Applications of intuitionistic fuzzy sets to assessment, decision making and to topological spaces

Michael Gr. Voskoglou

School of Engineering, University of Peloponnese, 26334 Patras, Greece; voskoglou@teiwest.gr

ABSTRACT: The intuitionistic fuzzy sets, in which the elements of the universe have their membership and non-membership degrees in [0, 1], are a generalization of Zadeh's fuzzy set. In this paper, intuitionistic fuzzy sets are used as tools for assessment and decision-making. This is useful in cases where one is not sure about the suitability of the linguistic characterizations assigned to each element of the universal set. Further, it is described how the notions of convergence, continuity, compactness, and of Hausdorff topological space are extended to intuitionistic fuzzy topological spaces. Applications illustrating our results are also presented.

KEYWORDS: fuzzy set; intuitionistic FS; soft set; fuzzy assessment; decision making in fuzzy environment; intuitionistic fuzzy topological space

AMS CLASSIFICATION: 03E72

1. Introduction

Assessment is a very important process connected to all human and machine activities because it helps to improve performance by avoiding mistakes or weak behaviors of the past. In many cases, the assessment is performed using qualitative grades (linguistic expressions) instead of numerical scores. This happens either because the existing data are not exact or for reasons of elasticity (e.g., from teacher to student). Obviously, in such cases, the mean performance of a group cannot be assessed by calculating the mean value of the individual scores of its members. For tackling this problem, we have used in earlier works either triangular fuzzy numbers (see Remark 1 iii of Section 2) or gray numbers (closed real intervals), and we have shown that these two methods are equivalent[1].

On the other hand, decision-making, which is a fundamental activity of human cognition, can be defined as the process of choosing a solution among two or more alternatives with the help of suitable criteria, aiming to obtain the best possible outcome for a given problem. Recent technological progress and the rapid changes happening in human society have increased the complexity of decision-making problems. Thus, since the 1950's, a systematic approach to decision-making has been known as statistical decision theory, which is based on a synthesis of principles from several scientific fields, including probability theory, statistics, economics, and pychology[2].

Frequently in everyday life situations, however, the decision-making process takes place under conditions of uncertainty, which makes the application of the traditional DM methods for the solution of the corresponding problems impossible. Several decision-making methodologies have been proposed for such cases, based on the principles of the theory of fuzzy sets[3] and their extensions[4–7].

Atanassov's theory of intuitionistic fuzzy sets, introduced in 1986[8], is a generalization of Zadeh's fuzzy sets. In fact, in an intuitionistic fuzzy set, all the elements of the universal set of the discourse are

characterized, apart from Zadeh's membership degree, by the degree of non-membership too, both of which take values in [0, 1].

Intuitionistic fuzzy sets have found important applications to practical everyday life problems in which the use of FSs has not been proven to be too effective for obtaining the required solutions. These applications include career determination[9], weather broadcasting[10], artificial intelligence[11], decision-making[12,13] medicine[14–17], logic[18,19], choice of discipline of studies[20], etc.

The target of this work is to present applications of intuitionistic fuzzy sets to assessment and decision-making in situations where one is not sure about the accuracy of the linguistic characterizations (parameters) assigned to each element of the universal set. Further, it is described how the fundamental concepts of classical topology can be extended to intuitionistic fuzzy topological spaces. More explicitly, Section 2 of this work includes information about fuzzy sets, intuitionistic fuzzy sets, and Molodtsov's soft sets[21], needed for the understanding of the paper. The application of intuitionistic fuzzy sets to assessment processes is developed in Section 3, whereas Section 4 introduces their application to decision-making. In Section 5, it is described how fundamental notions of classical topology are extended to intuitionistic fuzzy topological spaces. The last section, Section 6, includes the general conclusions and a short discussion for further research.

2. Background information

2.1. Fuzzy sets

Zadeh introduced in 1965 the notion of a fuzzy set for tackling mathematically the partial truths and the definitions having no clear boundaries in the following way[3]:

Definition 1. *A fuzzy set A in the universe U is of the form*

$$A = \{(x, m(x)): x \in U\} \tag{1}$$

In Equation (1) m: $U \rightarrow$ [0, 1] is the membership function of A, while the number $m(x)$ is called the membership degree of x in A. The closer $m(x)$ to 1, the better x satisfies the property of A. An ordinary set is obviously a fuzzy set, with the membership function being its characteristic function.

The definition of the membership function is not unique, depending on each observer's subjective criteria. Defining, for example, the FS of the good students of a class, one may consider all students with grades greater than 16 (in a scale from 0 to 20) as good and another one all those with grades greater than 17. As a result, the first observer will attach membership degree 1 to all students with grades between 16 and 17, whereas the second one will attach to them membership degrees <1. The only restriction for the membership function is to follow common sense so that the corresponding fuzzy set is compatible with reality. This does not happen, for instance, if, in the previous example, students with grades <10 have membership degrees ≥0.5.

Later, when membership degrees were reinterpreted as possibility distributions, fuzzy sets were also used for tackling the existing uncertainty in everyday life, which is caused by the shortage of knowledge about the corresponding situation. Until then, the unique tool for dealing mathematically with uncertainty was probability theory. Probability, however, is effective for tackling only the uncertainty which is due to randomness, like it happens with games of chance. Possibility is an alternative mathematical theory for dealing with uncertainty. Zadeh determined the difference between possibility and probability, stating that whatever is probable must be primarily possible[22]. Fuzzy sets are suitable in particular for tackling uncertainty due to vagueness, where one has difficulty distinguishing between two properties, like a mediocre and a good student.

