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Article

A novel decision-making approach based on interval-valued T-spherical fuzzy information with applications

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Abstract: Multi-attribute group decision-making (MAGDM) is a very significant technique for selecting an alternative from the provided list. But the major problem is dealing with the information fusion during the information. Aczel-Alsina t-norm (AATN) and Aczel-Alsina t-conorm (AATCN) are the most generalized and flexible t-norm (TN) and t-conorm (TCN), which are used for information processing. Moreover, the interval-valued T-spherical fuzzy set (IVTSFS) is the latest framework to cover the maximum information from the real-life scenarios. Hence, the major contribution of this paper is to deal with the information during the MAGDM process by introducing new aggregation operators (AOs). Consequently, the interval-valued T-spherical fuzzy (IVTSF), Aczel-Alsina weighted averaging (IVTSFAAWA), IVTSF Aczel-Alsina (IVTSFAA) ordered weighted averaging (IVTSFAAOWA), IVTSFAA weighted geometric (IVTSFAAWG), IVTSFAA ordered weighted geometric (IVTSFAAOWG), and IVTSFAA hybrid weighted geometric (IVTSFAAHWG) operators are developed. It is shown that the proposed operators are valid and the results obtained are reliable by discussing some basic properties. To justify the developed AOs, an example of the MAGDM is discussed. The sensitivity of these AOs is observed keeping in view of the variable parameter. To show the importance of the newly developed theory, a comparison of the proposed AOs is established with already existing operators.

Keywords: T-spherical fuzzy set; interval-valued T-spherical fuzzy set; Aczel-Alsina t-norm; decision making

1. Introduction

Improbability and incompleteness are constant problems when interpreting information. For instance, the concept of crisp sets holds that an object either belongs to or is not part of a specific scenario. However, many things in the true world could not be described in such detail. Zadeh [1] referred to this concept as FS and defined the membership of an element by membership degree (MD) in the range $[0, 1]$. By applying the FS, the ambiguity and the uncertainty reduce while describing any uncertain situation in a mathematical model. However, FS is limited due to the description of an object with the help of only MD. To describe the object with the help of MD as well as non-MD (NMD), Atanassov [2] introduced the intuitionistic FS (IFS). In IFS, the sum of the MD and NMD should be in the unit interval. IFS can give a better description of an object as compared to the FS. But it was still limited because the sum of MD and NMD exceeded 1 in some cases. To extend the range of the IFS, Yager [3] established the idea of Pythagorean FS (PyFS) by imposing lenient criterion by adding the square of the MD c and NMD v such that $c^2 + v^2 \in [0, 1]$. As a result of the flexible condition for objects, PyFS can reduce information loss and cover more information from the real-life scenarios. Though, some items with, $c, v \in [0, 1]$ could not yet be described by PyFS. Yager [4] proposed the concept of q-rung orthopair FS (qROFS) utilizing any positive real number raised by MD and NMD, i.e., $c^q + v^q \in [0, 1]$, in order to get around PyFS's flaw.

The developed methods in prior studies [1–4] handled the information with the help of only two degrees, i.e., MD and NMD. But there are various types of real-life scenarios that could be described with the help of only MD and NMD. For example, to describe the scenario of voting, Cuong [5] introduced the picture FS (PFS) with an additional degree called abstinence degree (AD). He imposed the condition that the sum of the MD, AD, and NMD should be in unit interval. PFS can describe an object to the part of the real-life scenarios with more certainty as compared to the previous frameworks. But sometimes, the sum of the MD, AD, and NMD did not part of the unit interval. After noticing these restrictions, Mahmood et al. [6] enlarged the idea of PFS to spherical FS (SFS), and then to T-spherical FS (TSFS), to relax the decision-makers to assign these MD, AD, and NMD from the unit interval as their individual preferences. To cover more information than the TSFS. Ullah et al. [7] introduced the IVTSFS by describing the information as the intervals of MD, AD, and NMD. Hence, the IVTSFS is the framework that deals with the information in the form of the intervals with the minimum level of uncertainty. It also has the capability to extract the maximum information from the real-life scenarios.

In many scientific domains, MAGDM is a significant topic, particularly when choosing one among a list of options based on certain criteria. Since the beginning of MAGDM, FS theory has been a key component. Several researchers developed AOs to solve the MAGDM problems by using IFS, PyFS, qROFS, PFS, SFS, TSFS, and IVTSFS. Ali et al. [8] applied AOs to solve the MAGDM problem for the assessments for the establishment of software outsourcing partnerships. For resolving the MAGDM problem, Hung et al. [9] developed AOs for IFSs. AOs in the context of PyFS with application in MAGDM were formalized by the study of Zhang and Xu [10]. The AOs were created for the qROFS environment for use in MAGDM. Based on qROFS, Yang and Pang [11] presented the three-way MAGDM. Wei [12] provided AOs for the PFS and then utilized them in MAGDM. Ullah et al. [13] created AOs for the situation of picture-hesitant FS to address the MAGDM complications. Ullah et al. [14] introduced the AOs for TSF based on the Hamacher and used them in MAGDM. Zeng et al. [15] developed the AOs for TSFS with their application to MAGDM problems. AOs applied in the MAGDM for the selection of the solar cell selection based on Einstein's operational laws are presented by Munir et al. [16]. AOs for TSFS based on the Frank operational laws are presented by Mahnaz et al. [17] and Riaz and Farid [18] introduced AOs for PFS. Ali et al. [19] introduced AOs for qROFS, Khan et al. [20] introduced AOs for TSFS, and so on.

Additionally, the literature has a large number of AOs based on TN and TCN that were first introduced to FS theory by Deschrijver et al. [21]. AOs for IF were created by Xia et al. [22] using Archimedean TN and TCN. AOs for IFS were created by Wang and Liu [23] utilizing Einstein TN and TCN. Based on Einstein TN and TCN, Wei and Zhao [24] presented the AOs for interval-valued IFS (IVIFS). Liu [25] created interval-valued IF (IVIFS) AOs using Hamacher TN and TCN. By utilizing Dombi TN and TCN, Ullah et al. [26] established AOs for the IVTSFS. If power AOs were created by Zhang et al. [27] using Frank TCN and TN. It shows that the role of the

TNs and TCNs is very important for the information fusion and in the development of the AOs. However, another type of TN and TCN, the AATN and AATCN, were initially introduced by Aczel and Alsina [28]. It has greater flexibility than the other TN and TCN previously cited and is helpful in the fusion of information. Due to the usefulness of the AATN and AATCN Senapati et al. [29] used in the development of the AOs for IFS. Moreover, the AATN and AATCN are utilized by Hussain et al. [30] for the development of the AOs for the TSFS. Senapati et al. [31] used AATN and AATCN to develop the AOs for a hesitant fuzzy environment. Senapati et al. [32] used AATN and AATCN in an IFS environment to introduce AOs. Senapati et al. [33] used AATN and AATCN in an IVIFS environment to introduce AOs. Senapati et al. [34] used AATN and AATCN in an IFS environment to introduce AOs. Senapati et al. [35] used AATN and AATCN in an interval-valued PFS (IVPFS) environment to introduce AOs. Senapati et al. [36] used AATN and AATCN in a PyFS environment to introduce AOs. Senapati et al. [37] used AATN and AATCN in q-ROFS environments to introduce AOs. Senapati [38] used AATN and AATCN in the PFS environment to introduce AOs.

Motivations behind this article are provided as follows.

1) AATN and AATCN are the most generalized forms of the operational laws based on the parameter. AATN and AATCN convert some basic TNs and TCN in special cases of the involved parameter.
2) Moreover, Farahbod and Eftekhari [39] did the comparison between different TNs and TCNs for the classification of the information. They found that the AATN and AATCN are the most reluctant, flexible, and reliable to deal with the fuzzy information. We can infer from the investigation above that the AOs used in MAGDM are complicated by actual phenomena. The information should be handled with more reliability to get the optimal alternative in MAGDM.
3) Additionally, IVTSFS operates the information with more certainty than IFS, IVIFS, PyFS, qROFS, PFS, SFS, and TSFS. We have not yet discovered the use of AATN and AATCN for the IVTSFS framework.
4) The information obtained from the real-life scenarios by the IVTSFS should be aggregated by using some of the of the latest operators. Hence, new AOs are developed in this article based on AATN and AATCN.

Hence, we are inspired by these factors to prepare this study. The following is how this article is organized:

Section 2 includes an introduction to basic terminology that makes the article easier to understand. In section 3, we define the TSFS, the IVTSFS, Aczel-Alsina sum, product, scalar multiplication, and power operation for IVTSF values (IVTSFVs). The IVTSFAAWA, IVTSFAAOWA, and IVTSFAAHA operators are created, and their basic properties are discussed in Section 4. With the aid of the Aczel-Alsina (AA) sum and AA product, we develop the IVTSFAAWG, IVTSFAAOWG, and IVTSFAAHG operators in Section 5 and observe their properties. The application of the IVTSFAAWA and IVTSFAAWG operators to the MAGDM problem is covered in Section 6. In section 6, we also examine how the IVTSFAAWA and IVTSFAAWG operators behave for various parameter values and conduct a comparison with other AOs. In Section 7, we conclude this study.

2. Preliminaries

We shall introduce some fundamental terms in this section. The terms TSFS, IVTSFS, score function, AATN, and AATCN are defined in this section.

