt{k_1}) - 1) = C_1(exp(-T\sqrt{k_1}) - 1)$, or $C_1 = -C_2 exp(T\sqrt{k_1})$. From Equation (14), we proof $b(0) = 0$, $b(T) = 0$. Also $a(T) = -a(0)$ and $a(T/2) = 0$, since we have (basing on Equation (13)): $a(0) = C_1 + C_2$, $a(T) = C_1 exp(-T\sqrt{k_1}) + C_2 exp(T\sqrt{k_1}) = -C2 - C1$,

$$a(T/2) = C_2(exp(T\sqrt{k_1}/2) - exp(T\sqrt{k_1}) exp(-T\sqrt{k_1}/2))$$

But $r0 = \sqrt{k_1}(C_1 - C_2) = C_1\sqrt{k_1}(exp(-T\sqrt{k_1}) + 1) > 0$, whence $C_1 > 0$, $C2 < 0$ (and $|C1| > |C2|$).

Structure analysis of optimal $a(t)$ shows $\dot{a} < 0$ in every instant $t \in [0, T]$, $\dot{a}$ is minimum at right and left bound points of trajectory: $\dot{a}(0) = \dot{a}(T) = -r_0$. Note see, that $\dot{a}(0) < 0, \dot{a}(T) < 0$, and $C1 > 0$, but $C2 < 0$, therefore $a(0)>0$, but $a(T)<0$. If $t<T/2$, then $a>0$ and $\ddot{a}> 0$, but $\dot{a} < 0$; within interval $t > T/2$ we have $a < 0$ and $\ddot{a} < 0$, but $\dot{a} < 0$. At instant $t = T/2$ we see that $\dot{a}$ is maximum (since $a(T/2) = 0$ and b has maximal value). Also, $\dot{a}(T/2) = -2C_1\sqrt{k_1}exp(-T\sqrt{k_1}/2) < 0$. Hence, $\dot{a} < 0$ during time period $t \in [0, T]$, time when $\dot{a}(t) = 0$ is absent. Concrete values of $C1$, $C2$ depend on the coefficients k_1, k_2 and integral.

$$Q = \int_0^T |L(t)| dt = \int_0^T b(t) dt \tag{15}$$

Its value not depends on behavior of function $b(t)$ (for motions according to Equations (9) and (11)); it is defined solely by attitude Λ_{in}, Λ_f, and spacecraft's moments of inertia [18] (Q is computed together with p_0).

Maximum rotary energy Ek and maximum angular momentum achieved at $t = T/2$; $E_{\max} = E_k(T/2)$; $L_{\max} = |\mathbf{L}(T/2)|$. If $k_1 = 0$, $a(t)$ is linear time function $a(t) = 2\sqrt{k_2}(1-2t/T)/C$

(modulus of spacecraft angular momentum is quadratic function of time, respectively), duration of reorientation is $T = \sqrt{6CQ/\sqrt{k_2}}$. Hence, $L\sqrt{3Q\sqrt{k_2}/(8C)}_{max}$ and $E\sqrt{k_2}_{max}$.

Figure 1 shows the form of optimum $b(t)$, $a(t)$ (depending of coefficient k_1 of optimized index (2.5) ($k_1^{(2)} > k_1^{(1)} > 0$, $k_1^{(3)} > k_1^{(2)}$); the dotted lines correspond to a case $k_1 \to 0$. The values $a(0)$ and $a(T)$ are fixed (they do not change if variate value k_1), $a(0) = 2\sqrt{k_2}/C$, $a(T) = -2\sqrt{k_2}/C$, $a(T/2) = 0$.

