Conservation laws, exact solutions and nonlinear dispersion: A lie symmetry approach

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ABSTRACT: In this study, we investigated a set of equations that exhibit compact solutions and nonlinear dispersion. We used the classical lie symmetry approach to derive ordinary differential equations (ODEs) that are well suited for qualitative study. By examining the dynamic behavior of these ODEs, we gained insights into the intricate nature of the underlying system. We also used a powerful multiplier approach to establish nontrivial conservation laws and exact solutions for these equations. These conservation laws provide essential information regarding the underlying symmetries and invariants of the system, and shed light on its fundamental properties.

KEYWORDS: lie symmetries; infinitesimals operator; conservation laws; Euler-Lagrangian operator; nonlinear dispersion; exact solutions; multipliers approach

1. Introduction

Nonlinear waves are waves that exhibit non-linear behavior, meaning that their amplitude and velocity are not linearly related. Solitons and compactons are two types of nonlinear waves. Solitons are stable pulse-like waves that can exist in some nonlinear systems. They can pass through each other without being destroyed, and they can retain their shape even after interacting with other waves. Compactons are a special type of soliton that does not have exponential tails. Solitons and compactons are used as building blocks to formulate the complex dynamical behavior of wave systems throughout science. They have been studied in a variety of fields, including hydrodynamics, nonlinear optics, plasmas, shock waves, tornadoes. Solitons have also acquired prominence in the fields of quantum mechanics and nanotechnology, particularly in the study of nano-hydrodynamics. The solitary wave dynamics[1] of the local fractional Bogoyavlensky

Konopelchenko model is a topic of active research in the field of nonlinear wave theory. The local fractional Bogoyavlensky Konopelchenko model is a partial differential equation (PDE) that describes the propagation of waves in a nonlinear medium. The model is a generalization of the classical Bogoyavlensky Konopelchenko model, and it takes into account the effects of fractional diffusion. It is well known that while conventional nonlinearity's influence does not significantly alter with spatial dimension, dispersive processes become more effective at disseminating information. As a result, a model that is well-balanced in one dimension becomes unbalanced in higher dimensions. As a result, strong solitonic structures are often far less common in higher spatial dimensions. Rosenau and Hyman[2] presented the compactons, solitons with a compact support, almost 20 years ago using the $C(l, p)$ model equation in its simplest form,

$$A_y + lA^{l-1}A_z + p(A^{p-1}A_z)_{zz} = 0 \quad l, p > 1 \tag{1}$$

and exact solutions as well as symmetry reductions were derived in the work of Bruzón and Gandarias[3], Bruzón et al.[4], and Anco and Bluman[5]. Naz[6] utilized the multiplier technique to construct the conservation laws for the equation. Because the authors believe that higher order multipliers determining equations are very complicated and cannot be manually separated, only multipliers of the kind $M(y,z,A)$ were considered in her work of Conservation laws for some compacton equations using the multiplier approach[6].

In their work, Rosenau and Oron[7] investigated how several symbolic forms of nonconvex convection affected the development of compact patterns. To do this, a basic model with cubic dispersion and numerous versions on a nonlinear modified dispersion are utilized which is of the form.

$$A_y + (A^3 - A^2)_z + [A(A^2)_{zz}]_z = 0$$
$$A_y + (A^3 - A^2)_z + 2(AA_z)_{zz} = 0$$

$$(2)$$

In contrast to the $C(n,n)$ compactons, the breadth of the current compactons varies on their velocity. In a recent study[8], Gandarias has successfully identified and formulated several conservation laws that are not simple or obvious. Furthermore, we have demonstrated that certain equations, which have solutions in the form of compactons and exhibit cubic dispersion, possess a unique property called nonlinear self-adjointness. This discovery is significant as it highlights the intricate dynamics and properties of these equations, providing valuable insights into their behavior and characteristics.

Conservation laws are widely recognized as crucial components in solving equations or systems of differential equations. While not all conservation laws in partial differential equations (PDEs) have direct physical interpretations, they serve a significant purpose in studying the integrability of PDEs. Understanding and identifying these conservation laws are vital steps in comprehending the behavior and properties of

PDEs and their solutions.