Definition 1. *The set* $T = \{(\rho, c(\rho), e(\rho), v(\rho) | \rho \in \mathbb{U})\}$ *is considered a TSFS where* $\mathbb{U}$ *is the universe and* c, e, v *are mappings from* $\mathbb{U}$ to $[0, 1]$. $c(\rho), e(\rho), v(\rho)$ *are MD, AD, and NMD respectively such that* $0 \leq c^r(\rho) + e^r(\rho) + v^r(\rho) \leq 1$ *where* $r \in z^+$. *Moreover,* $\pi = \sqrt[r]{1 - \left(c^r(\rho) + e^r(\rho) + v^r(\rho)\right)}$ *is known as RD of the T-spherical fuzzy value (TSFV) and* $\left(c(\rho), e(\rho), v(\rho)\right)$ *is called a TSF value (TSFV).*

Definition 2. *The set* $T = \{(\rho, c(\rho), e(\rho), v(\rho) | \rho \in \mathbb{U})\}$ *is considered a IVTSFS where* $\mathbb{U}$ *is the universe and* c, e, v *are mappings from* $\mathbb{U}$ *to* $[0, 1]$ *in the form of intervals. Moreover,* $c(\rho) = [c^l, c^u], e(\rho) = [e^l, e^u], v(\rho) = [v^l, v^u]$ *are MD, AD, and NMD respectively such that* $0 \leq c^{ur}(\rho) + e^{ur}(\rho) + v^{ur}(\rho) \leq 1$ *where* $r \in z^+$. *Moreover,* $\pi = \left[\sqrt[r]{1 - \left(c^{ur}(\rho) + e^{ur}(\rho) + v^{ur}(\rho)\right)}, \sqrt[r]{1 - \left(c^{lr}(\rho) + e^{lr}(\rho) + v^{lr}(\rho)\right)}\right]$ *is known as RD of the IVTSF value (IVTSFV).*

Definition 3. *Let* $sc(\alpha)$ *indicate the score value of IVTSFV* α. *Then score value can be specified as*

$$sc(\alpha) = \frac{(c^l)^r\left(1-(e^l)^r-(v^{g})^r\right)+(c^u)^r(1-(e^u)^r-(v^u)^r)}{3}.$$

The AATN and AATCN is the most flexible TN and TCN defined by Aczél and Alsina [28]. The definition of the AATN and TCN is provided as follows.

Definition 4. *The AATN is defined as*

$$T_B^M(\alpha, \beta) = \begin{cases} T_c(\alpha, \beta) & if\ \Gamma = 0 \\ \min(\alpha, \beta) & if\ \Gamma \to \infty \\ e^{-\left((-\ln\alpha))^{\Gamma} + (-\ln\beta)^{\Gamma}\right)^{\frac{1}{\Gamma}}} & \text{otherwise} \end{cases}.$$

Furthermore, the AATCN is defined by

$$S_B^M(\alpha, \beta) = \begin{cases} T_C(\alpha, \beta) & if\ \Gamma = 0 \\ \max(\alpha, \beta) & if\ \Gamma \to \infty \\ 1 - e^{-\left((-\ln(1-\alpha))^{\Gamma} + (-\ln(1-\beta))^{\Gamma}\right)^{1/\Gamma}} \end{cases}.$$

where $\Gamma \in [0, \infty]$.

3. Operational laws for IVTSFVS based on AATN and AATCN

This section deals with the introduction of some operations for IVTSFVs based on the AATN and AATCN. The AA sum $X \oplus_{AA} Y$ and product $X \otimes_{AA} Y$ between two IVTSFVs $X = (c_X, e_X, v_X)$ and $Y = (c_Y, e_Y, v_Y)$ are defined first as follows.

$$X \oplus_{AA} Y = \{(S(c_X, c_Y), T(e_X, e_Y), T(v_X, v_Y))\}$$
$$X \otimes_{AA} Y = \{(T(c_X, c_Y), S(e_X, e_Y), S(v_X, v_Y))\}$$

where T and S, respectively, are AATN and AATCN. As a result, we present Definition 5 and Theorem 1 in the following.

Definition 5. *Consider two IVTSFVs are* $T_1 = \left([c_1^l, c_1^u], [e_1^l, e_1^u], [v_1^l, v_1^u]\right)$ *and* $T_2 = \left([c_2^l, c_2^u], [e_2^l, e_2^u], [v_2^l, v_2^u]\right)$. *The AA sum and product are defined as follows.*

$$T_1 \oplus_{AA} T_2 = \left(\begin{array}{c} \left[\sqrt[r]{1-e^{-\left(\left(-\ln\left(1-c_1^{lr}\right)\right)^{\Gamma}+\left(-\ln\left(1-c_1^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \sqrt[r]{1-e^{-((-\ln(1-c_1^{ur}))^{\Gamma}+(-\ln(1-c_1^{ur}))^{\Gamma})^{\frac{1}{\Gamma}}}}\right], \\ \left[e^{-\left(\left(-\ln\left(e_1^{lr}\right)\right)^{\Gamma}+\left(-\ln\left(e_2^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\left(-\ln\left(e_1^{ur}\right)\right)^{\Gamma}+\left(-1\ \left(e_2^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}\right], \\ \left[e^{-\left(\left(-\ln\left(v_1^{lr}\right)\right)^{\Gamma}+\left(-\ln\left(v_2^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-((-\ln(v_1^{ur}))^{\Gamma}+(-1\ (v_2^{ur}))^{\Gamma})^{\frac{1}{\Gamma}}}\right] \end{array} \right), \Gamma \geq 1 \quad (1)$$

$$T_1 \otimes_{AA} T_2 = \left(\begin{array}{c} \left[e^{-\left(\left(-\ln\left(c_1^{lr}\right)\right)^{\Gamma}+\left(-\ln\left(c_2^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-((-\ln(c_1^{ur}))^{\Gamma}+(-\ln(c_2^{ur}))^{\Gamma})^{\frac{1}{\Gamma}}}\right] \\ \left[\sqrt[r]{1-e^{-\left(\left(-\ln\left(1-e_1^{lr}\right)\right)^{\Gamma}+\left(-\ln\left(1-e_2^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \sqrt[r]{1-e^{-((-\ln(1-e_1^{ur}))^{\Gamma}+(-\ln(1-e_2^{ur}))^{\Gamma})^{\frac{1}{\Gamma}}}}\right] \\ \left[\sqrt[r]{1-e^{-\left(\left(-\ln\left(1-v_1^{lr}\right)\right)^{\Gamma}+\left(-\ln\left(1-v_2^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \sqrt[r]{1-e^{-((-1\ (1-v_1^{ur}))^{\Gamma}+(-\ln(1-v_2^{ur}))^{\Gamma})^{\frac{1}{\Gamma}}}}\right] \end{array} \right) \quad (2)$$

We may be able to specify additional operations using the AATN and AATCN. The divisions and powers of the IVTSFV $T = ([c^l, c^u], [e^l, e^u], [v^l, v^u])$ *can be used to describe the operations for any real number* z.

$$zT = \left(\begin{array}{c} \left[\sqrt[r]{1-e^{-\left(z\left(-\ln\left(1-c^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \sqrt[r]{1-e^{-(z(-\ln(1-c^{ur}))^{\Gamma})^{\frac{1}{\Gamma}}}}\right]' \\ \left[e^{-\left(z\left(-\ln\left(e^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-(z(-\ln(e^{ur}))^{\Gamma})^{\frac{1}{\Gamma}}}\right], \left[e^{-\left(z\left(-\ln\left(v^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-(z(-\ln(v^{ur}))^{\Gamma})^{\frac{1}{\Gamma}}}\right] \end{array} \right), \Gamma \geq 1 \quad (3)$$

$$T^z = \left(\begin{array}{c} \left[e^{-\left(z\left(-\ln\left(c^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-(z(-\ln(c^{ur}))^{\Gamma})^{\frac{1}{\Gamma}}}\right] \\ \left[\sqrt[r]{1-e^{-\left(z\left(-\ln\left(1-e^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \sqrt[r]{1-e^{-(z(-\ln(1-e^{ur}))^{\Gamma})^{\frac{1}{\Gamma}}}}\right] \\ \left[\sqrt[r]{1-e^{-\left(z\left(-\ln\left(1-v^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \sqrt[r]{1-e^{-(z(-\ln(1-v^{ur}))^{\Gamma})^{\frac{1}{\Gamma}}}}\right] \end{array} \right), \Gamma \geq 1 \quad (4)$$

We then present Theorem 1 to demonstrate some characteristics of the AA sum and product between IVTSFVs T_1 and T_2, utilizing operations specified in Equations (1)–(4).

