(ε)-Kenmotsu manifold admitting Schouten-van Kampen connection

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ABSTRACT: The objective of this paper is to study some properties of quasi-conformal and concircular tensor on (ε)-Kenmotsu manifold admitting the Schouten-van Kampen connection. Expressions of the curvature tensor, Ricci tensor and scalar curvature admitting Schouten-van Kampen connection have been obtained. Locally symmetric (ε)-Kenmotsu manifold admitting Schouten-van Kampen connection and quasicon formally flat as well as quasi-conformally semisymmetric (ε)-Kenmotsu manifolds admitting Schouten-van Kampen connection are studied.

KEYWORDS: (ε)-Kenmotsu manifold; quasi-conformal curvature tensor; concircular curvature tensor; Schouten-van Kampen connection

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1. Introduction

De and Sarkar¹ introduced the concept of indefinite metrics on Kenmotsu manifold, and are called (ε)-Kenmotsu manifolds. They studied conformally flat, Weyl semisymmetric, ϕ- recurrent (ε)-Kenmotsu manifolds. The Schouten-van Kampen connection has been introduced for studying non-holomorphic manifolds. It preserves, by parallelism, a pair of complementary distributions on a differentiable manifold endowed with an affine connection (See Bejancu and Farran², Ianus³, Schouten and van Kampen⁴). Then, Olszak⁵ studied the Schouten-van Kampen connection to adapt it to an almost contact metric structure. He characterized some classes of almost contact metric manifolds with the Schouten-van Kampen connection and established certain curvature properties with respect to this connection. Recently, Ghosh⁶ and Yildiz⁷ have studied the Schouten-van Kampen connection in Sasakian manifolds and f-Kenmotsu manifolds, respectively. Some related developments can be found in many other works⁸-³³.

This paper is structured as follows: Section 2 gives a brief review of (ε)-Kenmotsu manifolds. In section 3, we obtain the expressions of the curvature tensor, Ricci tensor and scalar curvature admitting Schouten-van Kampen connection. In section 4, we study locally symmetric (ε)-Kenmotsu manifold admitting Schouten-van Kampen connection. In sections 5, we study quasiconformally flat and quasi-conformally semisymmetric (ε)-Kenmotsu manifold admitting Schouten-van Kampen connection. In section 6, we prove (ε) Kenmotsu manifold admitting Schouten-van Kampen connection satisfying \( \bar{Z}(X,Y,\bar{S}(U,W)) = 0 \) is an \( \eta \)-Einstein manifold.

2. Preliminaries

An almost contact structure on a differentiable manifold \( M^n \) is a triple \((ϕ,ξ,η)\), where \( ϕ \) is a tensor
field of type (1,1), \(\eta\) is a 1-form and \(\xi\) is a vector field such that
\[
\phi^2 = X_1 + \eta(X_1)\xi,
\]
\[
\eta(\xi) = 1, \phi\xi = 0, \eta\phi = 0
\]
A differential manifold with an almost contact structure is called an almost contact manifold. An almost contact metric manifold is an almost contact manifold endowed with a compatible metric \(g\). An almost contact metric manifold \(M\) is said to be an \((\epsilon)\)-almost contact metric manifold, if
\[
g(\xi, \xi) = \pm 1 = \epsilon,
\]
\[
\eta(X) = \epsilon g(X, \xi), \text{rank}(\phi) = n - 1,
\]
\[
g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y), \forall X, Y \in \Gamma(TM),
\]
where \(\xi\) is space-like or time-like but it is never a light like vector field. We say that \((\phi, \xi, \eta, g)\) is an \((\epsilon)\)-contact metric structure, if
\[
d\eta(X, Y) = g(X, \phi Y)
\]
In such case, \(M\) is an \((\epsilon)\)-contact metric manifold. An \((\epsilon)\)-contact metric manifold is called an \((\epsilon)\)-Kenmotsu manifold\(^1\), if
\[
\nabla X \phi Y = g(X, \phi Y)\xi - \epsilon \eta(Y)\phi X,
\]
where \(\nabla\) is the Riemannian connection of \(g\). An \((\epsilon)\)-almost contact metric manifold is a \((\epsilon)\)-Kenmotsu manifold if and only if
\[
\nabla X \xi = \epsilon (X - \eta(X)\xi).
\]
The following conditions hold in an \((\epsilon)\)-Kenmotsu manifold\(^1\):
\[
(\nabla_X \eta)(Y) = g(X, Y) - \epsilon \eta(X)\eta(Y),
\]
\[
\eta(R(X, Y, Z)) = \epsilon (g(X, Z)Y - g(Y, Z)X),
\]
\[
R(X, Y)\xi = \eta(X)Y - \eta(Y)X, R(\xi, X)Y = \eta(Y)X - \epsilon g(X, Y)\xi,
\]
\[
S(X, \xi) = -(n - 1)\eta(X), Q\xi = -\epsilon(n - 1)\xi,
\]
\[
S(\phi X, \phi Y) = S(X, Y) + \epsilon(n - 1)\eta(X)\eta(Y).
\]