As a general reference for fuzzy sets and the related to them uncertainty we propose[23].

2.2. Intuitionistic fuzzy sets

Following the introduction of fuzzy sets, a series of extensions and related theories have been proposed for tackling more effectively the existing uncertainty in each particular case[24].

Atanassov extended in 1986 the concept of fuzzy sets to that of intuitionistic fuzzy sets in the following way[8]:

Definition 2. *An intuitionistic fuzzy set A in the universal set U is of the form*

$$A = \{(x, m(x), n(x)): x \in U, m(x), n(x) \in [0, 1], 0 ℃ \leq m(x) + n(x) \leq 1\} \tag{2}$$

In Equation (2) m: U→[0, 1] is the *membership function* and n: U→[0, 1] is the non-membership function of A and $m(x)$, $n(x)$ are the degrees of membership and non-membership, respectively, for each x in A. Further, $h(x) = 1 - m(x) - n(x)$, is said to be the degree of *hesitation* of x in A. If $h(x) = 0$, then A is a fuzzy set.

The term intuitionistic fuzzy set is due to Atanassov's collaborator Gargov in analogy to the idea of intuitionism introduced by Brouwer at the beginning of the last century. It is recalled that intuitionism rejects Aristotle's law of the excluded middle by stating that a proposition is either true, or not true, or we do not know if it is true or not true. The first part of this statement corresponds to Zadeh's membership degree, the second to Atanassov's non-membership and the third to the degree of hesitation.

Example 1. *Let U be the set of all the players of a football team, let A be the IFS of the good players of the team, and let (x, 0.6, 0.3, 0.1) be in A. Then, there is a 60% belief that x is a good player, but also a 30% belief that he is not a good player and a 10% hesitation to characterize him as a good player or not.*

A fuzzy set with membership function $y = m(x)$ is obviously an intuitionistic fuzzy set with non-membership function $n(x) = 1 - m(x)$ and hesitation degree $h(x) = 0$, for all x in U. Intuitionistic fuzzy sets can be used everywhere ordinary fuzzy sets can be applied, but this is not always necessary. Suppose, for example, that one is interested in the degree of support that the governments of several countries have in their parliaments. For this, he calculates the quotient of the number of the ruling party's members of the parliament to the total number of its members. Therefore, the use of intuitionistic fuzzy sets is not needed here. In the case of an election, on the contrary, a candidate can have voted for (membership), voted against (non-membership), or been undecided (hesitancy) by the electoral vote. This is an example where the use of intuitionistic fuzzy sets is necessary.

Atanassov[25] proposed the geometrical representation of an intuitionistic fuzzy set shown in **Figure 1**, where E denotes the universe, $\mu_A(x)$ the membership and $\nu_A(x)$ the non-membership degree of the element x of E.
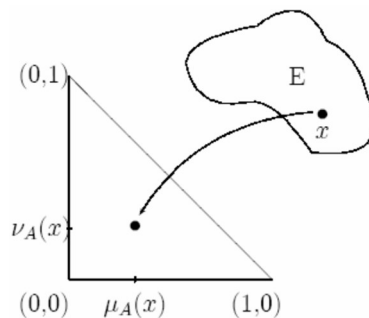

Figure 1. Geometrical representation of an IFS.

The intuitionistic fuzzy sets are suitable for tackling uncertainty due to *imprecision*, which appears frequently in human reasoning, and they have found various applications in many sectors of human activity[11]. An interesting special issue on intuitionistic fuzzy sets and their applications, consisting of 14 total papers, has been published in volumes 8 (2020) and 9 (2021) of the journal Mathematics[26].

Here, for simplicity, we denote an intuitionistic fuzzy set A by A = [m, n] and the elements of A in the form of intuitionistic fuzzy pairs (m, n), with m, n in [0, 1] and $0 \leq m + n \leq 1$. Considering the elements of an intuitionistic fuzzy set A as ordered pairs, we define addition and scalar multiplication in the usual way, i.e., as follows:

Definition 3. *Let* (m_1, n_1)*,* (m_2, n_2) *be in the IFS A and let r be a positive number. Then we define:*

$$\text{The } sum \ (m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2) \tag{3}$$

$$\text{The } scalar \ product \ \text{r}(m_1, n_1 = (\text{r}m_1, \text{r}n_1) \tag{4}$$

The sum and the scalar product of the elements of an intuitionistic fuzzy set A with respect to Definition 3 need not be elements of A, since it could happen that $(m_1 + n_2) + (m_1 + n_2) > 1$ or $\text{r}m_1 + \text{r}n_1 > 1$. The *mean value* of a finite number of elements of A, however, as defined below, is always in A

Definition 4. *Let A be an intuitionistic fuzzy set and let* (m_1, n_1)*,* (m_2, n_2) *…,* (m_k, n_k) *be a finite number of elements of A. Then the mean value of these elements is the element of A*

$$(m, n) = \frac{1}{k} [(m_1, n_1) + (m_2, n_2) + \dots + (m_k, n_k)] \tag{5}$$

For general facts on intuitionistic fuzzy sets we refer to the study by Atanassov[11].

Extensions of the concept of intuitionistic fuzzy set include[25]:
1) Interval-valued intuitionistic fuzzy set (Gargov and Atanassov);
2) u Intuitionistic L-fuzzy sets (Stoeva and Atanassov);
3) u intuitionistic fuzzy sets over different universes (Atanassov);
4) u intuitionistic fuzzy sets of type 2 and type n (Rangasamy, Vassilev and Atanassov);
5) u Temporal intuitionistic fuzzy sets (Atanassov);
6) u Intuitionistic fuzzy multidimensional sets (Szmidt, Kacprzyk and Atanassov), etc.