Theorem 1. *Let* $T = ([c^l, c^u], [e^l, e^u], [v^l, v^u])$, $T_1 = ([c_1^l, c_1^u], [e_1^l, e_1^u], [v_1^l, v_1^u])$ *and* $T_2 = ([c_2^l, c_2^u], [e_2^l, e_2^u], [v_2^l, v_2^u])$ *be three IVTSFVs*, $a > 0$, $a_1 > 0$ and $a_2 > 0$ *be real numbers. Then we can write:*

$$T_1 \oplus_{AA} T_2 = T_2 \oplus_{AA} T_1$$
$$T_1 \otimes_{AA} T_2 = T_2 \otimes_{AA} T_1$$
$$a(T_1 \oplus_{AA} T_2 = aT_1 \oplus_{AA} aT_2)$$
$$(a_1 + a_2)T = a_1 T \oplus_{AA} a_2 T$$

$$(T_1 \otimes_{AA} T_2)^a = T_1^a \otimes_{AA} T_2^a$$
$$T^{a_1} \otimes_{AA} T^{a_2} = T^{(a_1+a_2)}$$

Proof of Theorem 1. The proofs of these properties are skipped to avoid the extra length.

4. The proposed averaging AOS

We define the IVTSFAAWA, IVTSFAAOWA, and IVTSFAAHWA operators in this section and observe some of their fundamental characteristics. Keep in mind that we'll be using λ for the weight vector $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots . \lambda_n)^T$ with $\lambda_{g} \geq 0$ and $\sum_{g}^{n} \lambda_{g} = 1$. Further, $(g = 1, 2, 3, \ldots, n)$ will be used for indexing purposes.

Definition 6. *Let* $T_{g} = ([c_{g}^{l}, c_{g}^{u}], [e_{g}^{l}, e_{g}^{u}], [v_{g}^{l}, v_{g}^{u}])$ *be the set of IVTSFVs. Then, a IVTSFAAWA operator is a mapping* $IVTSFAAWA\colon T^n \rightarrow T$ *such that*

$$\text{IVTSFAAWA}(T_1, T_2, \ldots . T_n) = \bigoplus_{g=1}^{n}{}_{AA}\left(\lambda_{g} T_{g}\right).$$

In Theorem 2, we demonstrate that the value acquired by aggregation is an IVTSFV as well.

Theorem 2. *Let* $T_{g} = ([c_{g}^{l}, c_{g}^{u}], [e_{g}^{l}, e_{g}^{u}], [v_{g}^{l}, v_{g}^{u}])$ *represent the set of IVTSFVs. Consequently, the value obtained by the IVTSFAAWA operator following aggregate is also an IVTSFV, and*

$$\text{IVTSFAAWA}(T_1, T_2, \ldots . T_n)$$
$$= \left(\begin{array}{c} \left[\sqrt[r]{1 - e^{-\left(\sum_{g}^{n} \lambda_{g} \left(-\ln\left(1 - c_{g}^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \sqrt[r]{1 - e^{-\left(\sum_{g}^{n} \lambda_{g} \left(-\ln\left(1 - c_{g}^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}} \right], \\ \left[e^{-\left(\sum_{g}^{n} \lambda_{g} \left(-\ln\left(e_{g}^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\sum_{g}^{n} \lambda_{g} \left(-\ln\left(e_{g}^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right], \left[e^{-\left(\sum_{g}^{n} \lambda_{g} \left(-\ln\left(v_{g}^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\sum_{g}^{n} \lambda_{g} \left(-\ln\left(v_{g}^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right] \end{array} \right) \tag{5}$$

Proof of Theorem 2. By using the induction approach, we shall prove Equation (5) as follows:

We can write $n = 2$ as

$$\text{IVTSFAAWA}(T_1, T_2) = \lambda_1 T_1 \oplus_{AA} \lambda_2 T_2$$

$$= \left(\begin{array}{c} \left[\sqrt[r]{1 - e^{-\left(\lambda_1 \left(-\ln\left(1 - c_{g}^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \sqrt[r]{1 - e^{-\left(\lambda_1 \left(-\ln\left(1 - c_{g}^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}} \right] \\ \left[e^{-\left(\lambda_1 \left(-\ln\left(e_{g}^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\lambda_1 \left(-\ln\left(e_{g}^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right] \\ \left[e^{-\left(\lambda_1 \left(-\ln\left(v_{g}^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\lambda_1 \left(-\ln\left(v_{g}^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right] \end{array} \right) \oplus_{AA} \left(\begin{array}{c} \left[\sqrt[r]{1 - e^{-\left(\lambda_2 \left(-\ln\left(1 - c_{g}^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \sqrt[r]{1 - e^{-\left(\lambda_2 \left(-\ln\left(1 - c_{g}^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}} \right] \\ \left[e^{-\left(\lambda_2 \left(-\ln\left(e_{g}^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\lambda_2 \left(-\ln\left(e_{g}^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right] \\ \left[e^{-\left(\lambda_2 \left(-\ln\left(v_{g}^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\lambda_2 \left(-\ln\left(v_{g}^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right] \end{array} \right)$$

$$= \left(\begin{array}{c} \left[\sqrt[r]{1 - e^{-\left(\sum_{g}^{2} \lambda_g \left(-\ln\left(1 - c_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \sqrt[r]{1 - e^{-\left(\sum_{g}^{2} \lambda_g \left(-\ln\left(1 - c_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}} \right]', \\ \left[e^{-\left(\sum_{g}^{2} \lambda_g \left(-\ln\left(e_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\sum_{g}^{2} \lambda_g \left(-\ln\left(e_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right], \left[e^{-\left(\sum_{g}^{2} \lambda_g \left(-\ln\left(v_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\sum_{g}^{2} \lambda_g \left(-\ln\left(v_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right] \end{array} \right)$$

Consequently, for $n = 2$, the result of IVTSFAAWA is also an IVTSFV. Assuming that Equation (5) holds for $n = k$, we now get

$$\text{IVTSFAAWA}(T_1, T_2, \ldots T_k) = \left(\begin{array}{c} \left[\sqrt[r]{1 - e^{-\left(\sum_{g}^{k} \lambda_g \left(-\ln\left(1 - c_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \sqrt[r]{1 - e^{-\left(\sum_{g}^{k} \lambda_g \left(-\ln\left(1 - c_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}} \right], \\ \left[e^{-\left(\sum_{g}^{k} \lambda_g \left(-\ln\left(e_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\sum_{g}^{k} \lambda_g \left(-\ln\left(e_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right], \left[e^{-\left(\sum_{g}^{k} \lambda_g \left(-\ln\left(v_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\sum_{g}^{k} \lambda_g \left(-\ln\left(v_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right] \end{array} \right)$$

We must demonstrate that Equation (5). For $n = k + 1$, the following is true:

$$\text{IVTSFAAWA}(T_1, T_2, \ldots . T_k, T_{k+1}) = \left(\lambda_g T_g\right) \oplus_{AA} \left(\lambda_{k+1} T_{k+1}\right)$$

$$= \left(\begin{array}{c} \left[\sqrt[r]{1 - e^{-\left(\sum_{g}^{k} \lambda_g \left(-\ln\left(1 - c_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \sqrt[r]{1 - e^{-\left(\sum_{g}^{k} \lambda_g \left(-\ln\left(1 - c_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}} \right] \\ \left[e^{-\left(\sum_{g}^{k} \lambda_g \left(-\ln\left(e_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\sum_{g}^{k} \lambda_g \left(-\ln\left(e_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right] \\ \left[e^{-\left(\sum_{g}^{k} \lambda_g \left(-\ln\left(v_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\sum_{g}^{k} \lambda_g \left(-\ln\left(v_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right] \end{array} \right) \oplus_{AA} \left(\begin{array}{c} \left[\begin{array}{c} \sqrt[r]{1 - e^{-\left(\lambda_{k+1} \left(-\ln\left(1 - c_{k+1}^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \\ \sqrt[r]{1 - e^{-\left(\lambda_{k+1} \left(-\ln\left(1 - c_{k+1}^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}} \end{array} \right] \\ \left[e^{-\left(\lambda_{k+1} \left(-\ln\left(e_{k+1}^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\lambda_{k+1} \left(-\ln\left(e_{k+1}^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right] \\ \left[e^{-\left(\lambda_{k+1} \left(-\ln\left(v_{k+1}^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\lambda_{k+1} \left(-\ln\left(v_{k+1}^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right] \end{array} \right)$$

$$= \left(\begin{array}{c} \left[\sqrt[r]{1 - e^{-\left(\sum_{g}^{k+1} \lambda_g \left(-\ln\left(1 - c_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \sqrt[r]{1 - e^{-\left(\sum_{g}^{k+1} \lambda_g \left(-\ln\left(1 - c_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}} \right], \\ \left[e^{-\left(\sum_{g}^{k+1} \lambda_g \left(-\ln\left(e_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\sum_{g}^{k+1} \lambda_g \left(-\ln\left(e_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right], \left[e^{-\left(\sum_{g}^{k+2} \lambda_g \left(-\ln\left(v_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\sum_{g}^{k+2} \lambda_g \left(-\ln\left(v_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right] \end{array} \right)$$

which is again an IVTSFV. So, the proof is finished.

Every AO has certain fundamental characteristics, such as boundedness, monotonicity, and idempotency. We demonstrate the IVTSFAAWA operator's idempotency, boundedness, and monotonicity in Theorems 3 and 5, respectively, in the following.

Theorem 3. *(Idempotency) let* $T_g = \left(\left[c_g^l, c_g^u\right], \left[e_g^l, e_g^u\right], \left[v_g^l, v_g^u\right]\right)$ *be the set of IVTSFVs such that* $T_g = T$ *and* λ *be the weight vector. Then* $IVTSFAAWA(T_1, T_2, \ldots T_n) = T$.

