Article

Computer visuality in mathematics teaching

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Abstract: The purpose of this study is to look at the changes that computers have made in mathematics itself and in the mathematics curriculum. The aim of the study is to investigate the various applications of computers in education in general, especially in mathematics education, and their application in the mathematics curriculum and in the teaching and learning of mathematics. The primary use of educational tools for mathematical purposes is the quality verification of results. There are various tools for developing student logic based on interactivity. Mathematics education tools are designed for innovative, interactive, and dynamic learning in different areas of mathematics. It is undeniable that the use of computers and mathematical software has great benefits that have been proven and presented in their works by numerous researchers of effective learning. It is also indisputable that one of the main tasks of teaching mathematics is the development of students’ constructive thinking. This work aims to describe the application of educational tools with which it can develop interactivity and help pupils and students to better and more clearly understand mathematics and to understand that it is all around us, that it is our everyday life.

Keywords: mathematics; visualization; computers; tools for mathematics education; visual representation in mathematics

1. Introduction

Numerous studies on the effectiveness of mathematical learning have shown the justification and usefulness of the implementation of new teaching aids. They also showed that learning with educational software has a great impact on the achievement of students in the overall acquisition of mathematical knowledge during the school year as well as on the final exam at the end of primary education. Teaching using computers and software packages is interesting for students and increases their interest and active participation.

Students have historically struggled with learning how to solve mathematical problems, and mathematical development has long placed a strong emphasis on this topic. Successful problem solvers invest more time in analyzing the problem and looking for all possible solutions [1]. Carden and Klein [1] argue that encouraging visual imagery/visualization as a problem-solving technique would be beneficial for all students, even those who do not find word problems difficult. The likelihood that successful students will continue to succeed as they face more challenging challenges will be increased by using visualization techniques.

Children should learn these techniques at an early age and practice them often. The main reason why math is taught in schools from elementary to college is
because it is crucial to our daily lives. Because they find it difficult to answer questions, most children hate math. Teachers should look for ways to make their lessons simpler, easier, and more interesting for their students to get them to love mathematics. The use of visual representation is one method that can be used to teach mathematics. It is any method of producing images, diagrams, or animations to get a point across. Since the beginning of time, people have used visualization to convey both abstract and concrete ideas [2].

To be able to work on unique problems that require adaptation of well-known solution methods, as well as to solve well-known problems, mathematical solution techniques are a key requirement [3]. When you talk about visual representation in math, you might think of representing information in your head with a picture or information on a page with a diagram or graph. Fortunately, academics have concentrated on helping students strengthen their internal and external visual representations [4]. To improve mathematics learning for all types of questions, it is critical that students develop both internal and external visualization strategies.

2. Visualization in computer-assisted mathematics teaching

Teaching mathematics is among the most researched areas in education. There have been many different computer applications in teaching [5]. Math teachers are overwhelmed with the vast number of suggestions on how to teach math using computers. Teachers’ attitudes towards computers generally differ depending on the teacher’s age or years of work. The complete ‘ignorance’ attitude towards the computer still persists, although its size is weaker compared to previous years. This attitude is mainly shared by teachers who were trained before the beginning of the computer age, who have the most negative attitudes towards its pedagogical use, and who insist on using traditional teaching methods.

Systematic research on computer-assisted instruction suggests that there are few but positive effects compared to traditional instruction [6]. However, despite increasing availability and potential learning benefits, research shows that computers are underutilized in many schools and that the potential of computer technology is not being implemented [7]. Therefore, the main goal of the work is to improve the use of visual methods and computer-assisted teaching in primary and secondary schools, by providing information on available professional training for teachers in the use of new technology, workshops, and technical assistance.

The impact of the computer on visual reasoning in mathematics education is undeniable and the student uses it as a visualization tool through dynamic geometry software that facilitates visualization processes that have been discussed in many papers over the years. Expanding the effects of computer technology in mathematics visualization proves that this methodology has affective attributes in solving mathematical process problems (Figure 1). This approach is a powerful driver not only in mathematical topics such as geometry and trigonometry, but also in algebra, analysis, etc. Many specific advantages of using computer software that encourage dynamic visualization have also been reported [8].
Figure 1. Presentation of information technologies in education. “The goal of teaching and learning informatics and computer science is to enable students to manage information.”