Remark 1. *i) Smarandache, inspired by the various neutralities of everyday life, like win – draw - defeat, friend – neutral – enemy, etc., extended further the concept of intuitionistic fuzzy set in 1995 to that of neutrosophic set by introducing the degree of indeterminacy or neutrality in addition to those of membership and non-membership[27]. The information that we have for an element x of the universe U is said to be complete, if m(x) + i(x) + n(x) = 1, incomplete if m(x) + i(x) + n(x) < 1, and inconsistent if m(x) + i(x) + n(x) > 1, where m(x), i(x), and n(x) denote membership, indeterminacy, and non-membership, respectively. A neutrosophic set can contain elements simultaneously, leaving room for all the previous forms of information. In an intuitionistic fuzzy set, the indeterminacy coincides with the hesitation, whereas in a fuzzy set, n(x) = 1 – m(x) and i(x) = 0.*

ii) Zadeh in 1975 extended fuzzy set, which is now called type-1 fuzzy set, to the concept of type-2 fuzzy set[28] for handling more uncertainty connected to the membership function. The membership function of a type-2 fuzzy set is three - dimensional, its third dimension being the value of the membership function at each point of its two-dimensional domain, which is called the footprint of uncertainty. The footprint of uncertainty is determined by its two bounding functions, a lower membership function and an upper membership function, both of which are type-1 FSs. When no uncertainty exists about the membership function, then a type-2 fuzzy set reduces to a type-1 fuzzy set. Zadeh, in the same paper, generalized the type-2 fuzzy set to the type-n fuzzy set[28], n = 1, 2, 3,... In the late 1990s, Prof. Jerry Mendel and his group further cultivated the theory of the type-2 fuzzy set[29]. Since then, more and more papers have been written about type-2 fuzzy sets, e.g., Mohammadzadeh et al.[30], a recent very interesting work about Fourier-based, type-2 fuzzy neural networks effective for high dimensional problems. Also, some remarkable papers have been

written about type-3 fuzzy sets with important applications to renewable energy modelling prediction[31], to fractional learning algorithms[32], etc. (Remark 1 (ii) was transferred here from the last section.)

iii) A fuzzy number is a fuzzy set in the set R of real numbers with a piece-wise continuous membership function y = m(x), which is normal (i.e., there exists x in R such that m(x) = 1) and convex (i.e., all it's a-cuts $\{x \in R: m(x) \geq a\}$, $a \in [0, 1]$, are closed real intervals). The simplest form of fuzzy numbers are triangular fuzzy numbers. A triangular fuzzy number (a, b, c), with a, b, c in R, a < b < c, represents mathematically the linguistic statement "the value of b lies in the interval [a, c]," and its membership function is defined by

$$y = m(x) = \begin{cases} \dfrac{x-a}{b-a} & , \quad x \in [a,b] \\ \dfrac{c-x}{c-b}, & \quad x \in [b,c] \\ 0, & \quad x < a \ \text{or} \ x > c \end{cases}$$

For general facts on triangular fuzzy numbers we refer to the study by Kaufmann et al.[33].

2.3. Soft sets and parametric decision making

The difficulty of defining the membership function of a fuzzy set, which was described in Section 2.1, holds also for all their extensions involving membership functions, like intuitionistic fuzzy sets, neutrosophic sets, etc. Several solutions have been proposed for tackling this problem like:

1) The interval-valued fuzzy sets[34], in which the membership degrees are replaced by subintervals of [0, 1];
2) The rough sets[35], in which instead of a membership function, one introduces a pair of sets as the lower and upper bounds of the original set;
3) The grey numbers (closed intervals of real numbers)[36], where the definition of a membership function is not necessary;
4) The soft sets[21], tackling the uncertainty in a parametric manner, etc.

Molodtsov, introduced in 1999 the notion of soft set in the following way[21]:

Definition 5. *Let E be a set of parameters and let f be a map from E into the power set P(U) of the universal set U. Then the soft set (f, E) in U is defined as a parametrized family of subsets of U by*

$$(f, E) = \{(e, f(e)): e \in A\} \tag{6}$$

The term "soft" was introduced because the form of (f, E) depends on the parameters of E. A fuzzy set in U with membership function $y = m(x)$ is a SS in U of the form (f, [0, 1]), where $f(a) = \{x \in : m(x) > a\}$ is the a-cut of the fuzzy set, $\forall a \in [0, 1]$.

Maji et al.[37] introduced the *tabular representation* of a SS for storing it easily in a computer's memory, and they used it for parametric DM[38]. The following example illustrates their DM methodology.

Example 2. *A company wants to employ a person among the six candidates A_1, A_2, A_3, A_4, A_5 and A_6. The ideal qualifications for the new employee are to have satisfactory previous experience, to hold a university degree, to have a driving license and to be young. Assume that A_1, A_2, A_6 are the candidates with satisfactory previous experience, A_2, A_3, A_5, A_6 are the holders of a university degree, A_3, A_5 are the holders of a driving license and A_4 is the only young candidate. Find the best decision for the company.*

Solution: Set U = $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ and let F = $\{p_1, p_2, p_3, p_4\}$ be the set of the parameters p_1=well experienced, p_2=holder of a university degree, p_3=holder of a driving license and p_4=young.