The Noether theorem[2] is a powerful tool for deriving conservation laws in variational problems. It can be used to derive conservation laws for variational problems, which are problems that can be formulated in terms of a Lagrangian. However, for nonvariational situations, alternative methods are needed to construct conservation laws. Anco and Bluman[9] introduced an algorithmic technique that allows for the identification of all conservation laws for evolution equations. Ibragimov[10] presented a unique approach based on adjoint equations for nonlinear equations, which eliminates the need for function integrals and does not rely on Lagrangians. The concept of strictly self-adjoint equations[11–13] has been expanded upon, and Ibragimov's findings have sparked further research on self-adjointness and its relevance to partial differential equations (PDEs)[14–23]. This approach represents an extension of the previously described formula in work of direct construction of conservation laws from field equations[9], providing a broader framework for studying and applying conservation laws in PDEs.

In this research, we will solve the Equation (2) using the lie classical technique, as well as the multipliers approach to derive conservation laws for these equations.

2. Derivation of exact solutions from classical lie approach

In this part, we conduct a lie symmetry analysis for a specific system denoted in the Equation (2). We focus on exploring a one-parameter lie group consisting of infinitesimal transformations[24–26] in the variables (y, z, A). The transformations are expressed in a specific form, which we will investigate and analyze further.

$$y^* = y + \varepsilon\phi(y,z,A) + \vartheta(\varepsilon^2)$$
$$z^* = z + \varepsilon\psi(y,z,A) + \vartheta(\varepsilon^2)$$
$$A^* = A + \varepsilon\eta(y,z,A) + \vartheta(\varepsilon^2)$$

$$(3)$$

where ε is the group parameter. To ensure that

the transformation preserves the solutions of Equation (2), it is necessary to satisfy certain conditions. This leads to an overdetermined system of linear equations consisting of eleven equations involving the infinitesimals $\psi(y, z, A)$, $\phi(y, z, A)$, and $\eta(y, z, A)$. The collection of vector fields that satisfy these equations form the associated lie algebra of infinitesimal symmetries. These vector fields are expressed in the following specific form:

$$\mathbf{X} = \psi \frac{\partial}{\partial z} + \phi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial A}$$

(4)

where $\mathbf{X}$ is infinitesimal operator or generator of the group. After identifying the infinitesimals, the next step is to solving the invariant surface condition yields the symmetry variables. This condition ensures that the transformed equations remain invariant under the lie symmetry transformations.

$$\Upsilon = \psi \frac{\partial A}{\partial z} + \phi \frac{\partial A}{\partial y} - \eta = 0$$

(5)

By considering the determining system for the first formula of Equation (2), we find that the infinitesimals can be expressed as $\psi = \psi(y, z), \phi = \phi(y)$, and $\eta = \eta(y, z, A)$. These functions, namely ψ, ϕ, and η, need to satisfy the following system of equations:

$-3A\psi_z + \phi_y A + 2\eta = 0,$
$-3A\psi_{zz} + 3\eta_{Ax}A + 4\eta_z = 0,$
$-2A^2\psi_{zzz} - 3A^2\psi_z + 2A\psi_z - \psi_y + 3A_y A^2 + 6\eta_{Azz}A^2,$
$-2\phi_y A + 8\eta_{zz}A + 6\eta A - 2\eta = 0,$
$-12A\psi_z + 3\eta_{AA}A^2 + 4\phi_y A + 4\eta_A A + 4\eta = 0,$
$-4A\psi_{zz} + 3\eta_{AAx}A^2 + 8\eta_{Az}A + 3\eta_z = 0,$
$-3\psi_z + \eta_{AAA}A^2 + 4\eta_{AA}A + A_y + 2\eta_A = 0.$

(6)

Upon solving the determining equations for ψ, ϕ, and η, we are able to determine the lie point symmetry generators that form a two-dimensional lie algebra. These generators are obtained as a result of the solutions to the determining equations, and they characterize the symmetries admitted by the Equation (2).

$$\mathbf{X_1} = \frac{\partial}{\partial z}$$
$$\mathbf{X_2} = \frac{\partial}{\partial y}$$

(7)

