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Review

A new quantum computational set-up for algebraic topology via simplicial sets

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Abstract: In this paper, a quantum computational framework for algebraic topology based on simplicial set theory is presented. This extends previous work, which was limited to simplicial complexes and aimed mostly at topological data analysis. The proposed set-up applies to any parafinite simplicial set and proceeds by associating with it a finite dimensional simplicial Hilbert space, whose simplicial operator structure is studied in some depth. It is shown in particular how the problem of determining the simplicial set's homology can be solved within the simplicial Hilbert framework. Further, the conditions under which simplicial set theoretic algorithms can be implemented in a quantum computational setting with finite resources are examined. Finally a quantum algorithmic scheme capable of computing the simplicial homology spaces and Betti numbers of a simplicial set combining a number of basic quantum algorithms is outlined.

Keywords: algebraic topology; simplicial sets; homology; computational topology; quantum information; quantum computation

1. Introduction

Computational topology is a branch of computational mathematics seeking the evaluation of the topological invariants merging the methods and techniques of algebraic topology and computer science. It comprises areas as diverse as computational 3-manifold theory, computational knot theory and, with a broader scope, computational homotopy and homology theory.

The roots of computational topology lie in algebraic topology, particularly homology theory, as formalized by H. Poincaré. However, the application of these ideas to the computational study of large sampled topological spaces became feasible only in the early 2000s thanks to advances in algorithms and computing power. In particular, in the last three decades, there has been a growing interest in computational topology in relation to topological data analysis, an approach to the analysis of large, incomplete and noisy high dimensional sets of data through the determination of their intrinsic topological properties, as an alternative to traditional statistical procedures that focus on numerical summaries.

A breakthrough came in 2002 when Edelsbrunner et al. introduced persistent homology [1] as a method to quantify the stability of topological features across different scales. The approach was further refined by Zomorodian and Carlsson in [2]. This allowed researchers to distinguish between noise and significant structural patterns in data. In 2005, Carlsson's influential work [3] explained how topological data analysis could be used in practical settings, leading to broader adoption. About at the same time, the Mapper algorithm, developed by Singh et al. [4] made available a way to visualize high-dimensional data sets by constructing simplified topological networks making topological data analysis more intuitively accessible. We refer the reader to [5] for a review of the subject. In recent years, topological data analysis has found important applications in the analysis of multivariate time series [6], climate science [7], economic and financial modeling [8,9], biomedicine and genomic sequencing [10], medical and clinical neuroscience [11,12], neural network analysis [13] and deep learning [14].

Computational topology has the potential of obtaining meaningful and useful information about sampled topological spaces on one hand but poses formidable computational challenges on the other. With the advent of quantum computing, researchers have explored the possibility of devising quantum protocols capable of speeding up topological computations. The synthesis of quantum computing and computational topology became attainable with the development of quantum algorithms designed for linear algebra applications by Harrow et al. [15] and Gilyén et al. [16]. The quantum topological algorithms that have been proposed since these breakthroughs have focused mainly on topological data analysis, because of the issues that blight it and the useful practical applications it affords, but their scope is in principle broader.

The first seminal contribution in this direction was furnished by the work of Lloyd et al. [17], who proposed a quantum algorithm for computing Betti numbers combining quantum phase estimation and Hamiltonian simulation and exhibiting under certain conditions an exponential speedup compared to existing classical algorithms. Subsequent studies addressed strengths and weaknesses of the algorithm of Lloyd et al., beginning with the analysis of Neumann and Breeijen in Ref. [18] and Gyurik et al. in Ref. [19]. More efficient variants and adaptations of the algorithm were subsequently proposed by various research groups [20–23]. Quantum algorithms specifically designed for computing persistent Betti numbers were proposed by Hayakawa [24] and Ameneyro et al. [25]. There is also the potential for further improvement of the algorithmic efficiency by leveraging the efficient implementation of the boundary operator [26–28], exploiting Hodge theory of de Rham cohomology [29] and relying on the estimation of the density of states [30]. A critical evaluation of quantum topological data analysis from the perspective of complexity theory was presented in Refs. [31–33]. An interesting relationship between it and supersymmetric quantum mechanics was studied in Ref. [34].