Consider the map f: F→P(U) defined by $f(p_1) = \{A_1, A_2, A_6\}$, $f(p_2) = \{A_2, A_3, A_5, A_6\}$, $f(p_3) = \{A_3, A_5\}$, $f(p_4) = \{C_4\}$. Then the soft set defined with respect to F and f is equal to

$$(f, F) = \{(p_1, \{A_1, A_2, A_6\}), (p_2, \{A_2, A_3, A_5, A_6\}), (p_3, \{A_3, A_5\}), (p_4, \{A_4\})\} \tag{7}$$

The tabular representation of the soft set (f, F), shown in **Table 1**, is formed by assigning to each candidate the binary element 1, if he/she satisfies the qualification addressed by the corresponding parameter, or the binary element 0 otherwise. Then, the *choice value* of each candidate is determined by adding the entries of the row of the tabular representation of (f, F) where it belongs. Thus, the candidates A_1 and A_4 have choice value 1 and all the others 2. The company, therefore, must employ one of the candidates A_2, A_3, A_5 or A_6.

Table 1. Tabular matrix of the soft set (f, F) of Example 2.

	p_1	p_2	p_3	p_4
A_1	1	0	0	0
A_2	1	1	0	0
A_3	0	1	1	0
A_4	0	0	0	1
A_5	0	1	1	0
A_6	1	1	0	0

3. Intuitionistic fuzzy assessment

When the assessment is performed with qualitative grades, cases appear in which one is not sure about the accuracy of the qualitative grades assigned to the objects under assessment. In such cases, the use of IFSs as tools provides a potential way for evaluating a group's overall performance. This new methodology is applied in the next example.

Example 3. *The new coach of a football team is not sure yet about the quality of his players. He characterized, therefore, the good players as follows: p_1 and p_2 by (1, 0), p_3, p_4 and p_5 by (0.8, 0.1), p_6 and p_7 by (0.7, 0.2), p_8, p_9 and p_{10} by (0.6, 0.2), p_{11} by (0.5, 0.1), p_{12} and p_{13} by (0.5, 0.2), p_{14} by (0.3, 0.6), p_{15} and p_{16} by (0.2, 0.6), p_{17} by (0.1, 0.7), and the remaining three players by (0, 1). In other words, the coach believes that p_1 and p_2 are good players, he is 80% sure that p_3, p_4 and p_5 are good players too, but simultaneously he has a 10% belief that they may not be good players and a 10% hesitation to decide about it, and so on. For the last three players the coach is absolutely sure that they are not good players. Estimate the overall quality of the team according to the opinion of the coach about the individual quality of his players.*

Solution: The overall quality of the football team can be estimated by calculating the mean value, say $M = (m, n)$, of the elements of the intuitionistic fuzzy set of the good players of the team. For this, according to the given data, Equation (5) gives that

$$M = (m, n) = \tfrac{1}{20} [2 (1, 0) + 3 (0.8, 0.1) + 2 (0.7, 0.2) + 3 (0.6, 0.2) + (0.5, 0.1) + 2 (0.5, 0.2) \\ + (0.3, 0.6) + 2 (0.2, 0.6) + (0.1, 0.7) + 3 (0, 1)] \tag{8}$$

Therefore, Equations (3) and (4) give that

$$(m, n) = \tfrac{1}{20} (9.9., 7.3) = (0.495, 0.365) \tag{9}$$

Consequently, a random player of the team is a good player by 49.5%, but simultaneously there is a 36.5% chance to not be a good player and a 14% hesitation in deciding whether he is a good player or not. These outcomes give quite a good idea about the overall quality of the football team.

Remark 2. *i) As a consequence of the fact that there is no general rule for defining the membership and non-membership functions of an IFS, there is also no general rule for the assessment of the individual performance of the members of a group using IFSs. The individual assessment of the players' in Example 3 was therefore performed on the basis of the coach's personal criteria.*

ii) The assessment method applied in Example 3 for estimating the overall quality of the football team is of general character. This means that it can be applied to all cases of evaluating a group's overall performance when the qualitative grades assigned to its members are uncertain, provided that the evaluator is in a position to assess the individual performance of the group's members using IFSs (see Remark 2(i)). These cases may include, for example, students' or teachers' performance, the mean level of athletes' assessment, chess or bridge players' skills and even computer programs' effectiveness, industrial machines' or artificial intelligence's device proficiency, race cars' competitions, etc.

iii) In an earlier work[39] we introduced a similar assessment method by using neutrosophic sets as tools instead of intuitionistic fuzzy sets. The use of intuitionistic fuzzy sets, however, involves fewer calculations and provides equally reliable results. The advantage of using neutrosophic sets, on the other hand, is that they enable one to handle data providing incomplete or inconsistent assessment information; for example, 40% believe that a player is good, 30% doubt about it, and at the same time 20% (40%, respectively) believe that he is not a good player.

4. Intuitionistic fuzzy decision making

The decision-making method by Maji et al.[38] does not help very much in Example 2, since it excludes only two of the six candidates for employment. This is due to the fact that in the tabular matrix of the corresponding SS, the characterizations of the candidates by the corresponding parameters are replaced with the binary elements (truth values) 0, 1. In other words, although the method Maji et al.[38] starts from a fuzzy basis using soft sets, it then uses bivalent logic for making the required decision, e.g., young or not young in the example. This could lead to inadequate decisions if some of the parameters have a fuzzy texture, like the parameter "young" in Example 2. For tackling this problem, we have used grey numbers in the tabular representation of the corresponding soft set instead of the binary elements 0, 1[34] (Section 5.3). Decision-making cases appear frequently in everyday life, however, in which the decision-maker is not sure of the accuracy of the qualitative (fuzzy) parameters assigned to certain elements of the universe. For this, we have used neutrosophic sets as tools for DM in these cases instead of grey numbers[7]. The use of intuitionistic fuzzy sets instead of neutrosophic sets, however, provides equally good results, requiring fewer calculations. This new decision-making method is developed in Example 4.