There is an increasing number of different educational software that contribute to the easier learning of mathematics among students, especially when it comes to dynamic geometry software. One of them is the GeoGebra software, which is a multilingual software package dedicated to the study of Euclid’s geometry, algebra, analysis, statistics, and other mathematical fields, as well as connecting to other mathematical models of the natural sciences. GeoGebra is used for an interactive approach to mathematics education through modeling, selecting the right function, clicking checkboxes, moving sliders, etc., which makes this software adequate for strengthening mathematical skills and visualizing various abstract concepts.

**Modeling in GeoGebra**

To provide a brief but meaningful context, it is worth noting that modeling in terms of mathematics is a key feature embedded in the process and knowledge of mathematics. According to Dundar et al. [9], mathematical modeling involves a conversion process, which is a constant and inevitable part of realistic scenarios involving mathematics. Modeling is the process of converting incidents that actually happen and converting incidents based on mathematics into a real and realistic scenario.

Overall, GeoGebra can be considered a very creative tool for mathematical modeling\(^1\) [10]. Doerr and Pratt defined two different types of modeling based on each student’s task: exploratory modeling and expressive modeling. In exploratory modeling, a default model, built by a professional, is provided to each user, allowing the learner to demonstrate their skills in model construction. As the model is being constructed, the student may encounter a path that results in a better understanding of the relationship present in the model world and reality. According to Doerr and Pratt [10], the modeling element found in GeoGebra provides a brilliant way to show students a cyclic view.

GeoGebra includes most of the construction tools found in 2D computer-aided design software, which can help build dynamic models similar to those found in geometry facilities with specific parameters that can be changed.
3. Modern tools for dynamic visualization in mathematics teaching

Because many evaluations and implementations of visualization technology are based on student feedback and focused on a learning perspective, students are engaged in learning technology development. However, what is more troubling lies in the neglected perspective of the often leading role in the classroom—the teacher. Despite the tools designed for mathematics education, hardly any mathematics teacher uses technological teaching aids.

3.1. Visualization tools used in mathematics classes

According to the interviewed teachers, most classrooms are equipped with blackboards or whiteboards. Classroom computers that connect to projectors and a portable screen. Sometimes classrooms also have document cameras in the front of the classroom that can be used with projectors. The computers are connected to the Internet, and external tools such as personal computers or tablets can be connected to the computers in the classroom. With the variety of tools available in the classroom, the teacher often first chooses the whiteboard as his teaching visualization tool. Based on recorded interviews, the study compares and contrasts the pros and cons of whiteboards with other alternative tools [10].

Writing on the blackboard with chalk was a teaching convention in mathematics classroom education for several reasons. First of all, this is how people taught classroom settings in history, and instructors mostly taught math in the classroom by watching their teachers write with chalk on a slate board, which is made of a thin layer of black or dark gray slate stone. So, when instructors start teaching, they follow their teachers. However, this tendency to follow tradition could change. Quite a few of the teacher respondents said that they started using other visualization tools, especially digital visualization tools such as transparency, and document cameras, since the 1990s.

However, teachers also claim that there are disadvantages to using the blackboard. First of all, the notes on the board will only exist for a short time in class, and the students will not be able to see them after they are deleted. If you write on paper and project it onto an overhead screen, teachers are able to scan the notes later and share the notes with students electronically. If the teacher is writing electronically, he or she can immediately share it with students after class. Keeping electronic records not only helps students review and study but also teachers.

3.2. Types of visualization tools used in mathematics teaching

Various visualization tools have been developed and integrated into mathematics classrooms to enhance the learning experience. These tools can be grouped into several types, each offering unique advantages and insight into mathematical concepts. These visualization tools are often used in mathematics teaching.

Graphics software: Graphics software such as Desmos and GeoGebra have gained popularity in math classrooms. These tools allow students to create, manipulate, and explore graphs of various mathematical functions and equations. Instructors can use these tools to visually demonstrate concepts such as
transformations, intercepts, and asymptotes, making abstract ideas more concrete\textsuperscript{5} [11].

Interactive simulations: Interactive simulations allow students to experiment with mathematical phenomena in virtual environments. Websites such as PhET Interactive Simulations offer a variety of interactive mathematical simulations, allowing students to explore probability, geometry, and algebra through hands-on experimentation\textsuperscript{7} [12].

