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Original Research Article

Norm of the Hermite-Fejér interpolative operator with derivatives of variable order

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ABSTRACT: A new definition of a variable-order derivative is given. It is based on the interpolation of integer-order differentiation operators. An interpolation operator of the Hermite-Fejér type is built to jointly interpolate the function and its derivative of variable order. The upper estimate of the norm of this operator is obtained. This norm has been shown to be limited.

KEYWORDS: variable order derivative; Hermite-Fejér interpolation; Sobolev space; norm estimation

1. Introduction & observation

In the study of Fedotov[1], a new definition of a variable-order derivative was used to state a variable-order integro-differential equation and to prove the convergence of the quadrature-difference method for its solution. In another study by Fedotov[2], the norm of the Hermite-Fejér interpolative operator with integer-order derivatives is estimated. To develop the theory and practical use of the defined variable-order derivatives, an estimation of the norm of the Hermite-Fejér interpolative operator with derivatives of variable order is needed. Here, the results of Fedotov's[2] paper are generalized to the variable order derivatives defined in another of his paper[1].

2. Definition of the fractional order derivative

For the following, let us denote N the set of positive integers (we write N_0 if N is supplemented with the zero), Z the set of all integers, R the set of real numbers. Now let us fix $s \in R$ and denote H^s Sobolev space of order s, i.e., the closure of all 2π-periodic complex-valued functions of one variable with respect to the norm:

$$\|x\|_{H^s} = \left(\sum_{l\in Z} |\hat{x}(l)|^2 \underline{l}^{2s}\right)^{\frac{1}{2}}, \qquad \underline{l} = \begin{cases} |l|, & l \neq 0, \\ 1, & l = 0, \end{cases} \quad l \in Z.$$

where $\hat{x}(l) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(\tau)\bar{e}_l(\tau)d\tau,\ l \in Z$, are Fourier coefficients of the function $x \in H^s$ over the system of functions $e_l(\tau) = e^{il\tau},\ l \in Z$.

Hereinafter, we will suppose, that $s > \frac{1}{2}$, which[3] is sufficient for the embedding H^s in the space of continuous functions and H^{s+1} in the space of functions, which derivatives are continuous.

Let us assume that the function $\alpha(t), 0 \leq \alpha(t) \leq 1, t \in [-\pi, \pi)$, belongs to the space H^s and define for the functions of H^{s+1} a derivative of order α,

$$x^{(\alpha(t))}(t)=\big(1-\alpha(t)\big)x(t)+\alpha(t)x'(t), x\in H^{s+1}, t\in[-\pi,\pi).$$

In case $\alpha(t)\equiv 0, t\in[-\pi,\pi), x^{(\alpha(t))}(t)$ coincides with the function $x(t)$; in case $\alpha(t)\equiv 1, t\in[-\pi,\pi), x^{(\alpha(t))}(t)$ coincides with the derivative $x'(t)$. So, this definition should be considered as correct one. In the following we suppose that $0<\alpha(t)\le 1,\ t\in[-\pi,\pi)$.

Let us fix $n\in N_0$, define the grid of equidistant nodes

$$t_k=\frac{2\pi k}{2n+1}, |k|\le n \tag{1}$$

and denote $P_{\alpha,n}: H^{s+1}\to H^{s+1}$,

$$(P_{\alpha,n}x)(\tau)=\sum_{|k|\le n}(x(t_k)+i\frac{x^{(\alpha(t_k))}(t_k)-\big(1-\alpha(t_k)\big)x(t_k)}{\alpha(t_k)}(1-e_l(\tau-t_k)))\xi_n(\tau,t_k),$$

Hermite-Fejér interpolation operator which assigns, to each function $x\in H^{s+1}$, the trigonometric interpolation polynomial $P_{\alpha,n}x\in H^{s+1}$ at the Equation (1) of multiplicity α. Here

$$\xi_n(\tau,t_k)=\left(\frac{\sin((2n+1)(\tau-t_k)/2)}{(2n+1)\sin((\tau-t_k)/2}\right)^2=\frac{1}{(2n+1)^2}\sum_{|l|\le 2n}(2n+1-|l|)e_l(\tau-t_k), |k|\le n,$$

are normalized Fejér kernels at the Equation (1).