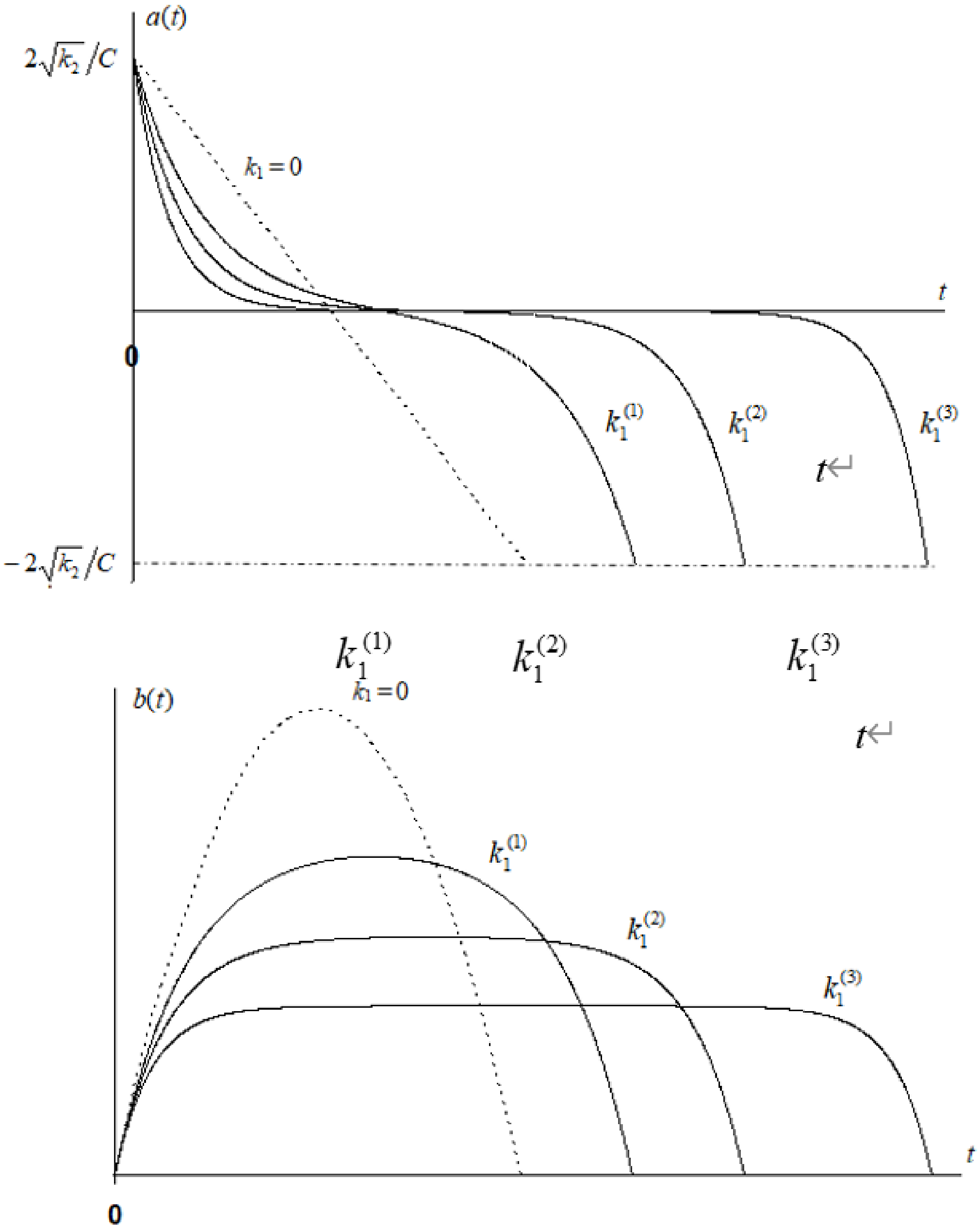

Figure 1. Optimum of $a(t)$ and $b(t)$.

If k_1 more, then function $b(t)$ more fast transfer to segment with $\dot{b} \approx 0$. When $b_{\max}$ decreases, a finish time of rotation maneuver increases, since the value (4.8) is invariable (it does not depend on the coefficients k_1, k_2). Thus, the function $b(t)$ more remotes from quadratic function (parabolic form) and is approximated to piecewise linear dependence when k_1 increases, it can approximate by function included the following segments: time interval with $\dot{b} \approx \sqrt{k_2}/C$, further $b \approx$ const, and then motion with $\dot{b} \approx -\sqrt{k_2}/C$. Under unbounded increasing of coefficients k_1 and k_2, maximal energy of rotation $E_{\max}$ approximates to the level $E_0 = k_2/(2k_1)$.