Example 4. *Revisiting Example 2, assume that the company has doubts about the accuracy of the binary characterizations 0, 1 assigned to each of the six candidates with respect to the parameters p_1 and p_4. It was decided, therefore, to replace the binary elements of the tabular representation of the corresponding soft set with intuitionistic fuzzy pairs. In this case, the tabular representation of the decision-making procedure takes the form shown in **Table 2**. Which will be the optimal decision in this case?*

Table 2. Tabular representation of the decision-making process.

	p_1	p_2	p_3	p_4
A_1	(1, 0)	(0, 1)	(0, 1)	(0.6, 0.1)
A_2	(1, 0)	(1, 0)	(0, 1)	(0.2, 0.6)
A_3	(0.5, 0.1)	(1, 0)	(1, 0)	(0.6, 0.2)
A_4	(0.5, 0.3)	(0, 1)	(0, 1)	(1, 0)
A_5	(0.5, 0.4)	(1, 0)	(1, 0)	(0.6, 0.1)
A_6	(1, 0)	(1, 0)	(0, 1)	(0.4, 0.2)

Solution: The four last columns of **Table 2** contain the elements of the intuitionistic fuzzy sets of the candidates who are well experienced, holders of a university degree, holders of a driving license and young respectively.

By Equations (3)–(5), the choice value of A_1 is equal to

$$\frac{1}{4}[(1, 0) + 2(0, 1) + (0.6, 0.1)] = \frac{1}{4}(1.6, 2.1) = (0.4, 0.525) \tag{10}$$

In the same way, one finds that the choice values of A_2, A_3, A_4, A_5 and A_6 are equal to (0.55, 0.4), (0.775, 0.075), (0.375, 0.575), (0.775, 0.125), and (0.6, 0.3), respectively.

The company, therefore, may choose either the candidate with the greatest membership degree or the candidate with the lower non-membership degree, i.e., either one of the candidates A_3 and A_5, or candidate A_3. A combination of the two criteria leads to the final choice of candidate A_3. Observe, however, that, since the hesitation degree of A_3 is 0.15 and of A_5 is 0.1, the choice of A_3 is connected with a slightly greater risk. In the final analysis, therefore, all the intuitionistic degrees assigned to each candidate give useful information about his suitability.

Remark 3. *i) Our DM algorithm applied in Example 4, which is an extension of the DM method of Maji et al.[38], provides better decisions than it in cases where the decision maker is not sure of the accuracy of the parameters. This algorithm can also be used in all the analogous cases of parametric multi-criteria DM involving fuzzy parameters. Examples that have already been presented in earlier authors' works (using GNs and NSs instead of IFSs) are related to decisions for buying a house[39], or a car[40] and for choosing a new player for a football team[7].*

ii) Weighted DM: In multiple DM processes, the decision-maker's goals are not always equally important. In such cases, weight coefficients are assigned to each parameter, whose sum is equal to 1. Assume, for instance, that the weight coefficients 0.4, 0.3, 0.2, and 0.1 have been assigned to the parameters p_1, p_2, p_3 and p_4 respectively, of Example 4. Then the weighted choice value of candidate A_1 is equal to

$$\frac{1}{4}[0.4(1, 0) + 0.3(0, 1) + 0.2(0, 1) + 0.1(0.6, 0.1)] = \frac{1}{4}(0.46, 0.51) = (0.115. 0.1275) \tag{11}$$

In the same way one finds that the choice values of the candidates A_2, A_3, A_4, A_5 and A_6 are (0.18, 0.065), (0.19, 0.015), (0.075, 0.115), (0.19, 0.0425) and (0.185, 0.055) respectively. The combination of the two criteria, therefore, shows again that the best decision for the company is to employ the candidate A_3.

5. Intuitionistic fuzzy topological spaces

The basic concepts and properties of ordinary sets are generalized for fuzzy sets and their extensions, including intuitionistic fuzzy sets[11,23]. Fuzzy sets and their extensions, apart from their various and interesting applications to everyday life situations, have also found important connections with areas of classical mathematics like algebra, analysis, geometry, topology, etc.

The notion of topological space[41], for example, has been generalized to fuzzy sets[42], to intuitionistic fuzzy sets[43], to neutrosophic sets[44], to soft sets[45], etc. It is recalled that a topology T on a set $U \neq \varnothing$ is a collection of subsets of U such that U and $\varnothing$ belong to T, and the intersection of any two elements of T and the union of any number of elements of T also belong to T. The elements of a topology T on U are called open subsets of U, and their complements are called closed subsets of U. The pair (U, T) is called a topological space on U. Topological spaces are the most general category of mathematical spaces, in which fundamental mathematical concepts can be defined[41].

In this section, we describe how one can generalize the concepts of convergence, continuity, compactness, and Hausdorff topological space from classical topology to intuitionistic fuzzy topological spaces. To introduce the concept of intuitionistic fuzzy topological space, one first needs the following two definitions:

Definition 6. *Let $A = [m_A, n_A]$ and $B = [m_B, n_B]$ be two intuitionistic fuzzy sets in U. Then:*

1) *The complement A^C of A is the intuitionistic fuzzy set $A^C = [n_A, m_A]$ in U.*
2) *A is said to be an intuitionistic fuzzy subset of B ($A \subseteq B$), if, and only if, $m_A(x) \leq m_B(x)$, and $n_A(x) \geq m_B(x)$, for all x in U. If $A \subseteq B$ and $B \subseteq A$, then A and B are equal intuitionistic fuzzy sets ($A = B$).*
3) *The union $A \cup B$ is the intuitionistic fuzzy set $[\max\{m_A, m_B\}, \min\{n_A, n_B\}]$ in U.*
4) *The intersection $A \cap B$ is the intuitionistic fuzzy set $[\min\{m_A, m_B\}, \max\{n_A, n_B\}]$ in U.*

It is straightforward to check that if A, B are crisp sets (fuzzy sets) then the previous definitions are reduced to the corresponding definitions of crisp sets (fuzzy sets).