Theorem 1. *For all $s\in R, s>1/2$, and $n\in N$, the following estimate is valid:* $\|P_{\alpha,n}\|_{H^{s+1}\to H^{s+1}}\le 2\sqrt{\zeta(2s)}$, *where $\zeta(t)=\sum_{j=1}^{\infty}j^{-t}$ is the Riemann zeta function bounded and decreasing for $t>1$.*

Proof. Let us fix $s\in R, s>1/2$, and $n\in N$ and take a function $x\in H^{s+1}$. We construct the polynomial

$$(P_{\alpha,n}x)(\tau)=\sum_{|k|\le n}(x(t_k)+i\frac{x^{(\alpha(t_k))}(t_k)-\big(1-\alpha(t_k)\big)x(t_k)}{\alpha(t_k)}(1-e_l(\tau-t_k)))\xi_n(\tau,t_k)$$

and replace in it the Fejér kernels and the value of the function x and its derivative $x^{(\alpha(\tau))}(\tau)$ at the Equation (1) by their Fourier series

$$x(\tau)=\textstyle\sum_{q\in Z}\hat{x}(q)e_q(\tau), x^{(\alpha(\tau))}(\tau)=\sum_{q\in Z}\hat{x}(q)e_q(\tau)(1+\alpha(\tau)(iq-1)),$$

$$(P_{\alpha,n}x)(\tau)=\frac{1}{(2n+1)^2}\sum_{|k|\le n}(\sum_{q\in Z}\hat{x}(q)e_q(t_k)+\sum_{q\in Z}q\hat{x}(q)e_q(t_k)(e_l(\tau-t_k)-1))$$
$$\times\sum_{|l|\le 2n}(2n+1-|l|)e_l(\tau-t_k)$$
$$=\frac{1}{(2n+1)^2}(\sum_{|l|\le 2n}(2n+1-|l|)e_l(\tau)\sum_{q\in Z}(1-q)\hat{x}(q)\sum_{|k|\le n}e_{q-l}(t_k)$$
$$+\sum_{|l|\le 2n}(2n+1-|l|)e_{l+1}(\tau)\sum_{q\in Z}q\hat{x}(q)\sum_{|k|\le n}e_{q-1-l}(t_k)).$$

Since

$$\sum_{|k|\le n}e_{q-l}(t_k)=\begin{cases}2n+1, & q-l=\mu(2n+1),\\ 0, & q-l\ne\mu(2n+1),\end{cases}\mu\in Z,$$

$$\sum_{|k|\le n}e_{q-l-1}(t_k)=\begin{cases}2n+1, & q-1-l=\mu(2n+1),\\ 0, & q-1-l\ne\mu(2n+1),\end{cases}\mu\in Z,$$

then

$$(P_{\alpha,n}x)(\tau)=\frac{1}{2n+1}\sum_{|l|\le 2n}(2n+1-|l|)e_l(\tau)\sum_{\mu\in Z}\hat{x}\big(l+\mu(2n+1)\big)(1-l-\mu(2n+1))$$

$$+\frac{1}{2n+1}\sum_{|l|\le 2n}(2n+1-|l|)e_{l+1}(\tau)\sum_{\mu\in Z}\hat{x}\big(1+l+\mu(2n+1)\big)(1+l+\mu(2n+1))$$

$$=-\frac{1}{2n+1}\sum_{|l|\le 2n}(2n+1-|l|)e_{l}(\tau)\sum_{\mu\in Z}\hat{x}\big(l+\mu(2n+1)\big)(l-1+\mu(2n+1))$$

$$+\frac{1}{2n+1}\sum_{|l-1|\le 2n}(2n+1-|l-1|)e_{l}(\tau)\sum_{\mu\in Z}\hat{x}\big(l+\mu(2n+1)\big)\big(l+\mu(2n+1)\big).$$