Proof of Theorem 3. Since $T_g = \left(\left[c_g^l, c_g^u\right], \left[e_g^l, e_g^u\right], \left[v_g^l, v_g^u\right]\right) = T$ we can acquire

$$\text{IVTSFAAWA}(T_1, T_2, \ldots T_n)$$

$$= \left(\begin{array}{c} \left[\sqrt[r]{1 - e^{-\left(\left(-ln\left(1-c_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \sqrt[r]{1 - e^{-\left(\left(-l\ \ \left(1-c_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}} \right], \\ \left[e^{-\left(\left(-ln\left(e_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\left(-l\ \ \left(e_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right], \left[e^{-\left(\left(-ln\left(v_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\left(-ln\left(v_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right] \end{array} \right)$$

$$= \left(\begin{array}{c} \left[\sqrt[r]{1 - e^{-\left(\left(-\ln\left(1-c_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \sqrt[r]{1 - e^{-\left(\left(-\ln\left(1-c_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}} \right], \\ \left[e^{-\left(\left(-\ln\left(e_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\left(-1\ \ \left(e_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right], \left[e^{-\left(\left(-\ln\left(v_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\left(-1\ \ \left(v_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right] \end{array} \right)$$

$$= \left(\begin{array}{c} \left[\sqrt[r]{1 - e^{-\left(\left(-\ln\left(1-c_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \sqrt[r]{1 - e^{-\left(\left(-\ln\left(1-c_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}} \right] \\ \left[e^{-\left(\left(-\ln\left(e_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\left(-1\ \ \left(e_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right], \left[e^{-\left(\left(-1\ \ \left(v_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\left(-\ln\left(v_g^{ur}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}} \right] \end{array} \right) = T$$

Because of this, the proof is finalized.

Theorem 4. *(Boundedness): Let* $T_g = \left(\left[c_g^l, c_g^u\right], \left[e_g^l, e_g^u\right], \left[v_g^l, v_g^u\right]\right)$ *be the set of IVTSFVs. Let* $T^- = min\{T_g\}$ and $T^+ = max\{T_g\}$. *Then* $T^- \leq IVTSFAAWA(T_1, T_2, \ldots, T_n) \leq T^+$.

Proof of Theorem 4. Let $T^- = min\{T_g\}$ and $T^+ = max\{T_g\}$ be the smallest and the greatest TSFVs respectively. Then we have $c_g^- = min\{c_g\}$, $c_g^+ = max\{c_g\}$. Similarly, $e_g^- = min\{e_g\}$, $e_g^+ = max\{e_g\}$ and $v_g^- = min\{v_g\}$, $v_g^+ = max\{v_g\}$. Hence

$$\sqrt[r]{1 - e^{-\left(\sum_g^n \lambda_g\left(-\ln\left(1-c_{g-r}^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}} \leq \sqrt[r]{1 - e^{-\left(\sum_g^n \lambda_g\left(-\ln\left(1-c_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}} \leq \sqrt[r]{1 - e^{-\left(\sum_g^n \lambda_g\left(-\ln\left(1-c_{g+r}^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}.$$

Similarly,

$$\sqrt[r]{e^{-\left(\sum_g^n \lambda_g\left(-\ln\left(e_{g-r}^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}} \leq \sqrt[r]{e^{-\left(\sum_g^n \lambda_g\left(-\ln\left(e_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}} \leq \sqrt[r]{e^{-\left(\sum_g^n \lambda_g\left(-\ln\left(e_{g+r}^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}.$$

And

$$\sqrt[r]{e^{-\left(\sum_g^n \lambda_g\left(-\ln\left(v_{g-r}^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}} \leq \sqrt[r]{e^{-\left(\sum_g^n \lambda_g\left(-\ln\left(v_g^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}} \leq \sqrt[r]{e^{-\left(\sum_g^n \lambda_g\left(-\ln\left(v_{g+r}^{lr}\right)\right)^{\Gamma}\right)^{\frac{1}{\Gamma}}}}.$$

Similarly, the upper value of the interval is also provable.

Therefore

$$T^- \leq \text{IVTSFAAWA}(T_1, T_2, \ldots T_n) \leq T^+.$$

The monotonicity of the IVTSFAAWA operators is stated as follows.

Theorem 5. *Let* $T_g = \left(\left[c_g^l, c_g^u\right], \left[e_g^l, e_g^u\right], \left[v_g^l, v_g^u\right]\right)$ *and* $T_g^a = \left(\left[c_g^a, c_g^a\right], \left[e_g^a, e_g^a\right], \left[v_g^a, v_g^a\right]\right)$ *be two families of IVTSFVs. If* $T_g \leq T_g^a$ *for all. Then*

$$\text{IVTSFAAWA}(T_1, T_2, \ldots, T_n) \leq \text{IVTSFAAWA}(T_1^a, T_2^a, \ldots, T_n^a).$$

The IVTSFAAOWA operator is now developed as follows.

Definition 7. *Let* $T_{g} = ([c_{g}^l, c_{g}^u], [e_{g}^l, e_{g}^u], [v_{g}^l, v_{g}^u])$ *be a set of IVTSFVs. An IVTSFAAOWA operator of dimension* n *is a function* $IVTSFAAOWA: T^n \rightarrow T$ *such that*

$$\text{IVTSFAAOWA}(T_1, T_2, \ldots. T_n) = \bigoplus_{g=1}^{n}{}_{AA}\left(\lambda_{g} T, \mathbb{U}(g)\right).$$

where $\left(\mathbb{U}(1), \mathbb{U}(2), \ldots. \mathbb{U}(n)\right)$ *are the permutations of such that* $\mathbb{U}(n-1) \geq \mathbb{U}(1)$.

In Theorems 6–8, we give the IVTSFAAOWA operator's fundamental characteristics.

Theorem 6. *(Idempotency) let* $T_{g} = ([c_{g}^l, c_{g}^u], [e_{g}^l, e_{g}^u], [v_{g}^l, v_{g}^u])$ *be the collection of IVTSFVs such that* $T_{g} = T$. *Then,* $IVTSFAAOWA(T_1, T_2, \ldots. T_n) = T$.

Theorem 7. *(Boundedness) let* $T_{g} = ([c_{g}^l, c_{g}^u], [e_{g}^l, e_{g}^u], [v_{g}^l, v_{g}^u])$ *be the set of IVTSFVs and* $T^- = min\{T_{g}\}$ and $T^+ = max\{T_{g}\}$. *Then* $T^- \leq IVTSFAAWA(T_1, T_2, \ldots, T_n) \leq T^+$.

Theorem 8. *(Monotonicity) let* $T_{g} = ([c_{g}^l, c_{g}^u], [e_{g}^l, e_{g}^u], [v_{g}^l, v_{g}^u])$ *and* $T_{g}^a = ([c_{g}^a, c_{g}^a], [e_{g}^a, e_{g}^a], [v_{g}^a, v_{g}^a])$ *be the two sets of IVTSFVs and if* $T_{g} \leq T_{g}^a$ *for all* g. *Then*

$$\text{IVTSFAAWA}(T_1, T_2, \ldots, T_n) \leq \text{IVTSFAAWA}(T_1^a, T_2^a, \ldots, T_n^a).$$

Definitions 5 and 7 make it very obvious that, respectively, the IVTSFAAWA and IVTSFAAOWA operators aggregate IVTSFVs by merely weighting them and by doing so in an ordered manner. Weights so display the various aspects for both IVTSFAAWA and IVTSFAAOWA operators. No operator addresses this shortcoming. In order to address the issue, we will define the IVTSFAAHWA operator in the following manner:

Definition 8. *Let* $T_{g} = ([c_{g}^l, c_{g}^u], [e_{g}^l, e_{g}^u], [v_{g}^l, v_{g}^u])$ *be the set of IVTSFVs. An IVTSFAAHWA operator of dimension is a mapping* $IVTSFAAHWA: T^n \rightarrow T$ *defined as*

$$\text{IVTSFAAHWA}(T_1, T_2, \ldots, T_n) = \bigoplus_{g=1}^{n}{}_{AA}\left(\lambda_{g} \Gamma, \mathbb{U}(g)\right).$$

where $\Gamma_{g} = k\lambda_{g} T_{g}$ *the permutation I weighted IVTSFVs are represented by* $\left(\Gamma_{\mathbb{U}(1)}, \Gamma_{\mathbb{U}(2)}, \ldots, \Gamma_{\mathbb{U}(n)}\right)$, *and* k *is the essential balance coefficient.*

The IVTSFAAHWA operator shares many of the same characteristics as the IVTSFAAWA operator that we covered in Theorems 2–5. Nonetheless, according to Theorem 9, the IVTSFAAHWA operator is superior to the IVTSFAAOWA operator.

Theorem 9. *The developed IVTSFAAHWA operator is a generalization of the IVTSFAAWA and IVTSFAAOWA operators.*

Proof of Theorem 9. Let $\lambda = \left(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{\lambda_n}\right)$. By Definition 8, we have

$$\text{IVTSFAAHWA}(T_1, T_2, \ldots, T_n) = \lambda_1 \Gamma_{\mathbb{U}(1)} \oplus_{AA} \lambda_2 \Gamma_{\mathbb{U}(2)}, \ldots, \oplus_{AA} \lambda_n \Gamma_{\mathbb{U}(n)}) =$$

$$\frac{1}{k}\left(\Gamma_{\mathbb{U}(1)} \oplus_{AA} \Gamma_{\mathbb{U}(2)}, \ldots, \oplus_{AA} \Gamma_{\mathbb{U}(n)}\right) = \lambda_1 T_1 \oplus_{AA} \lambda_2.$$

We can also demonstrate that IVTSFAAOWA is a unique instance of IVTSFAAHWA.