Definition 7. *The empty intuitionistic fuzzy set in the universe U is defined to be the intuitionistic fuzzy set $\varnothing_I = [m, n]$, with $m(x) = 0$ and $n(x) = 1$, for all x in U, whereas the universal intuitionistic fuzzy set in U is defined to be the intuitionistic fuzzy set $U_I = [m, n]$, with $m(x) = 1$ and $n(x) = 0$, for all x in U.*

Obviously, we have that $A \cup U_I = U_I$, $A \cap U_I = A$, $A \cup \varnothing_I = A$ and $A \cap \varnothing_I = \varnothing_I$, for all IFSs A in U.

Example 5. *Let $U = \{x_1, x_2, x_3\}$ be the universal set and let $A = \{(x_1, 0.3, 0.6), (x_2, 0.5, 0.4), (x_3, 0.7, 0.5)\}$, $B = \{(x_1, 0.6, 0.2), (x_2, 0.3, 0.5), (x_3, 0.3, 0.6)\}$ be two intuitionistic fuzzy sets in U. Then:*

- *It is not $A \subseteq B$ and $B \subseteq A$.*
- *$A^C = \{(x_1, 0.6, 0.3), (x_2, 0.4, 0.5), (x_3, 0.5, 0.7)\}$ and $B^C = \{(x_1, 0.2, 0.6), (x_2, 0.5, 0.3), (x_3, 0.6, 0.3)\}$.*
- *$A \cup B = \{(x_1, 0.6, 0.2), (x_2, 0.5, 0.4), (x_3, 0.7, 0.5)\}$.*
- *$A \cap B = \{(x_1, 0.3, 0.6), (x_2, 0.3, 0.5), (x_3, 0.3, 0.6)\}$.*

Definition 8. *An intuitionistic fuzzy topology T on a non-empty set U is defined to be a collection of intuitionistic fuzzy sets in U such that:*

- *U_I and $\varnothing_I$ belong to T;*
- *The intersection of any two elements of T and the union of any number of elements of T belong to T.*

Trivial examples are the *discrete* intuitionistic fuzzy topology of all intuitionistic fuzzy sets in U and the non-discrete intuitionistic fuzzy topology $\{U_I, \varnothing_I\}$.

The elements of an intuitionistic fuzzy topology on U are called *open* intuitionistic fuzzy sets in U and their complements are called *closed* intuitionistic fuzzy sets in U. The pair (U, T) is called an intuitionistic fuzzy topological space on U.

Example 6. *Set $U = \{x\}$ and let $A = \{(x, 0.5, 0.4)\}$, $B = \{(x, 0.4, 0.8)\}$, $C = \{(x, 0.5, 0.4)\}$, $D = \{(x, 0.4, 0.8)\}$ be IFSs in U. Then it is easy to show that the collection $T = \{\varnothing_N, U_N, A, B, C, D\}$ is an intuitionistic fuzzy topology on U.*

The concept of convergence is extended to intuitionistic fuzzy topological spaces as follows:

Definition 9. *Given two IFSs A and B of the IFTS (U, T), B is said to be a neighborhood of A, if there exists an open intuitionistic fuzzyset K such that $A \subseteq K \subset B$. We say then, that a sequence $\{A_n\}$ of intuitionistic fuzzy sets of (U, T) converges to the intuitionistic fuzzy set A of (U, T), if there exists a positive integer m such that, for each integer $n \geq m$ and each neighborhood B of A we have that $A_n \subset B$.*

According to Zadeh's extension principle[12], if U and V are crisp sets, U, V ≠ ∅, then f can extended to a function F mapping fuzzy sets in U to fuzzy sets in V. It is straightforward to check that the same principle holds for intuitionistic fuzzy sets too.

Definition 10. *Let (U, T) and (V, S) be two intuitionistic fuzzy sets and let f be a function f: U→V. According to the extension principle, f can be extended to a function F which maps intuitionistic fuzzy sets of U to intuitionistic fuzzy sets of V. We say then that f is an intuitionistic fuzzy-continuous function, if, and only if, the inverse image of each open intuitionistic fuzzy set of V through F is an open intuitionistic fuzzy set of U.*

Definition 11. *A family $A = \{A_i, i \in I\}$ of intuitionistic fuzzy sets of an intuitionistic fuzzy topological space (U, T) is called a cover of U, if $U = \bigcup_{i \in I} A_i$ If the elements of A are open intuitionistic fuzzy sets, then A is called an open cover of U. Also, each intuitionistic fuzzy subset of A which is also a cover of U is called a sub-cover of A. The intuitionistic fuzzy topological space (U, T) is said to be compact, if every open cover of U contains a sub-cover with finitely many elements.*

Definition 12. *An intuitionistic fuzzy topological space (U, T) is said to be a T_1-intuitionistic fuzzy topological space if, and only if, for each pair of elements x_1, x_2 of U with $x_1 \neq x_2$, there exist at least two open intuitionistic fuzzy sets K_1 and K_2 such that $x_1 \in K_1$, $x_2 \notin K_1$ and $x_2 \in K_2$, $x_1 \notin K_2$. Further, (U, T) is said to be a T_2-intuitionistic fuzzy topological space, if, and only if, for each pair of elements x_1, x_2 of U, with $x_1 \neq x_2$, there exist at least two open intuitionistic fuzzy sets K_1 and K_2 such that $x_1 \in K_1$, $x_2 \in K_2$ and $K_1 \cap K_2 = \emptyset_I$.*

A T_2-intuitionistic fuzzy topological space is also called a Hausdorff or a separable intuitionistic fuzzy topological space. Obviously a T_2-intuitionistic fuzzy topological space is always a T_1-intuitionistic fuzzy topological space.