In optimal solution (when $k_1 \neq 0$), constants C_1, C_2 and time T are calculated by the equations:

$$(exp(T\sqrt{k_1})+1)T=(QC\sqrt{k_1/k_2}+2/\sqrt{k_1})(exp(T\sqrt{k_1})-1) \tag{16}$$

$$C_1=2\sqrt{k_2}/(C(1-exp(-T\sqrt{k_1}))),C_2=2\sqrt{k_2}/C-C_1$$

Time of optimal turn T decreases with increasing of coefficient k_2 (if k_1 is invariable).

Boundary-value problem is to compute p_0 (or, p_{10}, p_{20}, p_{30}), and positive constant r_0, that lead to satisfaction of the terminal condition (2.4) for solution of equations system Equations (1) and (2), Equations (7)–(9) with initial condition (2.3) and dependence $r_i = r_0 p_i$ ($r_0 > 0$).

If take into account Equation (8), Equations (10) and (11) and optimal values of constants C_1 and C_2 for functions a, b, optimum control and optimum motion is described by expressions:

$$M_i=C_1[exp(-t\sqrt{k_1})-exp((t-T)\sqrt{k_1})]\,p_i/2 \tag{17}$$

$$L_i=C_1[1+exp(-T\sqrt{k_1})-exp(-t\sqrt{k_1})-exp((t-T)\sqrt{k_1})]p_i/(2\sqrt{k_1}) \tag{18}$$

p_i satisfy Equation (9) and $p=\tilde{\Lambda}\circ\Lambda_{in}\circ p_0\circ\tilde{\Lambda}_{in}\circ\Lambda$, $C_1=2\sqrt{k_2}/(C(1-exp(-T\sqrt{k_1})))$. Time T is calculated from Equation (16).

Optimum orientation program is determined by the system Equations (5) and (6), Equations (17) and (18); controlling functions M_i and angular momentum projections L_i variate according to Equations (17) and (18). The value p_0 as well as the integral Q, characterizing complexity of a turn, are computed in solution process of boundary-value problem of a turn. Program value of control M relates with attitude Λ

$$M=a(t)\tilde{\Lambda}\circ\Lambda_{in}\circ p_0\circ\tilde{\Lambda}_{in}\circ\Lambda/2$$

at which $a(t)$ changes by the law Equation (13).

Under optimal angular momentum $L(t)$, spacecraft rotation has symmetry properties (for $b(t)$, $a(t)$, primarily) as well as regularities:

$$a(0)=-a(T)>0,\ a(T-t)=-a(t),$$

$$b(t)\geq 0,\quad b(T-t)=b(t)$$

$$\int_0^{T/2}b(t),dt=\int_{T/2}^{T}b(t)dt,\int_0^{T/2}|a(t)|,dt=\int_{T/2}^{T}|a(t)|dt$$

$$\Lambda\circ M(T-t)\circ\tilde{\Lambda}=-\Lambda\circ M(t)\circ\tilde{\Lambda},\ \Lambda\circ L(T-t)\circ\tilde{\Lambda}=\Lambda\circ L(t)\circ\tilde{\Lambda}$$

$$\max_{0\leq t\leq T}|M(t)|=\sqrt{k_2}/C,L_{max}=\max_{0<t<T}\sqrt{L_1^2+L_2^2+L_3^2}=|L(T/2)|$$

$$L_{max}=\frac{r_0+\sqrt{r_0^2-\dfrac{4k_1k_2}{C^2}}}{(2k_1)\text{или}L_{max}}=(\sqrt{(C_1-C_2)^2/4-k_2/C^2}+(C_1-C_2)/2)/\sqrt{k_1}$$

Angular momentum and optimum control M satisfy the following relationships:

$$\Lambda\circ M(T-t)\circ\tilde{\Lambda}=-\Lambda\circ M(t)\circ\tilde{\Lambda},\ \Lambda\circ L(T-t)\circ\tilde{\Lambda}=\Lambda\circ L(t)\circ\tilde{\Lambda}$$

The conditions Equation (11) show: p is ort for angular momentum; optimum functions $\varphi_i(t)$, $p_i(t)$, $L_i(t)$ meet dependences Equations (10) and (11), where $p_i(t)$ satisfy the system (4.2). Optimal control is determined using (4.10); for any instant $t \in [0, T]$, M and L are collinear. At instant $t = T/2$, direction of torque M variates to an opposite, and modulus $|L|$ has maximal value $|L(T/2)| = L_{\max}$. Control Equation (17) is truly optimum, since this is single function satisfying to Equations (1) and (7), Equations (8) and (9). Optimal rotation (in sense of minimum index Equation (5)) occurs along "trajectory of free motion" [9]. Specify properties of optimum rotation follow from the system Equation (9) with equalities Equations (11) and (14). Original solution is determined by close equations system Equation (1), Equations (9) and (11), and conditions Equations (3) and (4) for function $\Lambda(t)$. Additionally, controlling torque M is smooth time function. Note: during rotation by inertia, body occupies positions Λ that form "trajectory of free motion" [9,24,26].