5. The proposed geometric AOS

In this part, we define the IVTSFAAWG, IVTSFAAOWG, and IVTSFAAHWG operators and observe some of their fundamental characteristics.

Definition 9. *Let* $T_g = ([c_g^l, c_g^u], [e_g^l, e_g^u], [v_g^l, v_g^u])$ *be a collection of IVTSFVs and* λ *be the weight vector. Then, an IVTSFAAWG operator is a function* $IVTSFAAWG: T^n \to T$ *such that*

$$\text{IVTSFAAWG}(T_1, T_2, \dots, T_n) = \bigotimes_{g=1}^{n}{}_{AA}\left(T_g^{\lambda_g}\right).$$

Theorem 8 states that the aggregate value of any number of IVTSFVs is also an IVTSFV, and the following Theorem 10 states that this is true using the procedures described above and Definition 9.

Theorem 10. *Let* $T_g = ([c_g^l, c_g^u], [e_g^l, e_g^u], [v_g^l, v_g^u])$ *be the set of IVTSFVs. Consequently, the value that the IVTSFAAWG operator obtained after aggregate is also an IVTSFV, and*

$$\text{IVTSFAAWG}(T_1, T_2, \dots, T_n) = \bigotimes_{g=1}^{n}{}_{AA}\left(T_g^{\lambda_g}\right)$$

$$\text{IVTSFAAWG}(T_1, T_2, \dots, T_n) = \left(\begin{matrix} \left[e^{-\left(\sum_g^n \lambda_g(-ln(c_g^{lr}))^{\Gamma}\right)^{\frac{1}{\Gamma}}}, e^{-\left(\sum_g^n \lambda_g(-ln(c_g^{ur}))^{\Gamma}\right)^{\frac{1}{\Gamma}}}\right] \\ \left[\sqrt[r]{1 - e^{-\left(\sum_g^n \lambda_g(-l\ \ (1-e_g^{lr}))^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \sqrt[r]{1 - e^{-\left(\sum_g^n \lambda_g(-ln(1-e_g^{ur}))^{\Gamma}\right)^{\frac{1}{\Gamma}}}}\right] \\ \left[\sqrt[r]{1 - e^{-\left(\sum_g^n \lambda_g(-ln(1-v_g^{lr}))^{\Gamma}\right)^{\frac{1}{\Gamma}}}}, \sqrt[r]{1 - e^{-\left(\sum_g^n \lambda_g(-ln(1-v_g^{ur}))^{\Gamma}\right)^{\frac{1}{\Gamma}}}}\right] \end{matrix}\right). \quad (6)$$

The IVTSFAAWG operator's idempotency, monotonicity, and boundedness are then stated.

Theorem 11. *(Idempotency) let* $T_g = ([c_g^l, c_g^u], [e_g^l, e_g^u], [v_g^l, v_g^u])$ *be the set of IVTSFVs such that* $T_g = T$. *Then*

$$\text{IVTSFAAWG}(T_1, T_2, \dots, T_n) = \bigotimes_{g=1}^{n}{}_{AA}\left(T_g^{\lambda_g}\right) = T.$$

Theorem 12. *(Boundedness) let* $T_g = ([c_g^l, c_g^u], [e_g^l, e_g^u], [v_g^l, v_g^u])$ *be the set of IVTSFVs. Let* $T^- = min\{T_g\} T^+ = max\{T_g\}$. *Then*

$$T^- \leq \text{IVTSFAAWA}(T_1, T_2, \dots, T_n) \leq T^+.$$

Theorem 13. *(Monotonicity) let* $T_g = ([c_g^l, c_g^u], [e_g^l, e_g^u], [v_g^l, v_g^u])$ *and* $T_g^a = ([c_g^a, c_g^a], [e_g^a, e_g^a], [v_g^a, v_g^a])$ *be two sets of IVTSFVs and* $T_g \leq T_g^a$ *for all* g. *Then*

$$\text{IVTSFAAWG}(T_1, T_2, \dots, T_n) \leq IVTSFAAWG(T_1^a, T_2^a, \dots, T_n^a).$$

The IVTSFAAOWG operator is generated as follows.

Definition 10. *Let* $T_g = ([c_g^l, c_g^u], [e_g^l, e_g^u], [v_g^l, v_g^u])$ *be the set of IVTSFVs, an IVTSFAAOWA operator of dimension* n *is a function* $IVTSFAAWG: T^n \to T$ such that

$$\text{IVTSFAAOWG}(T_1, T_2, \dots, T_n) = \bigotimes_{g=1}^{n}{}_{AA}\left(T_{\mathbb{U}(g)}^{\lambda_g}\right).$$

where $(\mathbb{U}(1), \mathbb{U}(2), \ldots, \mathbb{U}(n))$ *are the permutations of* $(g = 1, 2, 3, \ldots, n)$ *such that* $\mathbb{U}(n-1) \geq \mathbb{U}(1)$.

Next, we discuss the fundamental characteristics of the IVTSFAAOWG operator in Theorems 14–16.

Theorem 14. *(Idempotency) let* $T_g = ([c_g^l, c_g^u], [e_g^l, e_g^u], [v_g^l, v_g^u])$ *be the set of IVTSFVs such that* $T_g = T$. *Then*

$$\text{IVTSFAAOWG}(T_1, T_2, \ldots, T_n) = \otimes_{g=1}^{n}{}_{AA}\left(T_{\mathbb{U}(g)}^{\lambda_g}\right) = T.$$

Theorem 15. *(Boundedness) let* $T_g = ([c_g^l, c_g^u], [e_g^l, e_g^u], [v_g^l, v_g^u])$ *be the set of IVTSFVs. Let* $T^- = min\{T_g\}T^+ = max\{T_g\}$. *Then*

$$T^- \leq \text{IVTSFAAOWG}(T_1, T_2, \ldots, T_n) \leq T^+.$$

Theorem 16. *(Monotonicity) let* $T_g = ([c_g^l, c_g^u], [e_g^l, e_g^u], [v_g^l, v_g^u])$ *and* $T_g^a = ([c_g^a, c_g^a], [e_g^a, e_g^a], [v_g^a, v_g^a])$ *be two sets of IVTSFVs and* $T_g \leq T_g^a$ *for all* g. *Then*

$$\text{IVTSFAAOWG}(T_1, T_2, \ldots, T_n) \leq \text{IVTSFAAOWG}(T_1^a, T_2^a, \ldots, T_n^a).$$

Similar to Definition 8, we describe the IVTSFAAHG operator for the aggregate of weighted IVTSFVs.

Definition 11. *Let* $T_g = ([c_g^l, c_g^u], [e_g^l, e_g^u], [v_g^l, v_g^u])$ *be the set of IVTSFVs, an IVTSFAAHWG operator of dimension n is a mapping* $IVTSFAAHWG: T^n \to T$ *such that*

$$\text{IVTSFAAHWG}(T_1, T_2, \ldots . T_n) = \otimes_{g=1}^{n}{}_{AA}\left(T_{\mathbb{U}(g)}^{\lambda_g}\right).$$

where $\Gamma_g = k\lambda_g T_g$ *for* $g = 1, 2, 3, \ldots, n$. *The permutation of the weights of IVTSFVs are represented by* $(\Gamma_{\mathbb{U}(1)}, \Gamma_{\mathbb{U}(2)}, \ldots, \Gamma_{\mathbb{U}(n)})$, *and* k *is the essential balance coefficient.*

6. Application of proposed AOS

With the use of IVTSF data, we will construct a methodology in this part to apply the suggested operators in MAGDM and resolve a numerical example. We also examine how parameter variation affects behavior, and we contrast the suggested work with earlier methods already in use.

Let $F = \{f_1, f_2, \ldots, f_j\}$ represent j selection possibilities, $Y = \{y_1, y_2, \ldots, y_j\}$ represent j attributes with weight vector, and Q_k represents k decision-makers with weight vector w possessing the same criteria as mentioned before. The IVTSF information matrix should take the form $I = (s_k)_{m\times n}$, where s is the value of an attribute y_j that the decision-maker assigns for the alternative f_j in the form of IVTSFV, i.e., $s_k = ([c_k^l, c_k^u], [e_k^l, e_k^u], [v_k^l, v_k^u])$ indicates the alternative's evaluation value, where $0 \leq c_k^{ur} + e_k^{ur} + v_k^{ur} \leq 1$.

There are two types of criteria in MAGDM. The cost attribute is one of them, while the benefit attribute is the other. By taking its complements, we should change the values of cost attributes into benefit attribute values. Therefore, we get $I^c = [B_k]_{m\times n}$ such that

$$B_k = \begin{cases} s_k & \text{for benefit attribute} \\ (s_k)^c & \text{for cost attribute} \end{cases}$$

Then, using the suggested IVTSFAAWA/IVTSFAAWG operator, we may apply the proposed technique to MADM as shown in the subsequent steps:

Step 1: Investigate the IVTSFV for the value of parameter r. Consider the r integers for which $0 \leq c_k^{ur} + e_k^{ur} + v_k^{ur} \leq 1$.