In this section, we successfully derived the reduction of the first equation of the Equation (2) to ordinary differential equations (ODEs) using the generators $\gamma\mathbf{X_1} + \omega\mathbf{X_2}$. This reduction allows us to simplify the equation and express it in terms of ODEs, which are typically easier to analyze and solve. Additionally, we obtained the similarity variable and similarity solution.

$$\theta = \omega z + \gamma y$$
$$A = \rho(\theta)$$

(8)

Substituting Equation (8) into Equation (5), we obtain:

$$2\omega^3 \rho^2 \rho_{\theta\theta\theta} + 8\omega^3 \rho\rho_\theta\rho_{\theta\theta} + 2\omega^3(\rho_\theta)^3$$
$$+ 3\omega\rho^2\rho_\theta - 2\omega\rho\rho_\theta - \gamma\rho_\theta = 0$$

(9)

After performing the integration of the equation with respect to θ, we arrive at the following equation, which takes the form:

$$2\omega^3\rho^2\rho_{\theta\theta} + 2\omega^3\rho(\rho_\theta)^2 + \omega\rho^3 - \omega\rho^2 - \gamma\rho$$
$$+ \kappa_1 = 0$$

(10)

This reduced ordinary differential equation (ODE), given by Equation (10), exhibits a group corresponding to the generator $\mathbf{H} = \partial_\theta$. This group corresponds to a symmetry of the equation, indicating that there is a transformation along the θ direction that leaves the equation invariant.

By considering the invariants of the first prolongation and introducing the new variables as given in equations, namely:

$$\rho = \mathcal{Z}$$
$$\rho' = v(\mathcal{Z})$$
$$\rho'' = v(\mathcal{Z})\frac{dv}{d\mathcal{Z}}$$

(11)

we can further simplify Equation (10). This reduction allows us to express Equation (10) as a first-order ordinary differential equation (ODE).

$$2\omega^3 Z^2 v v_Z + 2\omega^3 Z v^2 + \omega Z^3 - \omega Z^2 - \gamma Z = 0 \tag{12}$$

whose implicit solution is,

$$12\omega^3 Z^2 v^2 + 3\omega Z^4 - 4\omega Z^3 - 6\gamma Z^2 + 12\kappa_1 Z \\ + \kappa_2 = 0 \tag{13}$$

where κ_1 and κ_2 are arbitrary constants. Following a similar procedure as before, we apply the same methodology to the second equation of the Equation (2). This leads to the derivation of the generators $\mathbf{X_1}$ and $\mathbf{X_2}$, as well as the determination of the similarity variable and similarity solution given by Equation (8). The corresponding reduced ordinary differential equation (ODE) is then obtained as equation, which takes the form:

$$2\omega^3 \rho \rho_{\theta\theta\theta} + 6\omega^3 \rho_\theta \rho_{\theta\theta} + 3\omega\rho^2 \rho_\theta - 2\omega\rho\rho_\theta \\ - \gamma\rho_\theta = 0 \tag{14}$$

By integrating Equation (14) once with respect to θ, we arrive at equation:

$$\kappa_1 + 2\omega^3 \rho \rho_{\theta\theta} + 2\omega^3 (\rho_\theta)^2 + \omega\rho^3 - \omega\rho^2 - \gamma\rho \\ = 0 \tag{15}$$

Equation (15) represents the reduced ODE, which admits the symmetry generator $\mathbf{H} = \partial_\theta$. By considering the invariants of its first prolongation and introducing the variables given by Equation (11), we can further simplify Equation (15) to obtain the first-order ODE.

$$2\omega^3 Z v v_Z + 2\omega^3 v^2 + k_1 + \omega Z^3 - \omega Z^2 - \gamma Z \\ = 0 \tag{16}$$

whose implicit solution is,

$$60\omega^3 Z^2 v^2 + 30 Z^2 \kappa_1 + 12\omega Z^5 - 15\omega Z^4 - 20\gamma Z^3 + \kappa_2 \\ = 0 \tag{17}$$

2.1 Qualitative study of ODEs

Equations (10) and (15), after setting $\kappa_1 = 0$, can be written as:

$$\rho'' + \frac{(\rho')^2}{\rho} + \frac{\rho}{2\omega^2} - \frac{1}{2\omega^2} - \frac{\gamma}{2\omega^3 \rho} = 0 \\ \rho'' + \frac{(\rho')^2}{\rho} + \frac{\rho^2}{2\omega^2} - \frac{\rho}{2\omega^2} - \frac{\gamma}{2\omega^3} = 0 \tag{18}$$