Virtual reality (VR): Virtual reality technology has begun to make its mark in mathematics education. VR allows students to immerse themselves in three-dimensional mathematical spaces, facilitating a deeper understanding of geometry and spatial reasoning\textsuperscript{4} [13].

Augmented reality (AR): Augmented reality tools overlay digital information onto the physical world, creating interactive and engaging learning experiences. AR applications such as AR math have been used to visualize geometric shapes and transformations, improving students’ spatial visualization skills\textsuperscript{5} [14].

Dynamic geometry software: Dynamic geometry software, including tools such as CabriGeometri and The Geometer’s Sketchpad, allows students to dynamically construct and manipulate geometric figures. These tools promote exploration and discovery, encouraging students to formulate assumptions and test hypotheses\textsuperscript{6} [15].

Data visualization tools: Data visualization tools help students understand large data sets and statistical concepts. Software such as Tableau and Excel allow students to create visual representations of data, facilitating the exploration of trends, distributions, and relationships\textsuperscript{7} [16].

4. Graphical solution of the system of equations

The method of graphically solving a system of equations is one of the simplest and most visually clear ways of finding a solution to a system of equations. It is based on drawing the equations involved in the system and finding their points of intersection. Thus, the graphical method allows you to visualize the general solution of the system and determine whether it exists, and if so, within what limits.

The main advantage of the graphic method is its simplicity and intuitiveness. Even without special mathematical knowledge, you can graph the equations and find out where they intersect relatively easily. This method is especially useful when solving two-variable systems when you can plot the graphs on the plane and visually see the points of intersection.

However, it should be noted that the graphical method has its limitations. It may not be applicable in cases where the system contains a large number of variables or non-linear equations. In addition, some systems may have an infinite number of solutions, and some may have no solutions at all. In such cases, the graphical method may be ineffective or even meaningless\textsuperscript{8} [17].

In general, the graphical method for solving a system of equations is a simple and convenient way to obtain a geometric interpretation of the system’s solution. Understanding its essence and putting it into practice will help you improve your equation-solving skills and your general understanding of mathematics.
4.1. A method for graphically solving a system of equations

To apply the method of graphical solution of the system of equations, it is necessary to construct the graphics of both equations of the system. The resulting graphs are straight on the plane. Then you need to find the point of intersection of these lines. The coordinates of this point represent the solution of the system of equations. If the graphs of two equations do not intersect, then the system of equations has no solution. If the graphs match (matching equations), then the system has infinitely many solutions, since all points are on a common solution line.

The method of graphically solving a system of equations is easy to use and intuitive, especially when the graphs of the equations have a simple shape, such as straight lines. This method can also be useful for checking the results obtained when solving the system using other methods. However, the graphical solution method has its limitations. It is not always applicable, especially when the system equations have a complex form or non-linear relationship. Also, the accuracy of the results may be limited by errors in plotting the graph and determining the intercept. In general, the graphical method of solving a system of equations is a convenient and visual way of solving a system of equations, which can be used in a number of practical tasks, especially in the case of simple equations.

4.2. Basic principles of the method of graphical solution of the system of equations

The basic principles of the method of graphical solution of the system of equations are:

Step 1. Stating a system of equations. It is necessary to write down the system of equations in general form. For example, for two equations and two unknowns it can be:

\[ ak + bi = c \]
\[ dk + i = f \]

Step 2. Graphical equations. A graph must be constructed for each equation in the system. To do this, you can use special programs or paper and pencil. A graph is a line or curve on a coordinate plane.

Step 3. Determining the intersection point of the graph. It is necessary to find the point where the graphs of the equations intersect. This point will be the solution to the system of equations.

Step 4. Solution check. The obtained solution should be substituted into the original system of equations and its correctness checked. If the solution satisfies all the equations, then it is correct.

