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Article

On wavelet type Chlodovsky Bézier operators

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Abstract: This paper mainly deals with Chlodovsky Bézier variants constructed via compactly supported Daubechies wavelets. We estimate the convergence rate of the aforementioned operators at a fixed point $x_0 > 0$ at which the one-sided limits exist of the target function f. It is evident that the class of operators under consideration encompasses at least the classical version of the Chlodovsky operators along with their B ézier and Kantorovich variants. Therefore, our findings broaden and build upon previous results on Chlodovsky, Chlodovsky Bézier, Chlodovsky-Kantorovich Bézier operators presented in the literature.

Keywords: wavelets; compactly supported Daubechies wavelets; Chlodovsky operator; Bézier basis; convergence rate; variation
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1. Introduction

Wavelet analysis is a comprehensive and interdisciplinary field that has seen significant contributions from a diverse range of experts, including mathematicians; engineers in fields such as computer, mechanical, and electrical engineering; as well as physicists and specialists in applied sciences like geology, geophysics, and statistics. Over the past four decades, substantial progress has been made in both the theoretical foundations and practical applications of wavelet analysis, leading to an extensive body of literature, including numerous research papers and textbooks.

This study focuses on the construction and impact of wavelets, specifically within the context of approximation theory. Key contributions to this area have been made by prominent researchers, including Daubechies, Agratini, Zhou, Gonska, and Karsli.

Let f be a measurable and locally bounded function on $[0,\infty)$ $(f \in M_{loc}[0,\infty))$. The Bernstein-Chlodovsky polynomials $C_n f$ are defined as

$$(C_n f)(x) := \sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right) P_{n,k}\left(\frac{x}{b_n}\right) \quad \text{for} \quad 0 \le x \le b_n \tag{1}$$

where $n \in \mathbb{N}$, $P_{n,k}(\bullet)$ is the Bernstein basis and the positive increasing sequence (b_n) satisfies

$$b_n \to \infty \text{ and } \frac{b_n}{n} \to 0 \text{ as } n \to \infty \tag{2}$$

([1]). Recall that $J_{n,k}(\bullet) = \sum\limits_{j=k}^{n} P_{n,j}(\bullet)$ be the Bézier basis given in [2]. The Bézier basis functions were originally introduced as part of the mathematical framework for constructing Bézier curves in computer graphics and geometric design, where they are used to create smooth curves that interpolate at some knot (or control) points. Specifically, for any $n-th$ degree Bézier curve, the basis functions satisfy the following conditions:

1) Non-negativity;
2) Partition of unity;
3) Boundary conditions, namely, interpolation at the endpoints.

Let $\alpha \geq 1$. This specifies that α is a positive constant. It likely serves as a parameter that will be used in modifying the operator, possibly in a scaling or smoothing capacity. The Bézier modification $C_{n,\alpha}f$ of operators (1), the notation suggests that this modification involves some form of Bézier-type modification of the original operator and is characterized by:

$$\left(C_{n,\alpha}f\right)(x)=\sum_{k=0}^{n}f(\frac{kb_n}{n})Q_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right)\qquad for\quad x\in[0,b_n] \tag{3}$$

where $Q_{n,k}^{(\alpha)}(\bullet)=J_{n,k}^{\alpha}(\bullet)-J_{n,k+1}^{\alpha}(\bullet)$ for $\bullet\in[0,1]$ ($J_{n,l}(\bullet)\equiv 0$ if $l>n$) (see [3,4]).

This would then be followed by a formal definition of how the operator $C_{n,\alpha}f$ modifies f, typically involving a combination of differential or integral operators adjusted by the parameters n and α. A complete definition would generally require more context from the broader mathematical framework.

If you're looking for the actual operator form or further specifics, these would typically be provided in the full mathematical text where the original equation is derived, particularly if it is associated with a specific approach within operator theory or Bézier curve manipulation in functional analysis or approximation theory.

When $\alpha=1$, the operators $C_{n,\alpha}f$ (3) simplify to the form given by Equation (1).

Since the discrete-type positive linear operators, such as Bernstein polynomials (1) and their Bézier variants (3), are not suitable for L^p approximation when $(1\leq p<\infty)$,, their Kantorovich and Durrmeyer modifications are employed to achieve positive approximation results within this framework (see [4–9]).