By introducing the change of variables $z = \rho$ and $\varphi = \rho' \rho$, the Equations (18) can be transformed into a system of the form,

$$\dot{z} = \frac{\varphi}{z} \\ \dot{\varphi} = \varrho(z) \tag{19}$$

where,

$$\varrho(z) = -\frac{1}{2\omega^2} z^2 + \frac{1}{2\omega^2} z + \frac{\gamma}{2\omega^3} \\ \varrho(z) = -\frac{1}{2\omega^2} z^3 + \frac{1}{2\omega^2} z^2 + \frac{\gamma}{2\omega^3} z \tag{20}$$

respectively.

The phase portrait of the Equation (19) is divided into two half-planes that are invariant, one for $z > 0$ and the other for $z < 0$. This means that the dynamics of the system in each half-plane remains confined within that respective half-plane. Equation (19) is conservative, meaning that there exist conserved quantities associated with it. These conserved quantities are defined by the differentiable functions $\mathbf{P}$, given by

$$\mathbf{P}(z,\varphi) = \frac{\varphi^2}{2} + \frac{z^4}{8\omega^2} - \frac{z^3}{6\omega^2} - \frac{\gamma z^2}{4\omega^2} \\ \mathbf{P}(z,\varphi) = \frac{\varphi^2}{2} + \frac{z^5}{10\omega^2} - \frac{z^4}{8\omega^2} - \frac{\gamma z^3}{6\omega^3} \tag{21}$$

These quantities remain constant along the trajectories of the system, meaning that $\frac{d\mathbf{P}}{dt} = \mathbf{P}_z \dot{z} + \mathbf{P}_\varphi \dot{\varphi} = 0$. Therefore, the trajectories lie on curves defined by $\mathbf{P}(z,\varphi)$ is equal to constant, and they exhibit symmetry relative to the z-axis. Importantly, $\mathbf{P}(z,\varphi)$ can be represented as

$$\mathbf{P}(z,\varphi) = \frac{\varphi^2}{2} + \Re(z) \tag{22}$$

where $\Re(z)$ is given by

$$\Re(z) = -\int_0^z v\varrho(v)dv \tag{23}$$

The equilibrium points $\mathcal{P}$ of the Equation (19), if they exist, are located on the z-axis and correspond to the critical points of $\mathbf{P}(z,\varphi)$. This

can be seen by analyzing the partial derivatives of $\mathbf{P}(z, \varphi)$ with respect to z and φ.

$$\frac{\partial \mathbf{P}}{\partial z} = -z\varrho(z) = 0$$
$$\Leftrightarrow \dot{\varphi} = 0$$
$$\frac{\partial \mathbf{P}}{\partial \varphi} = \varphi = 0$$
$$\Leftrightarrow \dot{z} = 0$$

(24)

The equilibrium point $\mathcal{P}(z^*, 0)$ is a fixed point of the Equation (19) if z^* is a critical point of $\mathfrak{R}(z)$, i.e., a zero of the polynomial function $\varrho(z)$ defined in Equation (21).

3. Multipliers approach

In their work, Anco and Bluman[5] presented a general method for deriving conservation laws for partial differential equations in a Cauchy-Kovaleskaya form, specifically for evolution equations of the form,

$$A_y = \mathrm{E}(z, A, A_z, A_{zz}, \dots, A_{nz})$$

(25)

The conservation laws are characterized by a multiplier Λ that does not depend on A_y and satisfies the following equation

$$\mathrm{F}[A]\left(\Lambda A_y - \Lambda G(z, A, A_z, A_{zz}, \dots, A_{nz})\right) = 0$$

(26)

where, the Euler-Lagrangian operator $\mathrm{F}[A]$ is defined as

$$\mathrm{F}[A] = \frac{\partial}{\partial A} - D_y \frac{\partial}{\partial A_y} - D_z \frac{\partial}{\partial A_z} + D_z^2 \frac{\partial}{\partial A_{zz}} + \cdots$$