The method of graphically solving a system of equations is particularly useful for a visual representation of solutions in two-dimensional space. However, its application is limited to systems with two equations and two unknowns. For a system with more than two equations, it is difficult to graph and manually find the point of intersection, so other solution methods are recommended for such systems.
4.3. When the method of graphical solution of the system of equations is used

One important application of the method is the analysis of economic models. Economic systems are often modeled by a system of equations for determining various parameters and predicting changes. The graphical solution method allows you to find the balance between supply and demand, as well as determine optimal strategies and decisions in the economic sphere. The graphical solution method is also used in physics and engineering. When modeling physical systems or designing various mechanisms and devices, it often becomes necessary to solve a system of equations to determine optimal parameters or operating conditions. The graphical method allows you to visualize the solution and interpret its results more clearly. The method of graphically solving systems of equations is also used in optimization and planning problems. For example, in logistics it can be used to determine the best delivery route or resource allocation. Similarly, in production planning or project management problems, this method can help determine the optimal allocation of resources and optimize production processes.

Thus, the method of graphically solving a system of equations is a universal tool that is used in various fields—from economics to physics and engineering. Thanks to a clear graphic display, this method allows you to find optimal solutions and explore different models in a convenient and understandable way.

4.4. Graphical solution of equations

Sometimes equations are solved graphically. To do this, you need to transform the equation so (if it is not already represented in transformed form) that there are expressions to the left and right of the equal sign for which you can easily graph functions [18]. For example, given the following equation:

\[ k^2 - 2k - 1 = 0 \]

If we have not yet studied solving quadratic equations algebraically, we can try to do so either by factoring or graphically (Figure 2). In order to solve such an equation graphically, we present it in this form:

\[ k^2 = 2k + 1 \]

It follows from this presentation of the equation that it is necessary to find such values of \( k \) for which the left side will be equal to the right side.
As you know, the graph of the function \( i = k^2 \) is a parabola, and \( i = 2k + 1 \) is a straight line. The coordinate \( k \) of the points of the coordinate plane that lie on both the first and second graphs (that is, the points of intersection of the graphs) are precisely those \( k \) values at which the left side of the equation will be equal to the right. In other words, the \( k \)-coordinates of the points where the graphs intersect are the roots of the equation\(^9\) [19].

Graphs can intersect at several points, at one point, or not at all. It follows that an equation can have several roots, one root, or none at all.

Let’s look at a simpler example:
\[
\begin{align*}
    k^2 - 2k &= 0 \\
    \text{or } k^2 &= 2k
\end{align*}
\]

Let’s draw the graphs of the functions \( i = k^2 \) and \( i = 2k \):

As can be seen from the drawing, the parabola and the line intersect at the points \((0; 0)\) and \((2; 4)\). The coordinates \( k \) of these points are respectively equal to \( 0 \) and \( 2 \). This means that the equation \( k^2 - 2k = 0 \) has two roots—\( k_1 = 0 \), \( k_2 = 2 \).

Let’s check this by solving the equation by taking the common factor out of the parentheses:
\[
\begin{align*}
    k^2 - 2k &= 0 \\
    k(k - 2) &= 0
\end{align*}
\]

A zero on the right can appear when \( k \) is 0 or 2.

The reason why we did not solve the equation \( k^2 - 2k - 1 = 0 \) graphically is that in most equations, the roots are real (fractional) numbers, and it is difficult to accurately determine the value of \( k \) on the graph. Therefore, for most equations, the graphical solution is not the best. However, knowing this method allows for a deeper understanding of the relationship between equations and functions.

4.5. Solving systems of linear equations by graphing

These equations are called linear because their graphs are straight lines in the \( x-y \) coordinate plane. The solution(s) of a set of linear equations corresponds to the point at which the graphs of the equations intersect. So, by graphing systems of
equations, and visually determining at which point (if any) every line in the system intersects, we can quickly and easily arrive at the solution (Figure 3).

Figure 3. The solution to this system is the point (0, 2). The point (0, 2) is the only solution to the system of linear equations that contains the equations.

Here we can see the graphs of these two equations:

The solution to this system is the point (0, 2).

The graphs of $5x + 5y - 10 = 0$ and $-16x - 2y + 4 = 0$, intersecting at the point (0, 2).

As expected, the graphs of these two equations are straight lines, and those lines intersect at the point (0, 2). So, the solution of this system of linear equations is $x = 0, y = 2$.

When solving systems by graphing, there are three possible outcomes: we could have exactly one solution, as seen in the previous example; we could have no solutions at all; or we could have infinitely many solutions. These are the only three possibilities. Think for a moment: why could there never be exactly two solutions to a system of linear equations? Since their graphs are straight lines, they will only ever intersect once, or not at all. That’s the reason!