To take care of the larger class $L^1[0,1]$ of functions integrable on $[0,1]$, Butzer and Karsli [10] created the Chlodovsky-Bernstein-Kantorovich operators as

$$\left(K_nf\right)(x):=\frac{n+1}{b_{n+1}}\sum_{k=0}^{n}P_{n,k}\left(\frac{x}{b_{n+1}}\right)\int\limits_{\frac{kb_{n+1}}{n+1}}^{\frac{(k+1)b_{n+1}}{n+1}}f(u)du\quad\text{if } 0\leq x\leq b_{n+1} \tag{4}$$

where $f\in L_{loc}[0,\infty)$ and (b_n) satisfies conditions (2). Keep in mind that the Bernstein-Kantorovich operators K_n^*f are defined as in Equation (4) in which $b_n=1$.

For $f\in L_{loc}[0,\infty)$, Pych-Taberska and Karsli [9] introduced the $(n-1)$ order Bézier variant of the Chlodovsky-Kantorovich operators (4) as

$$\left(K_{n-1,\alpha}f\right)(x):=\frac{n}{b_n}\sum_{k=0}^{n-1}Q_{n-1,k}^{(\alpha)}\left(\frac{x}{b_n}\right)\int\limits_{\frac{kb_n}{n}}^{\frac{(k+1)b_n}{n}}f(u)du\quad\text{if } 0\leq x\leq b_n,$$

where $\alpha>0$, $Q_{n-1,k}^{(\alpha)}(\bullet)=J_{n-1,k}^{\alpha}(\bullet)-J_{n-1,k+1}^{\alpha}(\bullet)$ and $J_{n-1,k}(\bullet)$ are the Bézier basis functions defined for $\bullet \in [0,1]$ as

$$J_{n-1,k}(\bullet) = \sum_{j=k}^{n-1} P_{n-1,j}(\bullet) \quad \text{if } k = 0,1,\ldots,n-1,$$

$J_{n-1,n}(\bullet) = 0$ (see [2]). Clearly, if $\alpha = 1$ then $K_{n-1,\alpha} f$ reduce to operators (4) with n replaced by $n-1$.

The primary focus of this study is to investigate the rate of pointwise convergence of the wavelet type operators when $f \in M_{loc}[0,\infty)$. We provide evaluations of the convergence rate of $(WC_{n,\alpha} f)(x)$ at the points where f is continuous, as well as at the first-kind discontinuity points for f. Our results will be presented for $WC_{n,\alpha} f$ with $\alpha \geq 1$.

In other words, leveraging the effects and interrelationships among various function spaces facilitated by wavelets, we aim to develop a generalized and extended framework for approximation theory. This will be achieved through the introduction of an integral operator, referred to as wavelet-type operators (see [11–16]). These newly proposed operators exhibit greater flexibility compared to their predecessors and serve as natural extensions of classical operators, including generalized Bézier operators and their Kantorovich modifications. For further insights and recent advancements in this area, we refer the reader to the studies conducted by the author (please see Karsli [17]).

In this study, a novel wavelet-based generalization, extension, and enrichment strategy is presented to enhance the computational accuracy and convergence rate of the classical Chlodovsky operators, which are defined for solving approximation problems on unbounded intervals. By incorporating wavelets into the approximation operators, the proposed method enables the effective reconstruction of functions with complex features, such as oscillations and discontinuities, that are difficult to capture using standard approaches.

In the current paper:

- **i)** We significantly improve the existing estimations.
- **ii)** We extend these results to functions f with bounded variation on $[0,\infty)$.
- **iii)** We broaden the scope to include a wide class of functions for all parameters $\alpha \geq 1$.

2. Auxiliary definitions and construction

In this preliminary section, we will review some important notation and background material, specifically focusing on Daubechies' compactly supported wavelets (refer to [18,19]), which are used extensively throughout the work.

We follow standard notation by letting $C[0,\infty)$ represent the space of all continuous and bounded functions defined on the interval $[0,\infty)$, equipped with the standard norm, making it a complete normed vector space.

Furthermore, we denote by $[a]$ the floor function, which gives the largest integer less than or equal to a given real number a.

Finally, we will now revisit the following definitions, which are central to the discussion that follows.

Definition 1. *Consider a function g that is bounded on a compact interval $[a,b] \subset$*

$\mathbb{R}$. *The concept of the modulus of variation, denoted as* $\nu_n(g;[a,b])$*, is introduced to quantify the extent of variation in g over the given interval. This modulus of variation is specifically defined for nonnegative integers n and provides a measure of how the function fluctuates within the interval* $[a,b] =: I$.

$$\nu_0(g;I) := 0$$

and for $k \geq 1$

$$\nu_k(g;I) := \sup_{\Delta_k} \sum_{j=0}^{k-1} |g(x_{2j+1}) - g(x_{2j})|\,,$$

here Δ_k *represents a collection of* k *mutually disjoint intervals* (x_{2j}, x_{2j+1})*,* $j = 0, 1, ..., k-1$ (*see* [4,9,20]).