3. \((\epsilon)\)-Kenmotsu manifolds admitting Schouten-van Kampen connection

The Schouten-van Kampen connection \(\nabla\) associated to the Levi-Civita connection \(\nabla\) is given by
\[
\nabla_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi
\]
for any vector fields \(X, Y\) on \(M\) (see Olszak\(^5\)). Using equations (8) and (9) in the above equation
\[
\nabla_X Y = \nabla_X Y - \epsilon \eta(Y)X - g(X, Y)\xi + 2\epsilon \eta(X)\eta(Y)\xi
\]
Putting \(Y = \xi\) and using (8) in (15), we obtain
\[
\nabla_X \xi = 0
\]
Let \(R\) and \(\tilde{R}\) denote the curvature tensor \(\nabla\) and \(\nabla\) respectively. Then
\[
\tilde{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[XY]} Z
\]
Using Equation (15) in Equation (17), we obtain
\[
\tilde{R}(X, Y)Z = R(X, Y)Z + \epsilon g(Y, Z)X - \epsilon g(X, Z)Y + (1 - \epsilon)\eta(X)g(Y, Z)\xi - (1 - \epsilon)\eta(Y)g(X, Z)\xi
\]
Putting \(Z = \xi\) and using (11) in (18), we obtain
\[
\tilde{R}(X, Y)\xi = 0
\]
On contracting (18), we obtain the Ricci tensor \(\tilde{S}\) of a \((\epsilon)\)-Kenmotsu manifold admitting Schouten-
Van Kampen connection $\bar{\nabla}$ as
$$\bar{S}(Y, Z) = S(Y, Z) + (en - 2e + 1)g(Y, Z) - \epsilon(1 - \epsilon)\eta(Y)\eta(Z)$$ (20)
This gives
$$\bar{Q}Y = QY + (en - 2e + 1)Y - (1 - \epsilon)\eta(Y)\xi$$ (21)
Contracting with respect to $Y$ and $Z$ in (20), we obtain
$$\bar{\tau} = r + n(en - 2e + 1) - (1 - \epsilon),$$ (22)
where $\bar{\tau}$ and $\tau$ are the scalar curvatures admitting Schouten-van Kampen connection $\bar{\nabla}$ and the Levi-Civita connection $\nabla$, respectively. From the above discussions we state the following:

**Theorem 1.** The curvature tensor $\bar{R}$, the Ricci tensor $\bar{S}$ and the scalar curvature $\bar{\tau}$ of an $(\epsilon)$-Kenmotsu manifold $M$ with respect to the Schouten-van Kampen connection $\bar{\nabla}$ are given by the Equations (18), (20), (21) and (22) respectively. Further, the curvature tensor $\bar{R}$ of $\bar{\nabla}$ satisfies the following:

(i) $\bar{R}(X, Y)Z = -\bar{R}(Y, X)Z$,
(ii) $\bar{R}(X, Y, Z, W) + \bar{R}(Y, X, Z, W) = 0$,
(iii) $\bar{R}(X, Y, Z, W) + \bar{R}(X, Y, W, Z) = 0$,
(iv) $\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = 0$,
(v) $\bar{S}$ is symmetric.

From Equation (20), the following result is immediate.

**Theorem 2.** An $(\epsilon)$-Kenmotsu manifold $M^n$ admitting the Schouten-van Kampen connection is Ricci flat admitting Schouten-van Kampen connection if and only if $M^n$ is an $\eta$-Einstein manifold with respect to Levi-Civita connection.