Ratio of angular momentum and kinetic energy E_k under optimum control is expression:

$$E_k = b^2\,(p_1^2/J_1 + p_2^2/J_2 + p_3^2/J_3)/2$$

Proportion is $E_k/|L|^2 = (p_1^2/J_1 + p_2^2/J_2 + p_3^2/J_3)/2 =$ const for any time within period $0 \le t \le T$. It is key property for spacecraft motion according to criterion Equation (5).

Problem of optimum control synthesis mainly consists in determination of such vector $p(0)$ under which the motion, according to the Equation (1), Equations (9) and (18) with condition Equation (3), satisfies the equality Equation (4). To solve this equations system is difficult in general case: its obtaining to compute $p(0)$, and $p(T)$ by Equations (9) and (11) and formula:

$$\Lambda_f \circ p(T) \circ \tilde{\Lambda}_f = \Lambda_{\text{in}} \circ p(0) \circ \tilde{\Lambda}_{\text{in}} \tag{19}$$

Significant difference of an offered result is applying of new optimality criterion which combines a contribution of control action (in sense of energy consumption) in motion of a spacecraft during a turn, integral of rotational energy, and maneuver time, with known ratio. Integral of the square form of controlling torques in Equation (5) provides limiting of controlling torques, and secondly, controlling variables are smooth (angular momentum is smooth time function, also). Introduction of time in a minimized indicator reduces duration of turn T. For any turn conditions Equations (3) and (4) (any Λ_{in}, Λ_{f}) and J_1, J_2, J_3, k_1, k_2, kinetic energy of rotation $E(t) \le k_2/(2k_1)$.

5. Particular cases of optimal turn

The determination of optimum rotation mode with minimum index Equation (5) is not trivial issue, $p(0)$ depends on respective attitude Λ_f and Λ_{in}, and characteristics J_1, J_2, J_3. It is non-easy to solve the problem of three-dimensional rotation for an arbitrary moments of inertia $J_1 \ne J_2 \ne J_3$ and attitude in initial and terminal time instants Λ_{in} and Λ_f. Difficulty is to find $p(T)$, $p(0)$ which satisfy Equation (19). Solution of system Equation (1), Equations (9) and (11) (in analytic form) is known for dynamically symmetric and spherical solid only. For body with spherically symmetry ($J_1 = J_2 = J_3$), function $p(t)$ is:

$$p_i(t) = const = p_{i\,0} = \nu_i / \sqrt{\nu_1^2 + \nu_2^2 + \nu_3^2}$$

$$M_j = C_1[exp(-t\sqrt{k_1}) - exp((t-T)\sqrt{k_1})]p_{i\,0}/2$$

$$L_i(t) = C_1\,[1 + exp(-T\sqrt{k_1}) - exp(-t\sqrt{k_1}) - exp((t-T)\sqrt{k_1})]p_{i0}/(2\sqrt{k_1}), i = \overline{1,3}$$

($\nu_0, \nu_1, \nu_2, \nu_3$ are the components of quaternion $\Lambda_t = \tilde{\Lambda}_{\text{in}} \circ \Lambda_f$); $Q = 2\,J_1$ accost ν_0.