Definition 2. *(Multiresolution analysis) A (MRA) is a structured sequence* $\{Vj\}_{j\in Z}$ *of closed subspaces within the function space* $L^2(\mathbb{R})([18,19])$. *This sequence is designed to systematically decompose and approximate functions at different levels of resolution. The subspaces* V_j *are constructed in such a way that they satisfy specific mathematical properties, ensuring a hierarchical and consistent representation of functions across varying scales. These properties define the fundamental framework of MRA and enable its application in areas such as wavelet theory and signal processing.*

i) *The subspace* V_j *consists of all functions* f *belonging to* $L^2(\mathbb{R})$ *that remain constant on each interval of length* 2^{-j}. *In other words, every function* f *in* V_j *exhibits a piecewise constant structure, where its value does not change within any subinterval of length* 2^{-j}*, ensuring a stepwise representation of functions at a given resolution level.*

$$...V_{-j-1} \subset V_{-j} \subset ... \subset V_j \subset V_{j+1} \subset ... \subset L_2(\mathbb{R}),$$

$$\overline{\bigcup_j V_j} = L_2(\mathbb{R}),$$

ii)

$$\bigcap_j V_j = \{0\}\,.$$

A wavelet basis family begins with orthogonal functions ϕ and φ, both of which belong to the space $L_2(\mathbb{R})$. These two functions are fundamental in wavelet theory: $\phi(\circ)$ is referred to as the scaling function (sometimes named father wavelet), and $\varphi(\circ)$ is known as the wavelet function (sometimes named mother wavelet). These satisfy specific mathematical conditions that enable them to form a basis for representing functions in terms of wavelets. The scaling function $\phi(\circ)$ is used to capture the low-frequency components of a function, while the wavelet function $\varphi(\circ)$ is responsible for capturing the high-frequency details. The relationship between these two functions is governed by certain equations, which allow them to form a complete and orthogonal set for decomposing functions across different scales and resolutions.

$$\int_{-\infty}^{\infty} \phi(\circ)d\circ = 1, \int_{-\infty}^{\infty} \varphi(\circ)d\circ = 0.$$

The wavelets refer to a family of orthonormal functions

$$\varphi_{a,b}(\circ)=\frac{1}{\sqrt{a}}\varphi\left(\frac{\circ-b}{a}\right),\quad a>0,b\in\mathbb{R} \tag{5}$$

with their special representation

$$\varphi_{j,k}(\circ)=2^{j/2}\varphi(2^{j}\circ-k) \tag{6}$$

is very well-known in Franklin-Strömberg and Littlewood-Paley Theories, where φ is the basic wavelet.

By considering specific values for the parameters a and b, one can derive various types of wavelets from Equations (5) and (6), including, but not limited to, the following examples: Haar, Radamacher, Franklin, Strömberg, Mallat, Meyer, Daubechies wavelets, etc.

This version emphasizes the process of deriving different wavelet types from particular choices of parameters and highlights that there are several possible wavelet forms resulting from those choices.

We now introduce the suitable wavelets that are considered in this paper. Let the scaling function, also referred to as the father wavelet, $\psi\in L_{\infty}(\mathbb{R})$,satisfy specific properties that are fundamental to the Daubechies wavelet construction. These functions are characterized by compact support, meaning they are non-zero only over a finite interval, which makes them well-suited for applications that require localized wavelet functions. The scaling function ψ holds significant importance in the construction of the wavelet basis, and it adheres to certain mathematical conditions, ensuring orthogonality and smoothness properties that are essential for efficiently decomposing and reconstructing signals or functions in wavelet analysis.

a) For a $\lambda>0$, supp $\psi\subset[0,\lambda]$;

b) $\int\limits_{\mathbb{R}}\psi(x)\,dx=1$;

c)

$$\int\limits_{-\infty}^{\infty}x^{j}\psi(x)dx=0,\quad j=1,...,N.$$

Now we define the new operators via compactly supported Daubechies wavelets as follows.

Definition 3. *Let $f\in M_{loc}[0,\infty)$ and $\psi\in L_{\infty}(\mathbb{R})$ be a father wavelet satisfies a)-c). Then the wavelet-Chlodovsky Bézier operators, constructed via suitable wavelets are defined by:*

$$(WC_{n,\alpha}f)(t):=\frac{n}{b_n}\sum_{k=0}^{n-1}Q_{n-1,k}^{(\alpha)}\left(\frac{t}{b_n}\right)\int\limits_{\mathbb{R}}f\left(x\right)\psi\left(\frac{nx}{b_n}-k\right)dx,\qquad(t\in\mathbb{R}) \tag{7}$$

specifying that supp$(\psi)\subseteq[0,\lambda]$, $0<\lambda$.