1.1. Simplicial approaches to computational topology

In computational topology, a wide range of topological spaces embedded in Euclidean spaces are analyzed by means of suitable abstract simplicial complexes associated with samplings of them, such as the Čech [35], Vietoris-Rips [36] and witness complexes [37] to mention the most popular.

A simplicial complex is a set of vertices, edges, triangles, tetrahedrons and higher dimensional polytopes called collectively simplices fitting with each other through their boundaries in a topologically meaningful manner [38]. The simplices of the complex build up a topological space and the complex realizes a generalized triangulation of such a space. The way the simplices of a simplicial complex join together is described combinatorially by an associated abstract simplicial complex [39]. An abstract simplicial complex is a family of nonempty finite subsets of a vertex set, called abstract simplices, that contains all the singleton sets of the vertices and is closed under subset taking. Upon identifying abstract simplices with simplices, an abstract simplicial complex turns into a combinatorial blueprint for a topological space, its topological realization.

Abstract simplicial complex theory turns out to be particularly effective in the study of embedded topological spaces. The representation of such spaces it provides is intuitive and reasonably simple. However, by its very nature, it falls short of satisfying key requirements and certain of its features limit its applicability to topological spaces of other types, as summarized in the following points.

i) The definitions of product of two simplicial complexes and quotient of a simplicial complex by a subcomplex are elaborate and involved reflecting the intricacy of making such operations enjoy desired compatibility properties with topological realization.

ii) Simplices with identified faces cannot occur in a simplicial complex, restricting the range of types of simplices available.

iii) Distinct simplices in a simplicial complex cannot share the same set of faces, limiting the combinatorial range of simplicial complex theory.

iv) The simplicial complexes that are employed to describe even relatively simple topological spaces contain as a rule a very large number of simplices.

v) The reduction methods that have been devised to curtail the size of these complexes while preserving their topological properties, such as Whitehead's simplicial contraction [40], are often of limited usefulness.

Alternative simplicial approaches to computational topology free of the shortcomings pointed out above would deserve to be considered. Simplicial set theory [41] is a generalization of simplicial complex theory that meets such demand. As simplicial complexes, simplicial sets have topological realizations and can therefore be employed as combinatorial models of topological spaces. However, simplicial sets enjoy the desirable properties listed below that simplicial complexes do not.

i) The product of two simplicial sets and the quotient of a simplicial set by a subset can be defined straightforwardly and the simplicial sets resulting from such operations are compatible with topological realization.

ii) Simplices with identified faces are allowed in a simplicial set.

iii) Distinct simplices sharing the same set of faces can occur in a simplicial set.

iv) Precisely because of the greater wealth of simplex types and simplex combinatorics, simplicial sets furnish leaner and more succinct simplicial models of topological spaces.

v) A wider range of reduction techniques, such as simplicial contraction previously mentioned and simplicial collapse, allow us to further streamline such models.

The essential feature distinguishing simplicial sets from simplicial complexes is the incorporation of degenerate simplices. These are simplices whose formal dimension is higher than their effective one. The nicer properties of simplicial sets as compared to simplicial complexes listed above are possible precisely thanks to the inclusion of degenerate simplices. The reasons for this are multiple.

i) A particular non degenerate simplex can have a degenerate face and be a face of a degenerate simplex.

ii) Degenerate simplices allow for the gluing of a non degenerate simplex through its boundary to another non degenerate simplex of arbitrarily lower dimension.

iii) Degenerate simplices enter in an essential way in the basic simplicial set theoretic operations already mentioned. In a product of two simplicial sets, a non degenerate simplex may have degenerate components. In a quotient of a simplicial set by a subset, non degenerate simplices are replaced by degenerate ones. Other examples could be mentioned.

The degenerate simplices are therefore indispensable constitutive elements of a simplicial set along with the non degenerate ones. This notwithstanding, degenerate simplices are hidden in the simplicial set's topological realization and further they do not contribute to its homology. This however does not imply that such simplices can simply be dropped. In fact, the indiscriminate removal of the degenerate simplices leaves in general an incomplete and/or inconsistent simplicial construct.