Denoting

$$\mathrm{sgn}l=|l|-|l-1|=\begin{cases}1, & l\ge 1,\\ -1, & l\le 0,\end{cases}$$

we obtain, that Fourier coefficients of the polynomial $P_{\alpha,n}x$ are equal to

$$\big(\widehat{P_{\alpha,n}x}\big)(l)=\textstyle\sum_{\mu\in Z}\hat{x}\big(l+\mu(2n+1)\big)(1+\mu\mathrm{sgn}l),\ -2n\le l\le 2n+1.$$

Now, according to the definition of the norm in H^{s+1}, we have

$$\|P_{\alpha,n}x\|_{H^{s+1}}^2=\sum_{l=-2n}^{2n+1}\left|\sum_{\mu\in Z}\hat{x}\big(l+\mu(2n+1)\big)(1+\mu sgnl)\right|^2\underline{l}^{2s+2}$$

$$=\sum_{l=-2n}^{2n+1}\left|\sum_{\mu\in Z}\hat{x}\big(l+\mu(2n+1)\big)\frac{\underline{(l+\mu(2n+1))}^{s+1}}{\underline{(l+\mu(2n+1))}^{s+1}}(1+\mu sgnl)\right|^2\underline{l}^{2s+2}$$

$$\le\sum_{l=-2n}^{2n+1}\sum_{\mu\in Z}\left|\hat{x}\big(l+\mu(2n+1)\big)\right|^2\underline{(l+\mu(2n+1))}^{2s+2}\sum_{\mu\in Z}\frac{(1+\mu sgnl)^2\underline{l}^{2s+2}}{\underline{(l+\mu(2n+1))}^{2s+2}}$$

$$\le 2\|x\|_{H^{s+1}}^2\max_{-2n\le l\le 2n+1}\sum_{\mu\in Z}\frac{(1+\mu sgnl)^2\underline{l}^{2s+2}}{\underline{(l+\mu(2n+1))}^{2s+2}}.$$

The function

$$y(l)=\sum_{\mu\in Z}\frac{(1+\mu sgnl)^2\underline{l}^{2s+2}}{\underline{(l+\mu(2n+1))}^{2s+2}}=1+\sum_{\mu=1}^{\infty}\left(\frac{(1+\mu sgnl)^2\underline{l}^{2s+2}}{\underline{(l+\mu(2n+1))}^{2s+2}}+\frac{(1-\mu sgnl)^2\underline{l}^{2s+2}}{\underline{(l-\mu(2n+1))}^{2s+2}}\right),l\in Z,$$

is even one, so

$$\max_{-2n\le l\le 2n+1}y(l)=\max_{0\le l\le 2n+1}y(l).$$

Let us estimate the values of the function $y(l),\ 0\le l\le 2n+1$,

$$y(l)=1+\sum_{\mu=1}^{\infty}\left(\frac{(1+\mu sgnl)^2\underline{l}^{2s+2}}{\underline{(l+\mu(2n+1))}^{2s+2}}+\frac{(1-\mu sgnl)^2\underline{l}^{2s+2}}{\underline{(l-\mu(2n+1))}^{2s+2}}\right)$$

$$=1+\sum_{\mu=1}^{\infty}\frac{(1+\mu)^2\underline{l}^{2s+2}}{\underline{l}^{2s+2}(1+\mu\frac{2n+1}{l})^{2s+2}}+\sum_{\mu=2}^{\infty}\frac{(1-\mu)^2\underline{l}^{2s+2}}{\underline{l}^{2s+2}(1-\mu\frac{2n+1}{l})^{2s+2}}$$

$$\le 1+\sum_{\mu=2}^{\infty}\mu^{-2s}+\sum_{\mu=1}^{\infty}\mu^{-2s}=2\zeta(2s).$$

Finally, we have

$$\max_{-2n\le l\le 2n+1}\sum_{\mu\in Z}\frac{(1+\mu sgnl)^2\underline{l}^{2s+2}}{\underline{(l+\mu(2n+1))}^{2s+2}}=2\zeta(2s),$$

and it is achieved at $l=2n+1$. Theorem is proven. □

Conflict of interest

The author declares no conflict of interest.

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