Throughout the paper, let

$$(WC_{n,\alpha}f)(t) := \int\limits_{\mathbb{R}} f\,(x)\, G_{n,k}^{(\alpha)}\left(\frac{x}{b_n}, \frac{t}{b_n}\right) dx, \qquad (t \in \mathbb{R})$$

where

$$G_{n,k}^{(\alpha)}\left(\frac{x}{b_n}, \frac{t}{b_n}\right) = \frac{n}{b_n}\sum_{k=0}^{n-1} Q_{n-1,k}^{(\alpha)}\left(\frac{t}{b_n}\right)\psi\left(\frac{nx}{b_n} - k\right) \tag{8}$$

Remark 1. *If we select the wavelet ψ to be the Haar function, defined as $\psi(x) = \chi_{[0,1]}(x)$, where $\chi_{[0,1]}(x)$ is the characteristic function of the interval $[0,1]$, then the wavelet-type operators considered in this context naturally simplify to the Kantorovich form of the Chlodovsky Bézier operators. Specifically, by choosing $\psi(x)$ in this way, the resulting wavelet-based approximation method becomes equivalent to a well-known class of operators, namely the Kantorovich modifications of the Chlodovsky Bézier operators. These operators are a generalization of the Bézier operators and are typically used to approximate functions while preserving certain desirable properties, such as positivity and smoothness.*

$$\begin{aligned}(WC_{n,\alpha}f)(t) &= \frac{n}{b_n}\sum_{k=0}^{n-1} Q_{n-1,k}^{(\alpha)}\left(\frac{t}{b_n}\right)\int\limits_{\mathbb{R}} f\,(x)\,\psi\left(\frac{nx-kb_n}{b_n}\right)dx\\ &= \sum_{k=0}^{n-1} Q_{n-1,k}^{(\alpha)}\left(\frac{t}{b_n}\right)\int\limits_0^1 f\left(\frac{(u+k)\,b_n}{n}\right)du\\ &= \frac{n}{b_n}\sum_{k=0}^{n-1} Q_{n-1,k}^{(\alpha)}\left(\frac{t}{b_n}\right)\int\limits_{\frac{kb_n}{n}}^{\frac{(k+1)b_n}{n}} f(u)du = (K_{n-1,\alpha}f)(t).\end{aligned}$$

This implies that the operators we have constructed using wavelets, denoted as Equation (7), serve as an intuitive extension of the Kantorovich-type Chlodovsky Bézier operators. These operators were originally introduced by Pych-Taberska and Karsli [9]. By incorporating wavelet functions into the construction of these operators, we generalize and extend the framework of the Kantorovich-type Bézier operators, which are known for their ability to approximate functions in a way that preserves certain properties, such as positivity and smoothness. This extension allows for a broader class of approximations, potentially enhancing the flexibility and applicability of these operators in various analytical contexts.

3. Main results

In this section, we will present several notations and structural assumptions that will be pivotal in the proofs of our convergence theorems. This part outlines the primary approximation results of the present work. The following key points will be discussed.

At this stage, we are prepared to present one of the initial major results of this study. This result establishes a robust connection between the Chlodovsky Bézier operators, as described in Equation (3), and the new operators (7) we have developed using wavelet functions.

Theorem 1. *Let* $\psi \in L_\infty(R)$ *satisfies a-c). For* $f \in M_{loc}[0,\infty)$, *the algebraic moments of wavelet-Chlodovsky Bézier operators and the Chlodovsky Bézier operators* (3) *are the same, namely*

$$(WC_{n,\alpha}x^s)(t)=\sum_{k=0}^{n-1}\left(\frac{kb_n}{n}\right)^s Q_{n-1,k}^{(\alpha)}\left(\frac{t}{b_n}\right).,\quad s=0,1,...,K$$

holds true.

Proof. In view of Equation (7), we have

$$\begin{aligned}(WC_{n,\alpha}x^s)(t) &= \frac{n}{b_n}\sum_{k=0}^{n-1}Q_{n-1,k}^{(\alpha)}\left(\frac{t}{b_n}\right)\int_{\mathbb{R}}x^s\psi\left(\frac{nx-kb_n}{b_n}\right)dx\\ &= \sum_{k=0}^{n-1}Q_{n-1,k}^{(\alpha)}\left(\frac{t}{b_n}\right)\int_{\mathbb{R}}\left(\frac{(u+k)\,b_n}{n}\right)^s\psi(u)du\\ &= \left(\frac{b_n}{n}\right)^s\sum_{k=0}^{n-1}Q_{n-1,k}^{(\alpha)}\left(\frac{t}{b_n}\right)\int_{\mathbb{R}}(u+k)^s\,\psi(u)du\\ &= \left(\frac{b_n}{n}\right)^s\sum_{k=0}^{n-1}Q_{n-1,k}^{(\alpha)}\left(\frac{t}{b_n}\right)\int_{\mathbb{R}}\left[\sum_{i=0}^{s}\begin{pmatrix}s\\ i\end{pmatrix}u^ik^{s-i}\right]\psi(u)du.\end{aligned}$$