We illustrate the points made above with a few examples. A two-dimensional torus can be represented as a simplicial set with one vertex, three edges with coinciding ends and two triangles with coinciding vertices and sharing the same edges, while a simplicial complex as simple as possible requires seven vertices, twenty-one edges and fourteen triangles. The minimal simplicial complex describing a three-dimensional sphere requires five vertices, ten edges, ten triangles, five tetrahedrons, while as a simplicial set, only one vertex and one tetrahedron with collapsed faces are sufficient. These examples, albeit elementary, indicate that in general the number of non degenerate simplices required to model a topological space as a simplicial set is indeed considerably smaller than as a simplicial complex. The infinitely many degenerate simplices accompanying the non degenerate ones in the simplicial set do not offset this advantage. The degenerate simplices are topologically invisible in the simplicial set and there are methods to effectively dispose of them.

It is important to realize that, in an appropriate sense, simplicial complexes are special cases of simplicial sets. A simplicial set describes a simplicial complex precisely when each non degenerate simplex has distinct vertices and no two non degenerate simplices share the same vertices. Instances of simplicial sets not satisfying these restrictive conditions are routine. Simplicial sets are therefore more general than simplicial complexes and for this reason have in principle a broader scope.

Our main proposition is that the investigation of the potential implementation of simplicial set theoretic algorithms in computational topology as a useful addition and complement to simplicial complex theoretic ones is a worthwhile endeavor.

i) Simplicial sets have the potential of providing a more efficient combinatorial codification of topological spaces ideally suited for algorithmic implementation.

ii) Techniques of computational topology employing simplicial sets and not just simplicial complexes may have a wider range of applications. Indeed, simplicial set theory can be used to describe and generalize a variety of combinatorial-topological structures such as directed multigraphs, partially ordered sets, categories and more generally ∞-categories.

This point of view has been forcefully advocated by Perry [42] and Zomorodian [43] based on the above and similar considerations. Independently, simplicial sets have found applications also in computational geometry, a discipline distinct from but overlapping with computational topology with relevant applications in geometric modeling, computer graphics, computer aided design and manufacturing, etc. Indeed computational geometry has topological and metric aspects, the first of which are amenable to the methods provided by simplicial set theory [44,45].

We conclude by observing that although most of its applications are concerned with homology computation, simplicial set theory enters noticeably also in homotopy computation; see, e.g., [46–48].

Basic notions and facts of simplicial set theory employed in this paper are reviewed in Section 2 to the reader's benefit and to fix notation and terminology.

1.2. A quantum framework for simplicial sets

As already anticipated, quantum computing may provide powerful new means of speeding up the algorithms of computational topology. The multiple quantum algorithms proposed in Refs. [17–30] indeed all achieve this goal to a varying extent. These algorithms rely on the description of sampled topological spaces as simplicial complexes. For the reasons explained in subsection 1.1 above, investigating whether it is possible to adapt and extend these algorithms to simplicial sets may be a worthy proposition. In the present paper, we explore this possibility.

In order to illustrate our formulation more formally, we provide a more precise elucidation of a number of generic terms used in our discussion up to this point as well as a precise definition of other terms employed in the following.

An algorithm is a finite sequence of precisely defined elementary mathematical operations conceived to systematically perform a computation. Concretely, an algorithm takes an input, handles it following a phased procedure and produces an output. A computer is a device capable of processing data and performing computations based on the instructions of an algorithm.

A classical algorithm is an algorithm designed to run on a classical computer, a device performing computations working according to the laws of classical physics. A quantum algorithm is an algorithm conceived to be implemented on a quantum computer, an implement that leverages basic elements of quantum mechanics such as superposition and entanglement to perform computations more efficiently than a classical computer.

In computational topology, a simplicial algorithm is an algorithm dedicated to performing computations involving simplicial complexes and sets, with reference in particular to their homology and homotopy. A simplicial computer is a computer with an architecture designed precisely for efficiently implementing simplicial algorithms. Simplicial quantum algorithms and computers are defined accordingly.