Step 2: On the given IVTSF decision information, the IVTSFAAWA/IVTSFAAWG operator is applied to find the aggregated values of each attribute individually.

Step 3: Apply the IVTSFAAWA/IVTSFAAWG operator again to aggregate the obtained values of the individually aggregated attributes collectively.

Step 4: To rank the possibilities, use the score function described in Definition 3.

Step 5: Choose the most suitable option.

Figure 1 shows the flowchart, which is helpful to understand the stepwise methodology in the following.

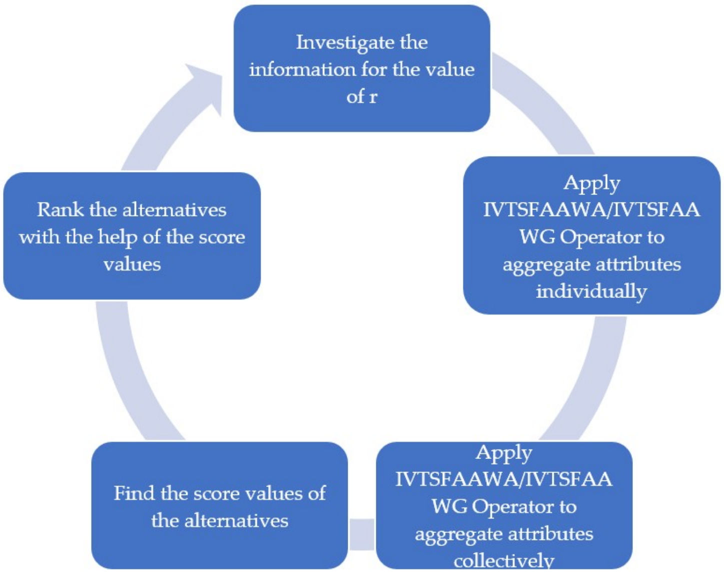

Figure 1. The stepwise method of MAGDM based on the IVTSF information.

Figure 1 shows the stepwise methodology of the MAGDM process. According to **Figure 1**, the first step is to investigate the information for the value of r. Secondly, we apply IVTSFAAWA/IVTSFAAWG operator to aggregate the attribute individually. In the third step, we aggregate the attributes collectively. Then we find the score values of the alternatives, and then we find the ranking of the alternatives finally. In the following, the example 1 describes the stepwise application of the MAGDM process.

Example 1. *An investment company wants to rank its projects and assess their success. To conduct the evaluation, they assemble a* $F_k = (1, 2, 3)$ *team of specialists. According to the weights* $w_e = (0.2, 0.1, 0.7)^t$ *are given to the designation by the experts. The specialists assess the performance using a few performance metrics (attributes), including (1) rate of return denoted by* C_1*, (2) time to completion denoted by* C_2*, (3) total cost of investment denoted by* C_3*, and (4) client feedback* C_4*. These qualities are given the following weights* $w_a = (0.2, 0.1, 0.35, 0.25)^t$ *only four projects* $a_g = (g = 1, 2, 3, 4)$ *are selected for the next round of examination.*

The following **Tables 1–3** present the decision matrices offered by experts.

Table 1. Decision matrix from expert I.

	a_1						a_2					
	c^l	c^u	e^l	e^u	v^l	v^u	c^l	c^u	e^l	e^u	v^l	v^u
C_1	0.46	0.6	0.4	0.7	0.3	0.8	0.2	0.8	0.24	0.48	0.29	0.52
C_2	0.4	0.7	0.34	0.9	0.48	0.6	0.3	0.46	0.45	0.9	0.39	0.76
C_3	0.55	0.6	0.54	0.61	0.5	0.65	0.33	0.59	0.34	0.76	0.49	0.65
C_4	0.65	0.73	0.6	0.65	0.34	0.76	0.37	0.7	0.11	0.78	0.44	0.66
	a_3						a_4					
	c^l	c^u	e^l	e^u	v^l	v^u	c^l	c^u	e^l	e^u	v^l	v^u
C_1	0.1	0.6	0.33	0.4	0.34	0.4	0.34	0.46	0.3	0.45	0.32	0.64
C_2	0.2	0.5	0.54	0.6	0.24	0.29	0.24	0.64	0.34	0.64	0.42	0.67
C_3	0.3	0.4	0.23	0.36	0.5	0.67	0.53	0.7	0.54	0.75	0.53	0.75
C_4	0.4	0.7	0.54	0.6	0.45	0.7	0.53	0.6	0.64	0.84	0.35	0.74

Table 2. Decision matrix from expert II.

	a_1						a_2					
	c^l	c^u	e^l	e^u	v^l	v^u	c^l	c^u	e^l	e^u	v^l	v^u
C_1	0.46	0.53	0.21	0.8	0.76	0.78	0.5	0.56	0.42	0.7	0.3	0.87
C_2	0.34	0.66	0.7	0.85	0.76	0.88	0.46	0.61	0.29	0.9	0.1	0.33
C_3	0.45	0.59	0.43	0.59	0.43	0.55	0.32	0.4	0.51	0.65	0.9	0.91
	c^l	c^u	e^l	e^u	v^l	v^u	c^l	c^u	e^l	e^u	v^l	v^u
C_4	0.44	0.5	0.48	0.57	0.21	0.9	0.55	0.79	0.58	0.76	0.34	0.44
	a_3						a_4					
	c^l	c^u	e^l	e^u	v^l	v^u	c^l	c^u	e^l	e^u	v^l	v^u
C_1	0.55	0.6	0.34	0.55	0.5	0.67	0.5	0.67	0.32	0.37	0.5	0.57
C_2	0.6	0.71	0.34	0.6	0.35	0.64	0.35	0.64	0.52	0.74	0.24	0.53
C_3	0.15	0.25	0.65	0.71	0.32	0.53	0.32	0.53	0.24	0.73	0.4	0.47
C_4	0.26	0.76	0.35	0.43	0.42	0.53	0.42	0.53	0.26	0.74	0.5	0.57

Table 3. Decision matrix from expert III.

	a_1						a_2					
	c^l	c^u	e^l	e^u	v^l	v^u	c^l	c^u	e^l	e^u	v^l	v^u
C_1	0.3	0.49	0.56	0.65	0.12	0.24	0.6	0.77	0.6	0.67	0.57	0.6
C_2	0.53	0.65	0.62	0.78	0.33	0.5	0.32	0.36	0.3	0.68	0.45	0.55
C_3	0.29	0.9	0.67	0.75	0.44	0.49	0.54	0.69	0.2	0.5	0.64	0.7
C_4	0.37	0.56	0.22	0.9	0.32	0.66	0.66	0.9	0.35	0.6	0.22	0.39
	a_3						a_4					
	c^l	c^u	e^l	e^u	v^l	v^u	c^l	c^u	e^l	e^u	v^l	v^u
C_1	0.55	0.6	0.34	0.55	0.5	0.67	0.5	0.67	0.32	0.37	0.5	0.57
C_2	0.6	0.71	0.34	0.6	0.35	0.64	0.35	0.64	0.52	0.74	0.24	0.53
C_3	0.15	0.25	0.65	0.71	0.32	0.53	0.32	0.53	0.24	0.73	0.4	0.47
C_4	0.26	0.76	0.35	0.43	0.42	0.53	0.42	0.53	0.26	0.74	0.5	0.57

6.1. IVTSFAAWA operator

In this sub-section, we will do MAGDM with the help of the IVTSFAAWA operator.

Step 1: Look at the r parameter's value for IVTSF information. Consider the r integers for which $0 \leq c_k^r + e_k^r + v_k^r \leq 1$ which is 3 here.

Step 2: On the IVTSF decision information, when the total IVTSF preference values are B_k and $B'_k = ([c'_k, c'_k], [e'_k, e'_k], [v'_k, v'_k])$, use the IVTSFAAWA operator as provided in **Table 4** in the following.

Table 4. Individual preference values by IVTSFAAWA.

	a_1						a_2					
	c^l	c^u	e^l	e^u	v^l	v^u	c^l	c^u	e^l	e^u	v^l	v^u
$\mathcal{C}_1$	0.3	0.49	0.56	0.65	0.12	0.24	0.6	0.77	0.6	0.67	0.57	0.6
$\mathcal{C}_2$	0.53	0.65	0.62	0.78	0.33	0.5	0.32	0.36	0.3	0.68	0.45	0.55
$\mathcal{C}_3$	0.29	0.9	0.67	0.75	0.44	0.49	0.54	0.69	0.2	0.5	0.64	0.7
$\mathcal{C}_4$	0.37	0.56	0.22	0.9	0.32	0.66	0.66	0.9	0.35	0.6	0.22	0.39
	a_3						a_4					
	c^l	c^u	e^l	e^u	v^l	v^u	c^l	c^u	e^l	e^u	v^l	v^u
$\mathcal{C}_1$	0.55	0.6	0.34	0.55	0.5	0.67	0.5	0.67	0.32	0.37	0.5	0.57
$\mathcal{C}_2$	0.6	0.71	0.34	0.6	0.35	0.64	0.35	0.64	0.52	0.74	0.24	0.53
$\mathcal{C}_3$	0.15	0.25	0.65	0.71	0.32	0.53	0.32	0.53	0.24	0.73	0.4	0.47
$\mathcal{C}_4$	0.26	0.76	0.35	0.43	0.42	0.53	0.42	0.53	0.26	0.74	0.5	0.57

Table 4 shows the aggregated values of information provided in **Tables 1–3** with the IVTSFAAWA operator.