6. Conclusions and future problems

In this paper, intuitionistic fuzzy sets were used as tools in assessment and decision-making processes. This is very useful when one is not sure about the accuracy of the qualitative grades or parameters assigned to the elements of the universal set. At the end of the paper, we described how fundamental concepts of topological spaces can be extended to intuitionistic fuzzy topological spaces.

In general, the advantage of using intuitionistic fuzzy sets as tools for assessment is that they enable one to estimate the overall performance of a group with respect to a certain activity when the qualitative grades assigned to its members are uncertain. Also, in decision-making, the advantage of using intuitionistic fuzzy sets in the tabular matrix is that they help the decision-maker make the right decision when some of the parameters involved are of fuzzy texture.

Probability, fuzzy sets, and the related theories[24] are not able to tackle each one alone effectively for all the types of uncertainty existing in everyday life and science. Each of them is more suitable for managing some special types of uncertainty, e.g., probability for randomness, fuzzy sets for vagueness, intuitionistic fuzzy sets for imprecision in human thinking, neutrosophic sets for ambiguity and inconsistency, etc. All these theories together, however, provide an adequate framework for dealing with uncertainty. In particular, suitable combinations of two or more of these theories give more effective ways of obtaining better results. For example, type-2 and type-3 fuzzy sets have been combined with intuitionistic fuzzy sets and interval valued fuzzy sets to produce the concepts of type-2[46] and type-3 intuitionistic fuzzy sets and interval valued fuzzy sets[47,48] respectively. This is, therefore, an interesting area for further research.

Conflict of interest

The author declares no conflict of interest.

Refrences

1. Voskoglou M. Methods for Assessing Human–Machine Performance under Fuzzy Conditions. Mathematics. 2019, 7(3): 230. doi: 10.3390/math7030230
2. Berger JO. Statistical Decision Theory. Springer New York; 1980. doi: 10.1007/978-1-4757-1727-3
3. Zadeh LA. Fuzzy sets. Information and Control. 1965, 8(3): 338-353. doi: 10.1016/s0019-9958(65)90241-x
4. Bellman RE, Zadeh LA. Decision-Making in a Fuzzy Environment. Management Science. 1970, 17(4): B-141-B-164. doi: 10.1287/mnsc.17.4.b141
5. Alcantud JCR. Fuzzy Techniques for Decision Making. Symmetry 2018; 10(1): 6. doi: 10.3390/sym10010006
6. Zhu B, Ren P. Type-2 fuzzy numbers made simple in decision making. Fuzzy Optimization and Decision Making. 2021, 21(2): 175-195. doi: 10.1007/s10700-021-09363-y
7. Voskoglou MGr. An Application of Neutrosophic Sets to Decision Making. Neutrosophic Sets and Systems. 2023, 53: 1-9.
8. Atanassov KT. Intuitionistic fuzzy sets. Fuzzy Sets and Systems. 1986, 20(1): 87-96. doi: 10.1016/s0165-0114(86)80034-3
9. Ejegwa PA, Akubo AJ, Joshua OM. Intuitionistic Fuzzy Set and its Application in Career Determination via Normalized Euclidean Distance Method, European Scientific Journal. 2014, 10(15): 529-536.
10. Annamalai C. Intuitionistic Fuzzy Sets: New Approach and Applications. Available online: https://www.researchgate.net/publication/360395248 (accessed on 19 December 2023).
11. Atanassov KT. Intuitionistic Fuzzy Sets. Physica-Verlag HD, 1999. doi: 10.1007/978-3-7908-1870-3
12. Szmidt E, Kacprzyk J. Intuitionistic fuzzy sets in group decision making. Notes on Intuitionistic Fuzzy Sets. 1996, 2(1): 11–14.
13. Szmitd E, Kacprzyk J. Remarks on some applications of intuitionistic fuzzy sets in decision making. Notes on Intuitionistic Fuzzy Sets. 1996, 2(3): 22-31.
14. De SK, Biswas R, Roy AR. An application of intuitionistic fuzzy sets in medical diagnosis. Fuzzy Sets and Systems. 2001, 117(2): 209-213. doi: 10.1016/s0165-0114(98)00235-8
15. Szmitd E, Kacprzyk J. Intuitionistic fuzzy sets in same medical applications. Notes on Intuitionistic Fuzzy Sets. 2001, 7(4): 58–64.
16. Davvaz B, Hassani Sadrabadi E. An application of intuitionistic fuzzy sets in medicine. International Journal of Biomathematics. 2016, 9(3): 1650037. doi: 10.1142/s1793524516500376
17. Mohamed Kozae A, Shokry M, Omran M. Intuitionistic Fuzzy Set and Its Application in Corona Covid-19. Applied and Computational Mathematics. 2020, 9(5): 146. doi: 10.11648/j.acm.20200905.11
18. Atanassov K, Gargov G. Intuitionistic fuzzy logic. Comptes Rendus de L'Academie Bulgare des Sciences. 1990, 43(3): 9–12.
19. Atanassov K, Georgiev C. Intuitionistic fuzzy prolog. Fuzzy Sets and Systems. 1993, 53(2): 121-128. doi: 10.1016/0165-0114(93)90166-f
20. Meena K, Ponnappen L. An Application of Intuitionistic Fuzzy Sets in Choice of Discipline of Study. Global Journal of Pure and Applied Mathematics. 2018, 14(6): 867-871.
21. Molodtsov D. Soft set theory—First results. Computers & Mathematics with Applications. 1999, 37(4-5): 19-31. doi: 10.1016/s0898-1221(99)00056-5
22. Zadeh LA. Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets and Systems. 1978, 1(1): 3-28. doi: 10.1016/0165-0114(78)90029-5
23. Klir GJ, Folger TA. Fuzzy sets, Uncertainty and Information. Prentice-Hall, London; 1988.
24. Voskoglou MGr. Generalizations of Fuzzy Sets and Related Theories. In: Voskoglou M (editor). An Essential Guide to Fuzzy Systems. Nova Science Publishers, NY; 2019. pp. 345-352.
25. Atanassov KT. 25 years of Intuitionistic Fuzzy Sets or: The most important mistakes and results of mine. In: Proceedings of the 7th International Workshop on Intuitionistic Fuzzy Sets and Generalizations, Warsaw, Poland, 2008.
26. Atanassova V. Intuitionistic Fuzzy Sets and Applications, Special Issue, Mathematics. Available online: https://www.mdpi.com/journal/mathematics/special_issues/intuitionistic_fuzzy_sets (accessed on 19 December 2023).
27. Smarandache F. Neutrosophy/Neutrosophic Probability, Set, and Logic, Proquest Information and Learning. Ann Arbor; 1998.