During optimal reorientation, spherical spacecraft rotates about the Euler axis, optimum trajectory $\Lambda(t)$ (in analytic form) is

$\Lambda(t) = \Lambda_{in} \circ e^{\boldsymbol{p}_0 s(t)/(2J_1)}$, $s(t) = \int_0^t b(t)dt$

For dynamically symmetric solid body (for example, $J_2 = J_3$), we can solve optimum control problem completely also (in this case p_1 = const = p_{10}): we have conic motion of spacecraft [10] angular momentum modulus is proportional to velocity about longitudinal axis, and angle between longitudinal axis and angular momentum ϑ is constant. Optimal $p(t)$ is:

$$p_1 = p_{10} = \text{const} = cos\,\vartheta,$$

$$p_2 = p_{20}\,cos\,\beta + p_{30}\,sin\,\beta,$$

$$p_3 = -p_{20}\,sin\,\beta + p_{30}\,cos\,\beta,$$

$$\beta = \frac{J_3 - J_1}{J_1 J_2}\int_0^t L_1(t)dt \qquad (20)$$

$$\Lambda_f = \Lambda_{in} \circ e^{\boldsymbol{p}_0 \beta/2} \circ e^{\boldsymbol{e}_1 \alpha/2}$$

where e_1 is unit vector of longitudinal axis of spacecraft; α and β are angles of turns about longitudinal axis, and about p, accordingly ($|\alpha| \le \pi$, $0 \le \beta \le \pi$).

For this case, the relations Equations (17) and (18), Equation (20) are solution for system of Equations (2) and (9) under condition Equation (8); p generates the cone about axis of body's symmetry in body frame (spacecraft coordinate system). To transfer body from attitude Λ_{in} in attitude Λ_f, it rotates simultaneously about its own longitudinal axis, and about c_E, that is constant in inertial coordinate system. Axial-symmetric solid rotates along "conical trajectory". Note, p_0 (corresponding to optimum solution) we can calculate by device [38]. Optimal functions $M_i(t)$ are written in analytical form:

$$M_1 = C_1[exp(-t\sqrt{k_1}) - exp((t-T)\sqrt{k_1})]p_{10}/2$$

$$M_2 = C_1[exp(-t\sqrt{k_1}) - exp((t-T)\sqrt{k_1})]\sqrt{1 - p_{10}^2}\,sin(\kappa + \gamma)/2$$

$$M_3 = C_1[exp(-t\sqrt{k_1}) - exp((t-T)\sqrt{k_1})]\sqrt{1 - p_{10}^2}\,cos(\kappa + \gamma)/2$$

Programmed values L_i (the components of angular momentum L) are:

$$L_1 = C_1\,[1 + exp(-T\sqrt{k_1}) - exp(-t\sqrt{k_1}) - exp((t-T)\sqrt{k_1})]p_{10}/(2\sqrt{k_1})$$

$$L_2 = C_1\,[1 + exp(-T\sqrt{k_1}) - exp(-t\sqrt{k_1}) - exp((t-T)\sqrt{k_1})]\sqrt{1-p_{10}^2}\,sin(\kappa+\gamma)/(2\sqrt{k_1})$$

$$L_3 = C_1\,[1 + exp(-T\sqrt{k_1}) - exp(-t\sqrt{k_1}) - exp((t-T)\sqrt{k_1})]\sqrt{1-p_{10}^2}\,cos(\kappa+\gamma)/(2J\sqrt{k_1})$$

where $\gamma = arcsin\left(p_{20}/\sqrt{1-p_{10}^2}\right)$, if $p_{30} \geq 0$, or $\gamma = \pi - arcsin\left(p_{20}/\sqrt{1-p_{10}^2}\right)$,если$p_{30} < 0$ ($|p_{10}| \neq 1$); case $|p_{10}| = 1$ corresponds to planar rotation around axis *OX*, and it is not considered.

Optimal trajectory $\Lambda(t)$ is:

$$\Lambda(t) = \Lambda_{in} \circ e^{\boldsymbol{p}_0\sigma/2} \circ e^{\mu\boldsymbol{e}_1/2}$$

where:

$$\sigma = C_1\,[(exp(-T\sqrt{k_1}) + 1)t/(2\sqrt{k_1}) + (exp(-t\sqrt{k_1}) + exp(-T\sqrt{k_1}) - exp((t-T)\sqrt{k_1}) - 1)/(2k_1)]/J_2$$

$\mu = p_{10}\sigma(J_2 - J_1)/J_1$ (for optimal p_0 and $t = T$, conditions Equations (3) and (4) are satisfied).