These are the basic steps involved in solving systems of linear equations by graphing.

No solutions

It is possible for a system of linear equations to have no solution. This only occurs when the two lines never intersect (Figure 4). Consider this system of linear equations:

$$4x - 6y = -106x - 9y = -12$$

If we graph this system of equations, we see the following:
Figure 4. Parallel lines never intersect. “Parallel lines never intersect or meet each other because they always have the same distance apart.”

These two equations correspond to lines that are parallel, so they never intersect. This system has no solutions. There is no single $x$-$y$ pair that will satisfy both equations. In fact, we can state a rule here: A system of linear equations has no solutions if and only if the graphs of those equations are distinct parallel lines.

5. Advantages of solving equations graphically

Elementary and middle school teachers often use graphs as part of their math curriculum. Graphs help students organize and analyze information in well-structured formats, making it easier to interpret the data. Visual learners respond especially well to graphs and often understand the information better without pages of text. Graphs do have a downside—students might jump to conclusions without carefully analyzing the limitations and parameters. Students might also rely on graphing calculators, without being able to solve equations or do the graphing themselves.

Visual graphs provide clues that words and equations don’t. For example, high school students may need a few minutes to read, try, interpret, and map a word problem. With a pictograph or pie chart, students can quickly draw conclusions. Graphs show trends, gaps, and clusters, and compare multiple data sets at once, often accommodating large sets of data. They make it easy for scientists and students alike to form hypotheses and draw conclusions.

Clarity and visibility: The graphical method allows you to see the geometric interpretation of the system of equations. The solution is presented in the form of a graph of functions, from which the intersection point can be determined, and thus the solution of the system can be found. This makes the method accessible and understandable to most people, even those without deep knowledge of mathematics.

Visualization of the solution: The graphical method allows you to visualize all possible solutions to a system of equations. If the graphs of the functions do not intersect, then the system has no solution. If the graphs intersect at one point, then the system has a unique solution. If the graphs match, then the system has an infinite
number of solutions.

Quick solution: In some cases, the graphical method may be faster and easier to perform, especially with a small number of equations. When using the method, simply find the point of intersection of the graph of the functions and determine its coordinates.

Versatility: The graphical method can be used to solve various types of systems of equations, including linear, nonlinear, and multivariable systems. This makes it a universal tool for solving mathematical problems.

In general, the graphical solution of a system of equations is a simple and visual method that allows you to get the exact solution and has many practical applications.

6. Conclusion

The use of visualization methods in the process of teaching mathematics among schoolchildren contributes to the development of the ability to solve mathematical problems. As a result, the effectiveness of mathematics teaching increases. It also encourages better and more complete assimilation of knowledge based on awareness of the methods used and contributes to the development and maintenance of interest in the subject.

The use of visualization methods develops students’ imaginative thinking, promotes the development of abstract thinking, and also contributes to the development of various forms of mental activity.

The graphical solution method allows you to visually represent the solution of the system in the form of a cross-section of the graph of the equations. The substitution method involves substituting one variable in one equation and then substituting that value into another equation. Each of these methods has its advantages and disadvantages, and the choice of method depends on the specific task. It is important to be able to apply different methods and choose the most suitable one for solving a particular system of equations.

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Conflict of interest: The authors declare no conflict of interest.

Notes

1. A useful overview of the basic steps involved in mathematical modeling of real-world problems.
2. Educational tools in teaching mathematics.
3. Integrating this technology into the course methods influenced beginning calculator teachers’ knowledge and comfort with using graphing calculator technology and teachers’ perspectives on using this technology.
4. Technology integration is the use of interactive virtual simulations, which have proven to be an effective way of placing students at the center of instruction, allowing them to explore scientific phenomena on their own.
5. Mobile devices can act as a portal to a wealth of data and resources, giving students access to learning opportunities that transcend conventional constraints of place and time.
6. The importance of improving the ability of teachers to produce AR applications at different educational levels.
7. Active learning is a broad concept that activates teaching methods and teacher-led learning processes.
The importance of nurturing teachers’ expertise in using AR applications for teaching mathematics, considering them as vital competencies that support the educational process.


References