Considering the result from part c), we can deduce the following when $i\neq 0$

$$\int_{\mathbb{R}}\left[\sum_{i=0}^{s}\begin{pmatrix}s\\ i\end{pmatrix}u^ik^{s-i}\right]\psi(u)du=0.$$

Additionally, when $i=0$, and based on the result from part b), we obtain the following

$$\begin{aligned}(WC_{n,\alpha}x^s)(t) &= \left(\frac{b_n}{n}\right)^s\sum_{k=0}^{n-1}Q_{n-1,k}^{(\alpha)}\left(\frac{t}{b_n}\right)\int_{\mathbb{R}}k^s\psi(u)du\\ &= \sum_{k=0}^{n-1}\left(\frac{kb_n}{n}\right)^sQ_{n-1,k}^{(\alpha)}\left(\frac{t}{b_n}\right).\end{aligned}$$

□

Remark 2. *The central moments of the wavelet-based Chlodovsky Bézier operators (denoted as Equation* (7)*) are identical to those of the classical Chlodovsky Bézier operators, as described in Equation* (3). *Certainly, following the approach outlined in Theorem 1 from earlier, we obtain the following result:*

$$\begin{aligned}
(WB_{n,\alpha}\,(x-t)^{\beta})(t) &= \frac{n}{b_n}\sum_{k=0}^{n-1} Q_{n-1,k}^{(\alpha)}\left(\frac{t}{b_n}\right)\int\limits_{\mathbb{R}} (x-t)^{\beta}\,\psi\left(\frac{nx-kb_n}{b_n}\right)dx \\
&= \sum_{k=0}^{n-1} Q_{n-1,k}^{(\alpha)}\left(\frac{t}{b_n}\right)\int\limits_{\mathbb{R}} \left(\frac{(u+k)\,b_n}{n}-t\right)^{\beta}\psi(u)du \\
&= \frac{1}{n^{\beta}}\sum_{k=0}^{n-1} Q_{n-1,k}^{(\alpha)}\left(\frac{t}{b_n}\right)\int\limits_{\mathbb{R}} ((u+k)\,b_n-nt)^{\beta}\,\psi(u)du \\
&= \frac{1}{n^{\beta}}\sum_{k=0}^{n-1} Q_{n-1,k}^{(\alpha)}\left(\frac{t}{b_n}\right)\int\limits_{\mathbb{R}} \left[\sum_{i=0}^{\beta}\left(\begin{array}{c}\beta\\ i\end{array}\right)u^i b_n^i\,(kb_n-nt)^{\beta-i}\right]\psi(u)du.
\end{aligned}$$

Using the properties outlined in parts b) and c), we can derive the following result:

$$(WC_{n,\alpha}\,(x-t)^{\beta})(t) = \frac{1}{n^{\beta}}\sum_{k=0}^{n-1} Q_{n-1,k}^{(\alpha)}\left(\frac{t}{b_n}\right)(kb_n-nt)^{\beta} \tag{9}$$

specifically,

$$\begin{aligned}
m_0(\psi) &:= (WC_{n,\alpha}1)(t) = 1, \\
m_1(\psi) &:= (WC_{n,\alpha}\,(x-t))(t) = 0, \\
m_2(\psi) &:= (WC_{n,\alpha}\,(x-t)^2)(t) = \alpha\frac{b_n}{n}t\left(1-\frac{t}{b_n}\right).
\end{aligned}$$

Moreover, for $q > 1$ and $n \geq 2$,

$$\mu_{2q}(\psi) \leq \alpha c_q\left(\frac{b_n}{n}\right)^q\left(t\left(1-\frac{t}{b_n}\right)+\frac{b_n}{n}\right)^q,$$

where $c_q > 0$ represents a constant that depends solely on the value of q $(c_1 = 1,\ c_2 < 5,\ c_3 < 61)$. In general

$$(WC_{n,\alpha}\,(x-t)^{\beta})(t) = O\left((n/b_n)^{-[(\beta+1)/2]}\right) \tag{10}$$

holds true.