For asymmetric body ($J_1 \neq J_2 \neq J_3$), the system Equation (1), Equations (9) and (11) can be solved using only numeric methods (for example, method of successive approximations, or iterations methods). One of such methods has been described in detail in the article [9]. We know that the solution $p(0)$ which satisfies the conditions $\Lambda(0) = \Lambda_{in}$, $\Lambda(T) = \Lambda_f$ and (4.12) for the system of Equation (1), Equations (9) and (11) does not depend on a type of changing the magnitude of angular momentum [18] (therefore, we take b = const $\neq 0$ in Equation (11) for search of $p(0)$). To compute the vector $p(0)$, we must solve the boundary problem $\Lambda(0) = \Lambda_{in}$, $\Lambda(T) = \Lambda_f$, taking into account the Equations (1) and (2) in which $M_i = 0$. As a result, vector of angular momentum at initial instant of time L_{cal}, for which a spacecraft rotates with its free motion ($M = 0$) from the state $\Lambda(0) = \Lambda_{in}$, $L(0) = L_{\text{cal}}$ to the state $\Lambda(T) = \Lambda_f$, will be found. The vector $p_0 = p(0)$ relates to L_{cal} as follows:

$$p_{i\,0} = \frac{L_{i\ \text{cal}}}{\sqrt{L_{1\text{cal}}^2 + L_{2\text{cal}}^2 + L_{3\text{cal}}^2}}$$

Other schemes of computing can be successful for some specifical or particular cases [39–41].

6. Example and results of mathematical modeling

Now, we demonstrate numeric example of optimal turn with minimum of Equation (5). Let us take spatial reorientation for 180 degrees from state Λ_{in}, corresponding to spatial position when spacecraft axis are parallel to the axis of inertial frame *I*, in an assigned attitude Λ_f, if angular rate is absent at instants $t = 0$, and $t=T$ (i.e., $L(0) = L(T) = 0$). The quaternion Λ_f has the following components:

$\lambda_0 = 0$, $\lambda_1 = 0.707107$, $\lambda_2 = 0.5$, $\lambda_3 = 0.5$

It is assumed that spacecraft has characteristics:

$J_1 = 63559$ kg·m^2, $J_2 = 192218.5$ kg·m^2, $J_3 = 176809$ kg·m^2

Further, after constructing the control program for optimum angular momentum variation during spacecraft rotation from state Equation (3) in the required state Equation (4), result of mathematic simulation of optimum turn are presented for the following coefficients: k_1 = 0.002 s^{-2} and k_2 = 0.04 J/s^2.

Decision of boundary-value problem of the turn from attitude $\Lambda(0) = \Lambda_{in}$ to orientation $\Lambda(T) = \Lambda_f$ give p_0 = {0.49535062; –0.11725655; 0.86074309} calculated. The method of iterations has been used, ensuring process of successive-approximation to true p_0 [9] (the method ensures asymptotic approaching in absolute most of cases). Maximum of control torque modulus is $|M(0)|$ = 70.2 N m, and Q = 401.63 kN m s^2.

7. Result and discussion

Mathematical calculations results and numerical simulation of optimum turn process are shown on **Figures 2–5** (in graphical form). **Figure 2** presents a variating of angular momentum as time function ($L_1(t)$, $L_2(t)$, $L_3(t)$ as projections on the body frame axes). Behavior of spacecraft attitude $\Lambda(t)$ during optimum maneuver is illustrated by **Figure 3** (functions λ_0, λ_1, λ_2, λ_3). Dynamics of p_1, p_2, p_3 for p is reflected on **Figure 4**. Finally, **Figure 5** give behavior of angular momentum modulus. Duration of optimal maneuver is T = 271.2 s. The rotational energy during a turn does not exceed E_{max} = 9.9 joules, spacecraft angular momentum has maximum magnitude L_{max} = 1562 N m s at time t = 135.6 s. It is seen that p_1 changes lot less than p_2, p_3. For optimum control, all p_i, L_i, λ_j are smooth time functions.