(27)

where, D_y and D_z are the total derivatives with to respect to y and z. The conserved vector is required to satisfy

$$\Lambda = \mathrm{F}[A]\Upsilon^y$$

(28)

and the flux Υ^z is given by Euler[27].

$$\Upsilon^z = -D_z^{-1}(\Lambda \mathrm{E}) - \frac{\partial \Upsilon^y}{\partial A_z} \mathrm{E} + \mathrm{E} D_z\left(\frac{\partial \Upsilon^y}{\partial A_{zz}}\right) + \cdots$$

(29)

The conservation law will be written as

$$D_y(\Upsilon^y) + D_z(\Upsilon^z) = 0$$

(30)

We get the following multipliers: for first equation of Equation (2).

$$\Lambda = 1$$
$$\Lambda = A$$
$$\Lambda = A^2 A_{zz} + \frac{A^3}{2} - \frac{A^2}{2} + AA_z^2$$

(31)

For second equation of the Equation (2),

$$\Lambda = 1$$
$$\Lambda = A^2$$

(32)

we have the equation, which represents the first equation of the Equation (2).

$$\mathrm{G} \equiv A_y + (A^3 - A^2)_z + [A(A^2)_{zz}]_z = 0$$

(33)

Equation (33) can be considered nonlinearly self-adjoint if there exists a nontrivial function $\rho(y, z, A, A_z, \dots)$, such that when we substitute $v = \rho(y, z, A, A_z, \dots)$ into the adjoint equation such that $\rho(y, z, A, A_z, \dots) \neq 0$, it becomes same as the original Equation (33); that is

$$\mathrm{G}^*_{|v=\rho} = \gamma \mathrm{G}$$

(34)

To do so, we consider its adjoint equation to Equation (33) is following, where v is a new dependent variable,

$$\mathrm{G}^* \equiv \frac{\delta(v\mathrm{G})}{\delta A} = 0$$

(35)

where,

$$\frac{\delta}{\delta A} = \frac{\partial}{\partial A} - D_y\left(\frac{\partial}{\partial A_y}\right) - D_z\left(\frac{\partial}{\partial A_z}\right) + D_z^2\left(\frac{\partial}{\partial A_{zz}}\right)$$

(36)

Equation (36) defines the variational derivative, also known as the Euler-Lagrangian operator. The variational derivative takes into account the total differentiations with respect to y and z, denoted by D_y and D_z, respectively.

Let us select nonlinearly self-adjoint equations from

$$G^* - \gamma\left(A_y + (A^3 - A^2)_z + [A(A^2)_{zz}]_z\right)$$
$$- \omega\left(A_y + (A^3 - A^2)_z\right.$$
$$+ [A(A^2)_{zz}]_z\Big)_z$$
$$- \epsilon\left(A_y + (A^3 - A^2)_z\right.$$
$$+ [A(A^2)_{zz}]_z\Big)_{zz} = 0$$

(37)

where γ, ω, and ϵ are undetermined coefficients. Setting $v = \rho(y, z, A, A_z, A_{zz})$, we can analyze the coefficients for the different derivatives of A in order to determine the requirements for the equation to be nonlinearly self-adjoint. We conclude that the following requirements must be met

$$\gamma = -\rho_A$$
$$\omega = -\rho_{A_z}$$
$$\epsilon = -\rho_{A_{zz}}$$

(38)

and by resolving the remaining equations, we obtain

$$\rho = \kappa_1 A^2 A_{zzz} + \kappa_1 A A_z^2 + \text{c}(A)A_z + \text{d}(A)$$

(39)

with

$$\text{c}(A) = \kappa_2 A^{\frac{4}{3}}$$
$$\text{d}(A) = \frac{1}{2}(\kappa_1 A^3 - \kappa_1 A^2) + \kappa_3 A + \kappa_4$$

(40)

The following are the outcome.

- In the given Equation (2), the first equation is stated to be nonlinearly self-adjoint.

$$\rho = 1$$
$$\rho = A$$
$$\rho = A^2 A_{zz} + \frac{A^3}{2} - \frac{A^2}{2} + A A_z^2$$

(41)

Using the same method on the second equation of Equation (2), we get the following conclusion.