Now, if $\bar{R}(X, Y)Z = 0$, then Equation (18) becomes
$$R(X, Y)Z + \epsilon(g(Y, X)Z - g(X, Z)Y) + (1 - \epsilon)(\eta(X)g(Y, Z)\xi - \eta(Y)g(X, Z)\xi) = 0$$ (23)
Thus, we have the following theorem.

**Theorem 3.** Let $M^n$ be a $(\epsilon)$-Kenmotsu manifold admitting the Schouten-van Kampen connection. The curvature tensor of $M$ admitting Schouten-van Kampen connection vanishes if and only if $M$ with respect to the Levi-Civita connection is isomorphic to the hyperbolic space $H^n(-1)$.

### 4. Locally symmetric $(\epsilon)$-Kenmotsu manifold admitting Schouten-van Kampen connection

**Theorem 4.** A locally symmetric $(\epsilon)$-Kenmotsu manifold $M^n$ admitting Schouten-van Kampen connection $\bar{\nabla}$ is an $\eta$-Einstein manifold.

**Proof.** Let $M^n$ be a locally symmetric $(\epsilon)$-Kenmotsu manifold admitting Schouten-van Kampen connection $\bar{\nabla}$. Then $(\bar{\nabla}_X R)(Y, Z)W = 0$. By contraction of the equation, we get
$$(\bar{\nabla}_X \bar{S})(Z, W) = \bar{\nabla}_X \bar{S}(Z, W) - \bar{S}(\bar{\nabla}_X Z, W) - \bar{S}(Z, \bar{\nabla}_X W) = 0$$ (24)
Putting $W = \xi$ in (24), we have
$$\bar{\nabla}_X \bar{S}(Z, \xi) - \bar{S}(\bar{\nabla}_X Z, \xi) - \bar{S}(Z, \bar{\nabla}_X \xi) = 0$$ (25)
Using (15) and (20) in (25), we obtain
$$S(X, Z) = Ag(X, Z) + B\eta(X)\eta(Y)$$ (26)
where $A = -\epsilon(n - 2) + 1$ and $B = -\epsilon(n - 2) + n$. □
5. Quasi-Conformally flat (e)-Kenmotsu manifold admitting Schouten van-Kampen connection

Theorem 5. A quasi-conformally flat (e)-Kenmotsu manifold admitting Schouten van-Kampen connection is an η-Einstein manifold.

Proof. An (e)-Kenmotsu manifold admitting a Schouten van-Kampen connection is said to be quasi-conformally flat if

\[ \bar{C}(X, Y)Z = 0 \]  \hspace{1cm} (27)

The quasi-conformal curvature tensor \( \bar{C} \) admitting a Schouten van-Kampen connection is (See Yano\[34\]):

\[ \bar{C}(X, Y)Z = aR(X, Y)Z + b\bar{S}(Y, Z)X - \bar{S}(X, Z)Y + g(Y, Z)\bar{Q}X \]
\[ -g(X, Y)\bar{Q}Y - \frac{a}{n}(2b + \frac{1}{n}) (g(Y, Z)X - g(X, Z)Y). \]  \hspace{1cm} (28)

In view of Equations (27) and (28), we have

\[ a\bar{R}(X, Y)Z = b\bar{S}(X, Z)Y - \bar{S}(Y, Z)X + g(X, Z)\bar{Q}Y - g(Y, Z)\bar{Q}X + \frac{a}{n}(2b + \frac{1}{n}) (g(Y, Z)X - g(X, Z)Y) \]  \hspace{1cm} (29)

Using Equations (18), (20), and (21) and taking inner product with \( \xi \) in Equation (29) we obtain