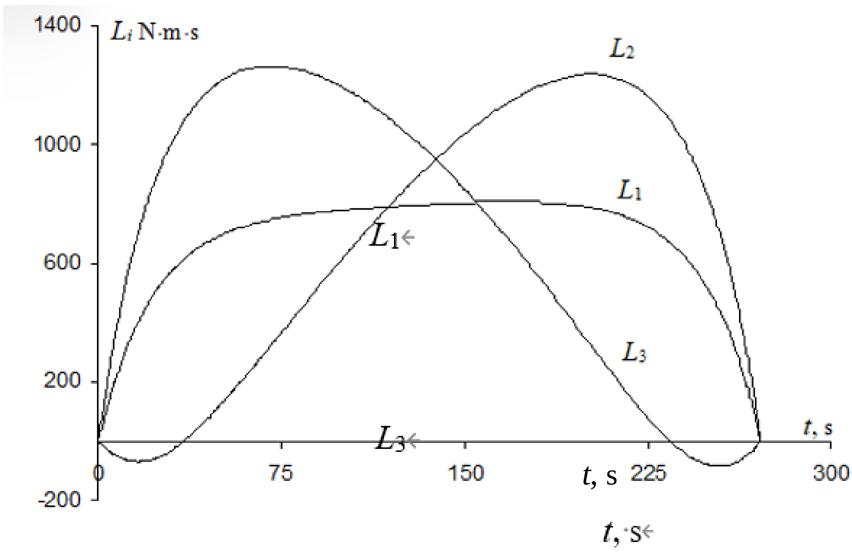

Figure 2. Angular momentum components during optimum turn.

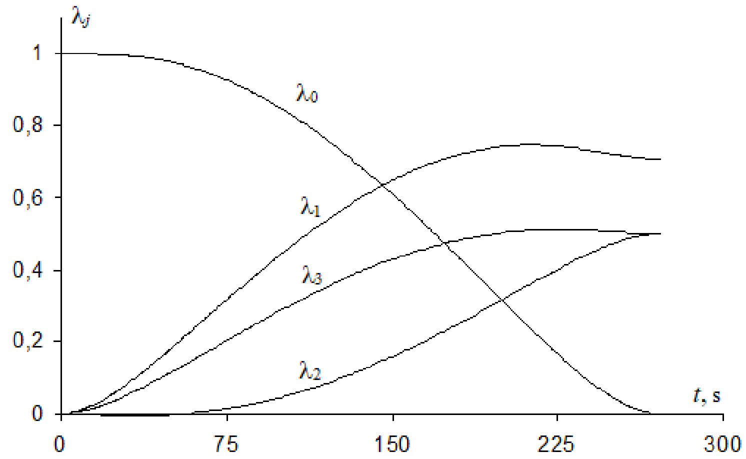

Figure 3. Quaternion (*t*) elements during optimum maneuver.

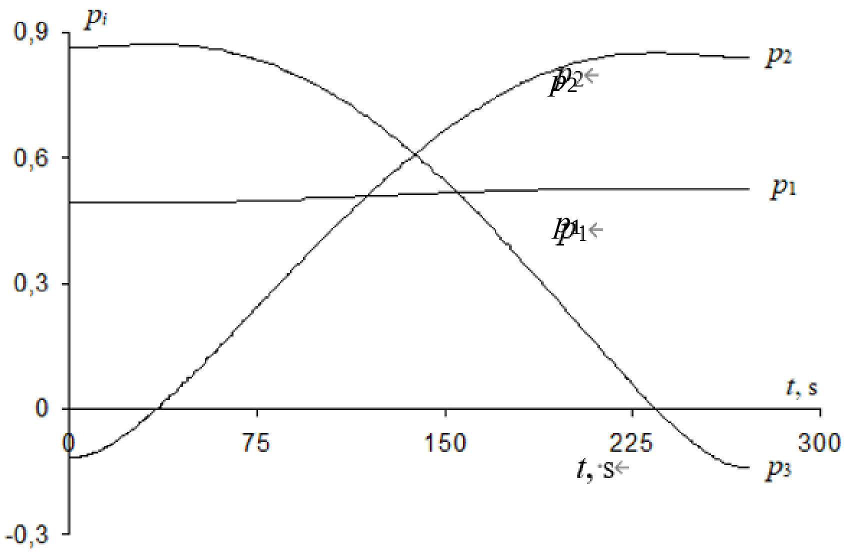

Figure 4. Variation of *p* under optimum control.