Theorem 2. *Let $f \in M_{loc}[0,\infty)$ and $x \in (0,\infty)$ be a first-kind discontinuity point for f. For $n \geq 2$ such that $b_n > 2x,\ n/b_n \geq 4$ we have*

$$\begin{aligned}
\left|(WC_{n,\alpha}f)\,(x) - \frac{1}{2^{\alpha}}f(x+) - \left(1-\frac{1}{2^{\alpha}}\right)f(x-)\right| &\leq 2v_1(g_x; H_x(x\sqrt{b_n/n})) \\
&+ \frac{\alpha\varphi n(t)}{t^2}\left[\sum j = 1^{m-1}\frac{v_j(g_x; H_x(jx\sqrt{b_n/n}))}{j^3} + \frac{v_m(g_x; H_x(x))}{m^2}\right] \\
&+ \frac{M(b_n; f)}{x^{2q}}\mu n, 2q(\psi) + 4\alpha\sqrt{\frac{bn}{n}}\sqrt{\frac{b_n}{x\,(b_n-x)}}\,|f(x+)-f(x-)|,
\end{aligned}$$

where $m := [\sqrt{n/b_n}]$, $M(b; f) := \sup\limits_{0 \le t \le b} |f(t)|$, $\varphi_n(x) = x\left(1 - \frac{x}{b_n}\right) + \frac{b_n}{n}$,

$$g_x(t) := \begin{cases} f(t) - f(x+) & if \quad t > x \\ 0 & if \quad t = x \\ f(t) - f(x-) & if \quad 0 \le t < x \end{cases} \tag{11}$$

q represents an arbitrary positive integer.

So, we get the following approximation theorems.

Corollary 1. *Let* $f \in M_{loc}[0, \infty)$ *together with the hypotheses of Theorem 2. Then we have*

$$\lim_{n \to \infty} \left(WC_{n,\alpha} f\right)(x) = \frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2^\alpha}\right) f(x-).$$

Corollary 2. *Suppose that* $f \in M_{loc}[0, \infty)$. *Then at every continuity point* $x \in (0, \infty)$ *of f, we have*

$$\lim_{n \to \infty} \left(WC_{n,\alpha} f\right)(x) = f(x).$$

4. Some lemmas

Lemma 1. *Let* $0 < x < \infty$, $\alpha \ge 1$ *and* $G_{n,k}^{(\alpha)}$ *as is defined in Equation* (8).

(i) *If* $0 < s < x$ *then*

$$\int_0^s G_{n,k}^{(\alpha)}(x, t)\, dt \le \frac{\alpha m_2(\psi)}{(x - s)^2},$$

(ii) *If* $x < s < \infty$ *then*

$$\int_s^\infty G_{n,k}^{(\alpha)}(x, t)\, dt \le \frac{\alpha m_2(\psi)}{(s - x)^2},$$

see also [9].

Since

$$Q_{n,k}^{(\alpha)} = J_{n,k}^\alpha - J_{n,k+1}^\alpha \le \alpha p_{n,k}, \quad \alpha \ge 1$$

the proofs follows.

Let $x > 0$. *Setting*

$$I_x(u) := [0, \infty) \cap [x + u, x] \quad when \quad u < 0,$$

$$I_x(u) := [x, x + u] \qquad when \quad u > 0.$$

Lemma 2. *Let* $f \in M_{loc}[0, \infty)$. *Consider a fixed point* $x \in (0, \infty)$, *where the one-sided limits of* $f(x+)$, $f(x-)$ *exist. We set* $d_n := \sqrt{b_n/n}$. *If* $h = -x$ *or* $h = x$, *then for all integers* n *such that* $n/b_n \ge 4$ *we have*

$$\left| \int_{I_x(h)} g_x(t) G_{n,k}^{(\alpha)}(x, t) dt \right| \le v_1(g_x; I_x(hd_n)) + \frac{8\alpha m_2(\psi)}{x^2} \left(\sum_{j=1}^{m-1} \frac{v_j(g_x; I_x(jhd_n))}{j^3} + \frac{v_m(g_x; I_x(h))}{m^2} \right),$$

where g_x *is as in Equation* (11), $m := [1/d_n]$, $\varphi_n(x) = x\left(1 - x/b_n\right) + b_n/n$. *See also*

[9].

Lemma 3. *Let $x > 0$, $\alpha \geq 1$ and let $k^* = [nx/b_n]$. Then for all $n \in N$ and for arbitrary k^* satisfying $0 \leq k^* \leq n$ there holds*

$$Q^{(\alpha)}_{n-1,k^*}\left(\frac{x}{b_n}\right) \leq \alpha\sqrt{\frac{b_n}{n}}\sqrt{\frac{b_n}{x\left(b_n - x\right)}},$$

where $n \geq 2$ is so large that $b_n > 2x$ [9].