Step 3: To derive the total preference values, all preference values are combined.

Table 5 shows the aggregated values of the attributes collectively with the help of the IVTSFAAWA operator

Table 5. Collective preference values by IVTSFAAWA.

	a_1						a_2					
	c^l	c^u	e^l	e^u	v^l	v^u	c^l	c^u	e^l	e^u	v^l	v^u
$\mathcal{C}_1$	0.5145	0.7866	0.3566	0.6186	0.3694	0.6887	0.4515	0.6877	0.3307	0.5984	0.2770	0.5388
	a_3						a_4					
	c^l	c^u	e^l	e^u	v^l	v^u	c^l	c^u	e^l	e^u	v^l	v^u
$\mathcal{C}_1$	0.5494	0.6619	0.3905	0.5025	0.3761	0.5131	0.491	0.6816	0.3672	0.5634	0.372	0.6076

Step 4: To rank the possibilities, use the score function described in Definition 3.

$sc(a_1) = 0.33568$, $sc(a_2) = 0.291432$, $sc(a_3) = 0.361207$ and $sc(a_4) = 0.295855$

Step 5: Choose the most suitable option.

Since, $\mathrm{sc}(\mathrm{a}_3) > sc(\mathrm{a}_1) > sc(\mathrm{a}_4) > sc(\mathrm{a}_2)$, $a_3 \succ a_1 \succ a_4 \succ a_2$, where "$\succ$" denotes superior to. Thus a_4 is the best alternative.

Hence, by using the IVTSFAAWA operator, we ranked the project success and found that the project a_3 is the most successful project. Now, we rank the project success by using the IVTSFAAWG operator.

6.2. IVTSFAAWG operator

In this sub-section, we will do decision-making with the help of the IVTSFAAWG operator.

Step 1: Look at the r parameter's value for IVTSF information. Consider the r integers for which $0 \leq c_k^r + e_k^r + v_k^r \leq 1$ which is 3 here.

Step 2: On the IVTSF decision information, when the total IVTSF preference values are B_k and $B'_k = ([c'_k, c'_k], [e'_k, e'_k], [v'_k, v'_k])$, use the IVTSFAAWG operator such that given in **Table 6**.

Table 6. Individual preference values by IVTSFAAWG.

	a_1						a_2					
	c^l	c^u	e^l	e^u	v^l	v^u	c^l	c^u	e^l	e^u	v^l	v^u
c_1	0.7575	0.3443	0.6521	0.5490	0.6684	0.5195	0.6	0.77	0.6	0.5666	0.6476	0.5237
c_2	0.8215	0.3463	0.433	0.3905	0.8748	0.4148	0.6763	0.3408	0.566	0.4707	0.5763	0.4920
c_3	0.585	0.3739	0.5023	0.4577	0.6876	0.845	0.8582	0.2408	0.351	0.5918	0.6514	0.4479
c_4	0.8486	0.494	0.7756	0.5220	0.7398	0.3843	0.5791	0.3516	0.6789	0.469	0.53690	0.41330
	a_3						a_4					
	c^l	c^u	e^l	e^u	v^l	v^u	c^l	c^u	e^l	e^u	v^l	v^u
c_1	0.6383	0.376	0.4606	0.4620	0.5385	0.6791	0.8160	0.7575	0.3443	0.6521	0.5490	0.6684
c_2	0.6843	0.2844	0.599	0.3251	0.579	0.3758	0.6330	0.8215	0.3463	0.433	0.3905	0.8748
c_3	0.5954	0.3050	0.6068	0.4715	0.6689	0.5201	0.717	0.5850	0.3739	0.5023	0.4577	0.6876
c_4	0.6207	0.3652	0.6899	0.6226	0.7714	0.6226	0.7217	0.8486	0.4945	0.7756	0.5220	0.7398

Table 6 shows the aggregated values of the attributes provided in T**ables 1–3** with the help of the IVTSFAAWG operator.

Step 3: To derive the total preference values, all preference values are combined.

Table 7 shows the aggregated values of attributes collectively with the help of the IVTSFAAWG operator.

Table 7. Collective preference values by IVTSFAAWG.

	a_1						a_2					
	c^l	c^u	e^l	e^u	v^l	v^u	c^l	c^u	e^l	e^u	v^l	v^u
c_1	0.5839	0.6010	0.8148	0.6456	0.7980	0.3780	0.5363	0.5068	0.7940	0.7750	0.7750	0.2672
	a_3						a_4					
	c^l	c^u	e^l	e^u	v^l	v^u	c^l	c^u	e^l	e^u	v^l	v^u
c_1	0.4545	0.5541	0.6238	0.4825	0.6384	0.3271	0.5604	0.5476	0.6989	0.617	0.7560	0.5839

Step 4: To rank the possibilities, use the score function described in Definition 3.

$sc(a_1) = 0.008402732$, $sc(a_2) = 0.0270360614$, $sc(a_3) = 0.0603637788$ and $sc(a_4) = 0.060868$.

Step 5: Choose the most suitable option.

Since, $sc(a_4) > sc(a_3) > sc(a_2) > sc(a_1)$, $a_4 \succ a_3 \succ a_2 \succ a_1 \succ$, where "$\succ$" denotes superior to. Thus, a_4 is the best alternative.

It is noticeable that the score values produced above show the order in which all alternatives are ranked. By using the IVTSFAAWA and IVTSFAAWG operators, we determine that the alternatives a_3 and a_4 are the best alternatives among the projects that were shortlisted. Additionally, we see that whereas the IVTSFAAWG operator is based on the geometric average, the IVTSFAAWA operator is based on the arithmetic mean and provides the average of group judgment. To test the applicability of these two AOs, we apply them. The IVTSFAAOWA (IVTSFAAOWG) and IVTSFAAHA (IVTSFAAHG) operators can also be used to obtain the corresponding findings. When the sequence of the information or its weight is significant, these four different types of operators stand out.

We shall discuss the implications of the parameters Γ and r on the described operations in the subsection that follows.

6.3. Impact of parameter Γ

In our numerical example, the parameter Γ's value is 3, as can be seen. The values of parameter Γ can, however, be changed by the decision-maker. By modifying the value of Γ, the hierarchy of the options can be modified. The following section shows how the IVTSFAAWA and IVTSFAAWG operators, as given in **Table 8**, show the ranking order of the alternatives.

Table 8. Variation of ranking in IVTSFAAWA and IVTSFAAWG with Γ.

Γ	IVTSFAAWA	IVTSFAAWG
3	$a_2 \succ a_1 \succ a_3 \succ a_4$	$a_4 \succ a_3 \succ a_2 \succ a_1$
5	$a_1 \succ a_2 \succ a_4 \succ a_3$	$a_3 \succ a_4 \succ a_1 \succ a_2$
7	$a_1 \succ a_2 \succ a_4 \succ a_3$	$a_3 \succ a_4 \succ a_1 \succ a_2$
9	$a_1 \succ a_2 \succ a_4 \succ a_3$	$a_3 \succ a_4 \succ a_1 \succ a_2$
11	$a_1 \succ a_2 \succ a_4 \succ a_3$	$a_3 \succ a_4 \succ a_1 \succ a_2$
21	$a_1 \succ a_2 \succ a_4 \succ a_3$	$a_3 \succ a_4 \succ a_1 \succ a_2$
25	$a_1 \succ a_2 \succ a_4 \succ a_3$	$a_3 \succ a_4 \succ a_1 \succ a_2$
35	$a_1 \succ a_2 \succ a_4 \succ a_3$	$a_3 \succ a_4 \succ a_1 \succ a_2$
41	$a_1 \succ a_2 \succ a_4 \succ a_3$	$a_3 \succ a_4 \succ a_1 \succ a_2$
45	$a_1 \succ a_2 \succ a_4 \succ a_3$	$a_3 \succ a_4 \succ a_1 \succ a_2$
49	$a_1 \succ a_2 \succ a_4 \succ a_3$	$a_3 \succ a_4 \succ a_1 \succ a_2$
51	$a_1 \succ a_2 \succ a_4 \succ a_3$	$a_3 \succ a_4 \succ a_1 \succ a_2$

Table 8 shows the ranking positions of the alternatives according to parameter Γ obtained by the IVTSFAAWA and IVTSFAAAWG operators. Be aware that the IVTSFAAWA operator selects a_3 as the best option when $\Gamma = 3$. However, if we increase the value of the parameter Γ to be more than 3, the best option is a_3. Until $\Gamma = 51$, we keep track of the ranking order. After $\Gamma = 3$, the IVTSFAAWA operator produces identical results. It should be observed that the IVTSFAAWA fails

and does not produce the desired outcome when we use any even Γ. Therefore, we advise using any odd number higher than 3. **Table 8** also displays the behavior of changing the values of Γ to generate different ranking orders of alternatives and preference orders using the IVTSFAAWG operator. After $\Gamma = 3$, we observe that the ranking of alternatives remains stable. The best alternative is a_4, which is obtained when $\Gamma = 3$. The best alternative is a_3, however, if we vary the amount of Γ and use Γ bigger than 3, we still get the same optimal alternative when we use $\Gamma \geq 5$. It should be observed that the IVTSFAAWG operator fails and produces no results when we utilize an even Γ. As a result, we advise choosing an odd number of $\Gamma \geq 5$. We show the variance in ranking orders of options in **Figure 2**.