28. Mendel JM. Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions. Prentice-Hall; 2001.
29. Zadeh LA. The concept of a linguistic variable and its application to approximate reasoning—I. Information Sciences. 1975, 8(3): 199-249. doi: 10.1016/0020-0255(75)90036-5
30. Mohammadzadeh A, Zhang C, Alattas KA, et al. Fourier-based type-2 fuzzy neural network: Simple and effective for high dimensional problems. Neurocomputing. 2023, 547: 126316. doi: 10.1016/j.neucom.2023.126316
31. Cao Y, Raise A, Mohammadzadeh A, et al. Deep learned recurrent type-3 fuzzy system: Application for renewable energy modeling/prediction. Energy Reports. 2021, 7: 8115-8127. doi: 10.1016/j.egyr.2021.07.004
32. Mohammadzadeh A, Sabzalian MH, Zhang W. An Interval Type-3 Fuzzy System and a New Online Fractional-Order Learning Algorithm: Theory and Practice. IEEE Transactions on Fuzzy Systems. 2020, 28(9): 1940-1950. doi: 10.1109/tfuzz.2019.2928509
33. Kaufmann AK, Gupta MM. Introduction to Fuzzy Arithmetic: Theory and Applications. Van Nostrand Reinhold Company; 1991.
34. Dubois D, Prade H. Interval-valued fuzzy sets, possibility theory and imprecise probability. In: Proceedings of the Joint 4th Conference of the European Society for Fuzzy Logic and Technology and the 11th Rencontres Francophones sur la Logique Floue et ses Applications; 7–9 September 2005; Barcelona, Spain. pp. 314-319.
35. Pawlak Z. Rough Sets: Aspects of Reasoning about Data. Kluer Academic Publishers; 1991.
36. Deng J. Control problems of grey systems. Systems and Control Letters. 1982, 1(5): 288-294.
37. Maji PK, Biswas R, Roy AR. Soft set theory. Computers & Mathematics with Applications. 2003, 45(4-5): 555-562. doi: 10.1016/s0898-1221(03)00016-6
38. Maji PK, Roy AR, Biswas R. An application of soft sets in a decision-making problem. Computers & Mathematics with Applications. 2002, 44(8-9): 1077-1083. doi: 10.1016/s0898-1221(02)00216-x
39. Voskoglou MGr. Fuzziness, Indeterminacy and Soft Sets: Frontiers and Perspectives. Mathematics. 2022, 10(20): 3909. doi: 10.3390/math10203909
40. Voskoglou MGr. Neutrosophic Assessment and Decision Making. Equations. 2023, 3: 68-72.
41. Willard S. General Topology. Dover Publications; 2004.
42. Chang CL. Fuzzy topological spaces. Journal of Mathematical Analysis and Applications. 1968, 24(1): 182-190. doi: 10.1016/0022-247x(68)90057-7
43. Luplanlez FG. On Intuitionistic Fuzzy Topological Spaces. Kybernetes. 2006, 35(5): 743-747.
44. Salama AA, Alblowi SA. Neutrosophic Sets and Neutrosophic Topological Spaces. IOSR Journal of Mathematics. 2013, 3(4): 31-35.
45. Shabir M, Naz M. On soft topological spaces. Computers & Mathematics with Applications. 2011, 61(7): 1786-1799. doi: 10.1016/j.camwa.2011.02.006
46. Dan S, et al. Intuitionistic Type-2 Fuzzy Set and Its Properties. Symmetry. 2019, 11: 808.
47. Castillo O, Melin P. Towards Interval Type-3 Intuitionistic Fuzzy Sets and Systems. Mathematics. 2022, 10(21): 4091. doi: 10.3390/math10214091
48. Castillo O, Castro JR, Melin P. Interval Type-3 Fuzzy Control for Automated Tuning of Image Quality in Televisions. Axioms. 2022, 11(6): 276. doi: 10.3390/axioms11060276