Lemma 4. *Let $\alpha \geq 1$ and let*

$$sgn^{(\alpha)}_x(t) := \begin{cases} 2^\alpha - 1 & when \quad t > x, \\ 0 & when \quad t = x, \\ -1 & when \quad 0 \leq t < x. \end{cases}$$

Then one has

$$\left|\left(WC_{n,\alpha} sgn^{(\alpha)}_x\right)(x)\right| \leq 2^{\alpha+2}\alpha\sqrt{\frac{b_n}{n}}\sqrt{\frac{b_n}{x\left(b_n - x\right)}}.$$

see also [9].

5. Proofs of the main results

At this point, we are prepared to demonstrate the proofs of our key theorems.

Proof of Theorem 2. We decompose $f(t)$ into

$$\begin{aligned} f(t) =& \frac{1}{2^\alpha}f(x+) + \left(1 - \frac{1}{2^\alpha}\right)f(x-) + g_x(t) + \frac{f(x+) - f(x-)}{2^\alpha}sgn^{(\alpha)}_x(t) \\ &+ \delta_x(t)\left[f(x) - \frac{1}{2^\alpha}f(x+) - \left(1 - \frac{1}{2^\alpha}\right)f(x-)\right] \end{aligned} \tag{12}$$

where

$$\delta_x(t) = \begin{cases} 1 \; if \; x = t, \\ 0 \; if \; x \neq t. \end{cases}$$

From Equation (12) we have

$$\begin{aligned} \left|\left(WC_{n,\alpha}f\right)(x) - \frac{1}{2^\alpha}f(x+) - \left(1 - \frac{1}{2^\alpha}\right)f(x-)\right| \leq& \left|\left(WC_{n,\alpha}g_x\right)(x)\right| \\ &+ \left|\frac{f(x+) - f(x-)}{2^\alpha}\left(WC_{n,\alpha} sgn^{(\alpha)}_x\right)(x)\right| \end{aligned} \tag{13}$$

Let us write the interval $I = [0, \infty)$ into three parts as

$$[0, \infty) = I_x(-x) \cup I_x(x) \cup (2x, \infty).$$

At this stage, the expression $\left|\left(WC_{n,\alpha}g_x\right)(x)\right|$ can be bounded or approximated in the following manner:

$$
\begin{aligned}
\left|\int\limits_{0}^{\infty} g_x(t)G_{n,k}^{(\alpha)}(x,t)\,dt\right| &\leq \left|\int\limits_{I_x(-x)} g_x(t)G_{n,k}^{(\alpha)}(x,t)\,dt\right| + \left|\int\limits_{I_x(x)} g_x(t)G_{n,k}^{(\alpha)}(x,t)dt\right| \\
&\quad + \left|\int\limits_{2x}^{\infty} g_x(t)G_{n,k}^{(\alpha)}(x,t)dt\right| \\
&=: |E_{1,\alpha}| + |E_{2,\alpha}| + |E_{3,\alpha}|
\end{aligned} \tag{14}
$$

The bounds for $|E_{1,\alpha}|$ and $|E_{2,\alpha}|$ are provided in Lemma 2, where we substitute $h = -x$ and $h = x$, respectively.

Set $H_x(u) = [x-u, x+u],\ 0 < u \leq x$. By applying the straightforward inequality,

$$
v_j(g_x; I_x(-u)) + v_j(g_x; I_x(u)) \leq 2v_j(g_x; H_x(u))
$$

this condition is satisfied, and as a result, we can derive the following conclusion:

$$
|E_{1,\alpha}| + |E_{2,\alpha}| \leq 2v_1(g_x; H_x(x/\sqrt{b_n/n})) + \frac{8\alpha m_2(\psi)}{x^2}\left[\sum_{j=1}^{m-1} \frac{v_j(g_x; H_x(jx/\sqrt{b_n/n}))}{j^3} + \frac{v_m(g_x; H_x(x))}{m^2}\right] \tag{15}
$$

for all $n \geq 4$.

The estimate of $|E_{3,\alpha}|$ follows from Theorem 1, Lemma 1 (ii), Remark 2 and Equation (9).

$$
\begin{aligned}
|E_{3,\alpha}| &\leq \int\limits_{2x}^{\infty} |g_x(t)|\, G_{n,k}^{(\alpha)}(x,t)\,dt 2M(b_n; f)\frac{n}{b_n}\sum_{k=0}^{n-1} Q_{n-1,k}^{(\alpha)}\left(\frac{x}{b_n}\right)\int\limits_{2x}^{b_n} \chi_{n,k}(t)dt \\
&\leq \frac{2M}{x^{2q}}(WC_{n,\alpha}\,(x-t)^{2q})(t) = \frac{2\alpha M m_{2q}(\psi)}{x^{2q}}
\end{aligned}
$$

with arbitrary positive integer q. Observing that Equation (10)

$$
(WC_{n,\alpha}\,(x-t)^m)(t) = O\left((n/b_n)^{-[(m+1)/2]}\right), n \to \infty,
$$

then consequently, we get

$$
|E_{3,\alpha}| = \frac{\alpha M m_{2q}(\psi)}{x^{2q}} = O\left((n/b_n)^{-q}\right) \tag{16}
$$