- For the second equation of the Equation (2) to be nonlinearly self-adjoint, we are given the following choices for the function $\rho(y, z, A, A_z, A_{zz})$,

$$\rho = 1$$
$$\rho = A^2$$

(42)

The functions $\rho(y, z, A, A_z, A_{zz})$ derived from the condition of nonlinear self-adjointness in the equations correspond to the multipliers used in the Anco and Bluman method[5] for the direct construction of conservation laws.

4. Conservation laws

We obtain the conserved quantities (vectors) and fluxes associated with the multipliers from Equations (28) and (29). For the first equation of Equation (2):

4.1 First conserved vector

$$\Lambda = 1$$
$$\eta^y = A$$
$$\eta^z = A(A^2 + (-1 + 2A_{zz})A + 2A_z^2)$$

(43)

4.2 Second conserved vector

$$\Lambda = A$$
$$\eta^y = \frac{A^2}{2}$$
$$\eta^z = \frac{3}{4}A^4 - \frac{2}{3}A^3 + 2A^3 A_{zz} + A^2 A_z^2$$

(44)

4.3 Third conserved vector

$$\Lambda = A^2 A_{zz} + \frac{A^3}{2} - \frac{A^2}{2} + A A_z^2$$
$$\eta^y = -\frac{1}{2}A^2 + \frac{1}{8}A^4 - \frac{1}{6}A^3$$
$$\eta^z = \frac{1}{4}\left(A^4 + (4A_{zz} - 2)A^3 + \left(4A_z^2\right.\right.$$
$$+ 4\left(A_{zz} - \frac{1}{2}\right)^2\Big)A^2\Big)$$
$$+ \frac{1}{4}\left(8\left(A_{zz} - \frac{1}{2}\right)A_z^2 A + 4A_z^4\right.$$
$$+ 4A_y A_z\Big)A^2$$

(45)

The following multipliers, conserved densi-

ties and fluxes are obtained for the second equation of Equation (2):

4.4 First conserved vector

$$\Lambda = 1$$

$$\eta^y = A$$

$$\eta^z = 2AA_{zz} + 2(A_z)^2 - A^3 + A^2$$

(46)

4.5 Second conserved vector

$$\Lambda = A^2$$

$$\eta^y = \frac{A^3}{3}$$

$$\eta^z = \frac{1}{10}A^3(20A_{zz} + 6A^2 - 5A)$$

(47)

Applying the theorem on conservation laws derived from the generators $\mathbf{X}_1$ and $\mathbf{X}_2$ in the work of Ibragimov[10] may lead to trivial conservation laws in this case. Trivial conservation laws are those that do not provide new information about the system and are often associated with symmetries that are not physically relevant. However, it is worth noting that in the Conservation laws of scaling-invariant field equations[28], a method is presented specifically for deriving conservation laws associated with scaling symmetries. This method may provide more meaningful conservation laws for the system. If scaling symmetries are present in the system described by Equation (2), applying the method described by Anco[28] could yield nontrivial conservation laws.

5. Conclusions

We used the classical lie approach to solve two partial differential equations (PDEs) with nonlinear dispersion and compacton solutions. Because these equations have symmetries, we were able to reduce them further into first-order ordinary differential equations (ODEs). This reduction gave useful insights into their dynamic behaviour and qualified them for qualitative analysis. We used infinitesimal operator of the group and Euler-lagrangian operator to get system of determining equations. The multipliers approach helps us to find exact solutions of our system of differential equations. We also investigated that Equation (2) is nonlinear selfadjointness. When studying the translation generators, we discovered that the conservation laws generated using the conservation laws theorem[10], which removes the necessity for integrating functions, result in some conservation laws that do not give additional information (trivial conservation laws). Using the multipliers technique, we were able to generate nontrivial conservation laws using integral formulae, which improved our knowledge of the system's conservation properties.

Author contributions

Conceptualization, AS and ZA; methodology, AS; software, QM; formal analysis, ZA; investigation, AS; resources, ZA; data curation, AS; writing—original draft preparation, AS and QM; writing—review and editing, AS and QM; visualization, AS; supervision, AS; funding acquisition, ZA and QM. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

The authors declare no conflict of interest.

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