\[
\begin{align*}
&\alpha(g(R(X, Y)Z, \xi) + \epsilon g(Y, Z)g(X, \xi) - \epsilon g(X, Z)g(Y, \xi) + \\
&\quad (1 - \epsilon)\eta(Y)g(X, Z)g(X, \xi) - (1 - \epsilon)\eta(X)g(Y, Z)g(U, \xi)) \\
&\quad = b((S(X, Z)g(Y, \xi)) + (en - 2e + 1)g(Y, Z)g(Y, \xi)) - \epsilon(1 - \epsilon)\eta(Y)\eta(Z)g(Y, \xi) \\
&\quad - (S(X, Z)g(X, \xi)) + (en - 2e + 1)g(Y, Z)g(X, \xi) + \epsilon(1 - \epsilon)(\eta(Y)\eta(Z)g(X, \xi)) \\
&\quad + g(X, Z)(g(QY, \xi)) + (en - 2e + 1)g(Y, \xi) + (1 - \epsilon)\eta(Y)g(\xi, \xi) \\
&\quad - g(Y, Z)(g(QX, \xi)) + (en - 2e + 1)g(X, \xi) + (1 - \epsilon)\eta(X)g(\xi, \xi) \\
&\quad + \frac{\bar{r}}{n}(\frac{a}{n} + 2b)g(Y, Z)g(X, \xi) - g(X, Z)g(Y, \xi) \hspace{1cm} (30)
\end{align*}
\]

Putting \( X = \xi \) and using Equations (3), (4), (11) in Equation (30) we obtain

\[ S(Y, Z) = Cg(Y, Z) + D\eta(Y)\eta(X), \]  \hspace{1cm} (31)

where

\[ C = \frac{1}{eb} \left( (1 - \epsilon)(a + 2b) + \epsilon \frac{\bar{r}}{n}(\frac{a}{b(n - 1)} + 2b) \right) \]

and

\[ D = \frac{1}{eb} \left( (1 - \epsilon)(a + 2b) - \frac{\bar{r}}{n}(\frac{a}{b(n - 1)} + 2b) \right). \]  \hspace{1cm} □

6. Quasi-Conformally semisymmetric (e)-Kenmotsu manifold admitting Schouten van-Kampen connection

Theorem 6. A quasi-Conformally semisymmetric (e)-Kenmotsu manifold admitting Schouten van-Kampen connection is an η-Einstein manifold.

Proof. An (e)-Kenmotsu manifold admitting a Schouten van-Kampen connection is said to be quasi-conformally semisymmetric if

\[ \bar{R} \cdot \bar{C}(U, V)W = 0 \]  \hspace{1cm} (32)

which implies that

\[ \bar{R}(\xi, Y)\bar{C}(U, V)W - \bar{C}(\bar{R}(\xi, Y)U, V)W - \bar{C}(U, \bar{R}(\xi, Y)V)W - \bar{C}(U, V)\bar{R}(\xi, Y)W = 0 \]  \hspace{1cm} (33)

In view of Equation (18) in Equation (33), we have
(1 - \varepsilon)g(Y, \tilde{C}(U,V)W)\xi - \varepsilon(1 - \varepsilon)\eta(Y)\eta(\tilde{C}(U,V)W)\xi \\
-(1 - \varepsilon)g(Y, U)\tilde{C}(\xi,V)W + \varepsilon(1 - \varepsilon)\eta(Y)\eta(\tilde{C}(\xi,V)W) \\
-(1 - \varepsilon)g(Y, V)\tilde{C}(U,\xi)W + \varepsilon(1 - \varepsilon)\eta(Y)\eta(\tilde{C}(U,\xi)W) \\
-(1 - \varepsilon)g(Y, W)\tilde{C}(U,V)\xi + \varepsilon(1 - \varepsilon)\eta(Y)\eta(W)\tilde{C}(U,V)\xi = 0 \quad (34)

Replacing Y = U and taking inner product with \xi in (34), we have

\varepsilon g(U, \tilde{C}(U,V)W) - \eta(\tilde{U})\eta(\tilde{C}(U,V)W) - (g(U, U) - \varepsilon\eta(U)\eta(U))\eta(\tilde{C}(\xi,V)W) \\
-(g(U, V) + \varepsilon\eta(U)\eta(V)\tilde{C}(U,\xi)W) - (g(U, W) - \varepsilon\eta(U)\eta(W))\eta(\tilde{C}(U,\xi)W)\xi = 0 \quad (35)

provided (1 - \varepsilon) \neq 0.