The inequalities presented above, specifically Equations (14)–(16) lead to the following estimation for $|(WC_{n,\alpha}g_x)\,(x)|$. □

6. Conclusion

In this study, a novel wavelet-based generalization, extension, and enrichment strategy is presented to enhance the computational accuracy and convergence rate of the classical Chlodovsky operators, which are defined for solving approximation problems on unbounded intervals. By incorporating wavelets into the approximation operators, the proposed method enables the effective reconstruction of functions with complex features, such as oscillations and discontinuities, that are difficult to capture using standard approaches. This advancement extends the applicability of function reconstruction problems, making the method a valuable tool in fields such as computer graphics, remote sensing, and the numerical solution of partial differential equations.

The core innovation of this work can be described as the formulation and analysis of a version of the Chlodovsky operators defined via wavelets, offering flexible control over the reconstruction (approximation) operator. Compared to other methods, including the authors' previous contributions, this enrichment provides a robust theoretical foundation that ensures improved local approximation accuracy and computational efficiency. Furthermore, this approach is expected to be extended to other problems within a framework where wavelet functions can be effectively employed.

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References

1. Chlodovsky I. On the development of functions defined in an infinite interval in series of polynomials by M. S. Bernstein (French). Compositio Mathematica. 1937; 4: 380–392.
2. Bézier P. Numerical Control Mathematics and Applications. J. Wiley Publishing; 1972.
3. Pych-Taberska P. Some properties of the Bé zier-Kantorovich type operators. Journal of Approximation Theory. 2023; 123(2): 256–269.
4. Pych-Taberska P, Karsli H. On the rates of convergence of Bernstein-Chlodovsky polynomials and their Bézier-type variants. Applicable Analysis. 2011; 90(3–4): 403–416.
5. Altomare F. Korovkin-type theorems and approximation by positive linear operators. Surveys in Approximation Theory. 2010; 5: 92–164.
6. Altomare F, Campiti M. Korovkin-Type Approximation Theory and its Applications. De Gruyter Publishing; 1994.
7. Butzer PL, Nessel RJ. Fourier Analysis and Approximation. Academic Press; 1971. Volume 1.
8. Lorentz GG. Bernstein Polynomials. University of Toronto Press; 1953.
9. Pych-Taberska P, Karsli H. On the rates of convergence of Chlodovsky-Kantorovich operators and their Bézier variant. Commentationes Mathematicae. 2009; 49(2): 189–208.
10. Butzer PL, Karsli H. Voronovskaya-type theorems for derivatives of the Bernstein-Chlodovsky polynomials and the Sz ász-Mirakyan operator. Commentationes Mathematicae. 2009; 49(1): 33–58.
11. Agratini O. Construction of Baskakov-type operators by wavelets. Journal of Numerical Analysis and Approximation Theory. 1997; 26(1–2): 3–11.
12. Burrus CS, Gopinath RA, Guo H. Introduction to wavelets and wavelet transforms. A primer. Prentice Hall Publishing; 1998.
13. Gonska HH, Zhou DX. Using wavelets for Szász-type operators. Journal of Numerical Analysis and Approximation Theory. 1995; 24(1–2): 131–145.
14. Kelly S, Kon M, Raphael L. Pointwise convergence of wavelet expansions. Bulletin (new series) of Theamerican Mathematical Society. 1994; 30(1): 87–94.
15. Meyer Y. Ondelettes: Wavelets and Operators. Cambridge University Press; 1993. p. 223.
16. Walter GG. Pointwise convergence of wavelet expansions. Journal of Approximation Theory. 1995; 80(1): 108–118.
17. Karsli H. On Wavelet Type Bernstein Operators. Carpathian Mathematical Publications. 2023; 15(1): 212–221.
18. Daubechies I. Orthonormal bases of compactly supported wavelets. Communications on Pure and Applied Mathematics. 1988; 41(7): 909–996.
19. Daubechies I. Ten Lectures on Wavelets, CBMS-NSF Series in Applied Mathematics 61. SIAM Publishing; 1992.
20. Chanturiya ZA. Modulus of variation of functions and its application in the theory of Fourier series. Doklady Akademii Nauk SSSR. 1974; 214(1): 63–66.