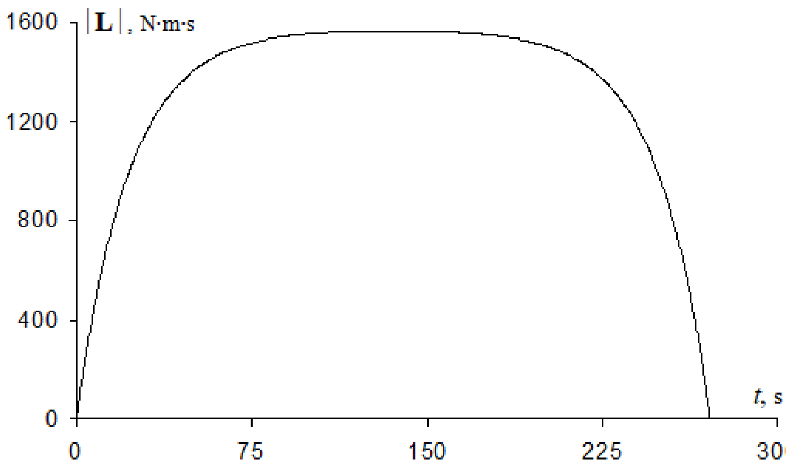

Figure 5. Variation of angular momentum modulus during optimal process.

8. Conclusion

Problem of optimum control of rotation from the initial attitude to the prescribed final attitude was researched in detail. In the considered investigation, new control method of spacecraft was designed; chosen optimality criterion is unique. The developed method for motion control was described in details. For optimization, the selected index of quality combines time, integral of the kinetic energy of rotation and control contribution (as energy consumption), necessary for turn realization. A solved problem is very topical. The issue of economical control of rotation is relevant very march, therefore, a task considered above is very important practically. Importance (implication, or significance) of the made research is that optimization of the adopted quality index minimizes energy costs (for example, electric energy consumption if spacecraft orientation control is performed by EJE-engines).

Main properties of reorientation and trajectory type corresponding to a chosen index of quality Equation (5) have been discovered. The maximum principle and quaternion models were used. As was shown, ratio of kinetic rotary energy to a squared modulus of angular momentum is constant. It was illustrated, mode of control when an angular momentum and a controlling moment are parallel over entire time period of rotation is optimal turn. The Hamilton and the conjugate equations system as well as analytical expressions for optimum controlling functions were written for optimization problem formulated in article. The structure of optimal control was defined, basing on the necessary optimality conditions. Relations for determining a spatial rotation were given, analytical formulas were written. The found optimal solution is unique.

Algorithm for orientation control is an essential element of attitude control system (for spacecrafts, and orbital stations, in particular). Analytical solving of a proposed problem has been presented. Computing expressions (and equations formalized) have been written for optimum program of controlled turn. Formulas to estimate maximum rotational energy and calculate maximum control torque magnitude were written analytically. Expressions of temporal characteristics of optimum process were presented. Implementation of controlled turn was described, also. Presence of calculated formulas given in explicit form, does a made work practically suitable and significant for direct applying in spaceflights practice. Optimum algorithms designed for spacecraft motion control improves a control system efficiency, originating spacecraft performance more economic during orbital flight.

Principal difference of solution, presented in this research, is new quality index used, and control torques cannot be unboundedly large even restrictions in absence for control. It is the original point of this article. Presence of energy integral in Equation (5) limits spacecraft's kinetic energy of rotation. Also, in an obtained mode, controlling torque modulus is not constant. Single admissible version of optimal solution is motion with variable control torque modulus (rotation period with constant control modulus are absent absolutely). How flat change the modulus of an angular momentum modulus, is defined by the coefficient of proportion k_1. The coefficients k_1 and k_2 (the ratio k_2/k_1) specify maximal energy of rotation.

Example of mathematic modeling which demonstrated a behavior of attitude parameters was given. For dynamically symmetric spacecraft (as special particular case), design of optimum control has been completed: the equations system has been written analytically (in direct form) that allows to solve two-point boundary value problem directly (to calculate the constants required for control law). The developed method of spacecraft control differs from all known publications. Its usefulness lies in significantly saving of control resources that will increase possibilities of spacecraft. In particular, interest causes use our method in spacecraft with EJE (or ion engines) because modern EJE has very large value of specific impulse (6000–6500 s approximately [42]) that require less fuel costs, and first integral in Equation (5) estimates electrical consumption. Second integral in Equation (5) limits kinetic energy of rotation (it is important in spaceflight, also). Necessity to reduce energy consumption is relevant problem, therefore research and optimization of optimal turn using on Equation (5) is topical.

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