Putting U = \xi and using Equations (28), (18), (20) and (21), we obtain

S(V,W) = E g(V,W) + F\eta(V)\eta(W) \quad (36)

where

E = -\frac{1}{b}\left[\alpha(1 - 3\varepsilon) + b(\varepsilon n - 2\varepsilon + 1) + 2\varepsilon(1 - \varepsilon) + \frac{\tilde{r}}{n} \left(\frac{a}{n - 1} + 2b\right)\right]

and

F = -\frac{1}{b}\left[-\varepsilon\alpha(1 - \varepsilon) + b(n - 1) - b\varepsilon(\varepsilon n - 2\varepsilon + 1) - b(1 - \varepsilon) - \frac{\tilde{r}}{n} \left(\frac{a}{n - 1} + 2b\right)\right] \quad \square

7. (\varepsilon)-Kenmotsu manifold admitting Schouten van-Kampen connection satisfying

\tilde{Z}(X,Y \cdot \tilde{S}(U,W)) = 0

Theorem 7. An (\varepsilon)-Kenmotsu manifold admitting Schouten van-Kampen connection satisfying \tilde{Z}(X,Y \cdot \tilde{S}(U,W)) = 0 is an \eta-Einstein manifold.

Proof. An (\varepsilon)-Kenmotsu manifold admitting Schouten van-Kampen connection satisfies

\tilde{Z}(X,Y \cdot \tilde{S}(U,W)) = 0 \quad (37)

which implies that

\tilde{S}(\tilde{Z}(\xi,Y)U,W) + \tilde{S}(U,\tilde{Z}(\xi,Y)W) = 0 \quad (38)

The concircular curvature tensor \tilde{Z} admitting Schouten van-Kampen connection is given by (see Yano\cite{35})

\tilde{Z}(X,Y)Z = \tilde{R}(X,Y)Z - \left(\frac{\tilde{r}}{n(n - 1)}(g(Y,Z)X - g(X,Z)Y)\right) \quad (39)

In view of Equations (3), (4), (18), (38), and (39), we have

\tilde{S}R(\xi,Y)U,W) + \varepsilon g(Y,U)\tilde{S}(\xi,W) - \eta(U)\tilde{S}(Y,W) + (1 - \varepsilon)g(Y,U)\tilde{S}(\xi,W) \\
-\varepsilon(1 - \varepsilon)\eta(Y)\eta(U)\tilde{S}(\xi,W) - \left(\frac{\tilde{r}}{n(n - 1)}\right)g(Y,U)\tilde{S}(\xi,W) + \varepsilon\left(\frac{\tilde{r}}{n(n - 1)}\right)\eta(U)\tilde{S}(Y,W) \\
+\tilde{S}(U,R(\xi,Y)W) + \varepsilon g(Y,W)\tilde{S}(\xi,U) - \eta(W)\tilde{S}(Y,U) + (1 - \varepsilon)g(Y,W)\tilde{S}(\xi,U) \\
-\varepsilon(1 - \varepsilon)\eta(Y)\eta(W)\tilde{S}(\xi,U) - \left(\frac{\tilde{r}}{n(n - 1)}\right)g(Y,W)\tilde{S}(\xi,U) + \varepsilon\left(\frac{\tilde{r}}{n(n - 1)}\right)\eta(W)\tilde{S}(Y,U) = 0. \quad (40)

Using Equation (20) and putting U = \xi in Equation (40), we obtain

S(Y,W) = G g(Y,W) + H\eta(Y)\eta(W), \quad (41)

where

G = -\frac{1}{2}\left[\varepsilon(n - 1) + (1 - \varepsilon) + (\varepsilon n - 2\varepsilon + 1)\left(\frac{\tilde{r}\varepsilon}{n(n - 1)} - 1\right)\right]

and
\[ H = -\frac{1}{2} \left( -\epsilon(1 - \epsilon) - \epsilon(1 - \epsilon) \left( \frac{r_{\epsilon}}{n(n-1)} - 1 \right) \right) \]

**Author contributions**

Conceptualization, SGB, RR, PSKR and NP; methodology, SGB, RR and PSKR; validation, SGB, RR, PSKR and NP; formal analysis, SGB and NP; investigation, SGB, RR, PSKR and NP; resources, SGB, RR and PSKR; data curation, SGB, RR and PSKR; writing—original draft preparation, SGB, RR and PSKR; writing—review and editing, RR and PSKR; visualization, RR and PSKR; supervision, RR and PSKR. All authors have read and agreed to the published version of the manuscript.

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**Conflict of interest**

The authors declare no conflict of interest.

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