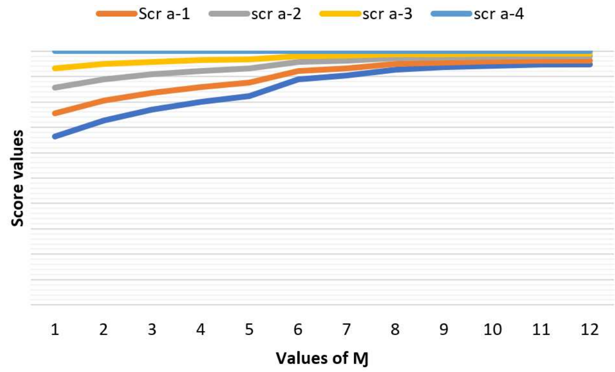

Figure 2. Ranking of the IVTSFAAWA operator with the values of the parameter Γ.

In **Figure 2**, we display how the IVTSFAAWA operator's ranking orders of alternatives change as Γ values are changed. Because the IVTSFAAWA operator does not produce results for even values of Γ, we plot the graph from $\Gamma = 3$ to $\Gamma = 51$ without using any even Γ.

Figure 2 shows the variation of the ranking results obtained at the different values of Γ discussed above in **Table 8**. The interesting ranking can be observed in **Figure 2**. Similarly, the ranking obtained by the IVTSFAAWG operator in **Table 8** can be plotted graphically.

Figure 3 shows the tendency of the ranking of the alternatives produced by the IVTSFAAWG operator as discussed in **Table 8**. Use the IVTSFAAWA operator in **Table 9** to compare options. We explore the characteristics of the ranking trends of alternatives by adjusting the values of r in the IVTSFAAWG operator. This is the same as how the IVTSFAAWA operator lets its output results change.

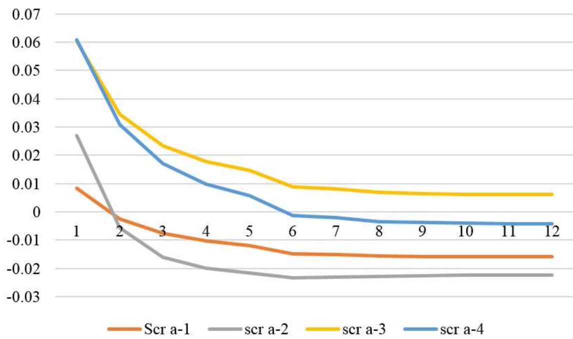

Figure 3. Ranking of the IVTSFAAWG operator with the values of the parameter Γ.

Table 9. Variation of ranking in IVTSFAAWA and IVTSFAAWG with r.

r	**IVTSFAAWA**	**IVTSFAAWG**
3	$a_2 \succ a_4 \succ a_3 \succ a_1$	$a_4 \succ a_2 \succ a_3 \succ a_1$
4	$a_2 \succ a_1 \succ a_3 \succ a_4$	$a_4 \succ a_3 \succ a_2 \succ a_1$
5	$a_2 \succ a_1 \succ a_4 \succ a_3$	$a_4 \succ a_2 \succ a_3 \succ a_1$
6	$a_2 \succ a_1 \succ a_3 \succ a_4$	$a_4 \succ a_2 \succ a_3 \succ a_1$
7	$a_2 \succ a_1 \succ a_4 \succ a_3$	$a_4 \succ a_2 \succ a_1 \succ a_3$
8	$a_1 \succ a_2 \succ a_4 \succ a_3$	$a_4 \succ a_2 \succ a_1 \succ a_3$
9	$a_1 \succ a_2 \succ a_4 \succ a_3$	$a_4 \succ a_2 \succ a_1 \succ a_3$
10	$a_1 \succ a_2 \succ a_4 \succ a_3$	$a_4 \succ a_2 \succ a_1 \succ a_3$
15	$a_1 \succ a_2 \succ a_4 \succ a_3$	$a_4 \succ a_1 \succ a_2 \succ a_3$
20	$a_1 \succ a_2 \succ a_4 \succ a_3$	$a_4 \succ a_1 \succ a_2 \succ a_3$
30	$a_1 \succ a_2 \succ a_4 \succ a_3$	$a_1 \succ a_4 \succ a_2 \succ a_3$
50	$a_1 \succ a_2 \succ a_4 \succ a_3$	$a_1 \succ a_4 \succ a_2 \succ a_3$

6.4. The impact of r

The values of the parameter r are changed in the next section, and the reordering of the options is observed. We utilize r values ranging from 3 to 50, and for each number we achieve the same ordering of the options. a_4 remains the optimal solution for all values of r. Moreover, it should be noted that when r is appreciably large, the score values of alternatives approach zero. We demonstrate several ranking order alternatives. **Table 9** shows the effects of the parameter r obtained by the IVTSFAAWA and IVTSFAAWG operators. We variated the values of the r up to 51 and obtained interesting results from both developed operators.

6.5. Comparison with other operators

Given that Ullah et al. [7] constructed AOs in an IVTSFS context employing "Hamacher TN and TCN" and explored potential implementation in the assessment of robot performance, Hussain et al. [40] proposed AOs IVTSFS adopting Frank TN and TCN. We compare the order in which the AOs were ranked with the operators we suggested for IVTSFAAWA and IVTSFAAWG. The IVTSFAAWA and IVTSFAAWG operators should be preferred given that the AATN and TCN are more flexible than the TN and TCN that were used in earlier AOs (**Table 10**). The following are some intriguing findings we've made:

1) Each of the recommended operators in this article is based on two parameters known as Γ and r, and because of this, it is up to the decision-makers to determine what values these parameters should be given.
2) By utilizing the (generic) IVTSF weighted average (IVTSFWA) AOs, we determine that a_1 is the optimal alternative. Once the IVTSFDWA and IVTSFAAWA are applied. As a result, a_3 is the finest choice. When we use IVTSFDWG and IVTSFWG operators, we obtain the optimal alternative a_1.
3) Using the IVTSFAAWG AOs, we can produce a_4, which is the best alternative.

Table 10. Comparison with other operators.

IVTSFDWG	$\boldsymbol{a_1 \succ a_4 \succ a_2 \succ a_3}$
IVTSFWG	$a_1 \succ a_4 \succ a_3 \succ a_2$
IVTSFAAWG	$a_4 \succ a_3 \succ a_2 \succ a_1$
IVTSFDWA	$a_3 \succ a_4 \succ a_2 \succ a_1$
IVTSFWA	$a_1 \succ a_4 \succ a_3 \succ a_2$
IVTSFAAWA	$a_3 \succ a_1 \succ a_4 \succ a_2$

Figures 4 and **5** depict the comparison between the score values achieved employing the AOs stated in this section.

Figure 4. Displays the comparison between the developed IVTSFAAWG operator and traditional AOs.

In **Figure 4**, the comparison of different geometric operators with existing is displayed graphically. We can observe the ranking of the alternatives in **Figure 4**. It is cleared that the ranking is linear in case of each operator.

Figure 5. Displays the comparison between the developed IVTSFAAWA operator and existing AOs.

In **Figure 5**, the comparison of different geometric operators with existing is displayed graphically. We can observe the ranking of the alternatives in **Figure 5**. It is cleared that the ranking is linear in case of each operator.

7. Conclusion

In this study, we started by outlining some fundamental about IVTSFS, AATN, and AATCN. Then, we introduced four different AO types: IVTSFAAWG, IVTSFAAOWG, IVTSFAAHWG, and IVTSFAAWG operators. We demonstrated several intriguing characteristics of these AOs, such as monotonicity, idempotency, and boundedness. With the aid of an example, we also used the IVTSFAAWA and IVTSFAAWG operators to resolve MADM problems. By altering the values of the r and Γ involved parameters, we further analyzed how these operators behave. We compared the suggested operators to the IVTSFWA, IVTSFHWA, and IVTSFEHWA operators as well as the IVTSFWG, IVTSFHWG, and IVTSFEHWG operators. The findings include the results below.

1) First off, thanks to AATN and AATCN, due to which, the IVTSFAAWA and IVTSFAAWG operators are more adaptable than the other equivalent operators.
2) Additionally, by varying the values of the two parameters r and Γ, the IVTSFAAWA and IVTSFAAWG operators provide a distinct ranking of possibilities.
3) The IVTSFAAWG operator based on the weighted geometric mean of alternatives typically provides a superior option to obtain the best alternative than the IVTSFAAWA operator based on the weighted average of the alternatives.

We intend to work on the theory and applications of Archimedean norms Wang and Garg [41] inside the framework of IVTSFSs shortly. We also intend to investigate how the proposed works might be applied in the manufacturing sector [42] and to the framework defined in the studies of Ullah [43] and Mahmood [44]. The current work can also be generalized for the frameworks defined in the studies of Al-Quran [45] and Al-Quran [46].

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