Our work essentially proposes *a model of a simplicial quantum computer designed to run simplicial quantum algorithms based on simplicial sets*. We explain next its constitutive principles and illustrate its layout and main features.

A simplicial set can be described as a collection of simplices subdivided according to their dimension and of face and degeneracy maps which indicate which simplices are the faces and degeneracies of any given simplex. In the quantum set-up we are going to present, a given parafinite simplicial set is inscribed in a *finite dimensional simplicial Hilbert space* by viewing the simplices as simplex vectors forming a distinguished orthonormal basis and the face and degeneracy maps as face and degeneracy operators acting on simplex vectors in a way that precisely correlates to how the face and degeneracy maps act on simplices.

The dagger structure of the simplicial Hilbert space set-up brings in the adjoints of the face and degeneracy operators. These encode relevant features of the underlying simplicial set and can be used to obtain alternative reformulations of standard problems of computational topology.

In Section 3, the simplicial homology with complex coefficients of a parafinite simplicial set is shown to be isomorphic to that of the associated simplicial Hilbert space. The problem of its determination is recast as that of the computation of the kernels of certain simplicial Hilbert Laplacians along the lines of Ref. [17]. The degenerate simplex subspaces, which are homologically irrelevant by the homological normalization theorem, can furthermore be effectively disposed of in our formulation by resorting to normalized simplicial Hilbert homology, which is equivalent to simplicial Hilbert homology but computationally more convenient. Its determination can be reduced again to the computation of the kernels of certain normalized simplicial Hilbert Laplacians.

The simplicial Hilbert framework is broad enough to afford, in addition, the derivation of a number of technical results and leads to novel theoretical constructs. Also in Section 3, we introduce in particular the notion of simplicial quantum circuit, a special kind of quantum circuit specifically designed to perform homological computations, and analyze its general properties. A related approach was presented by Schreiber and Sati in Ref. [49].

In Section 4, we examine some of the problems that may arise in the implementation of simplicial set theoretic topological algorithms in a realistic quantum computer relying on the quantum computational set-up of Section 3. The issues analyzed here comprise the truncation and skeletonization of simplicial sets, as a means of modeling finite storage resources capable of assembling simplicial data up to a certain finite dimension, the digital encoding of a truncated simplicial set, the counting and parametrization of simplices to their translation into a quantum computational framework. Mainly for illustrative purposes, we also outline a quantum algorithmic scheme capable in principle of computing the simplicial homology spaces and Betti numbers of a simplicial set along the lines of that worked out in [17] combining a number of basic quantum algorithms.

1.3. Scope and limitations of the present work

We conclude this introductory section with some remarks clarifying the scope and limitations of the present work.

This paper is interdisciplinary in that it combines elements from computational algebraic topology and quantum computation scattered in the literature proposing a unified view and a synthesis. It hopefully is of some interest to algebraic topologists, willing to know how their discipline may find application in quantum computational topology, and quantum computational topology specialists, wishing to have an understanding of their field from a more foundational perspective.

The paper has a theoretical and mathematical outlook. As already stated in Subsection 1.2, it illustrates a quantum computational framework for algebraic topology based on simplicial set theory, laying the foundations of an abstract model of a simplicial quantum computer. No new quantum algorithms solving specific problems of algebraic topology with a better performance than classical ones are presented, but hopefully the ground is prepared for the future development and study of such algorithms. The paper also contains original research work, but the results presented have mostly a speculative bearing for the time being. Time will tell whether these ideas will turn out to be useful in practice.

The paper focuses on homology for its practical relevance and as an illustration of the efficacy of the formal apparatus devised, though application of it to homotopy is conceivable and presumably attainable. The simplicial complex based quantum algorithm of Ref. [17] employed for homological computations in topological data analysis and its derivations and refinements elaborated in later literature, however, will likely remain the most competitive option in the foreseeable future.

In Section 5, we list the problems that are still open and discuss the outlook of our work.

Conventions: Following a widely used convention of algebraic topology and computer science, we denote by $\mathbb{N}$ the set of all non negative integers. So, $0 \in \mathbb{N}$. We indicate by $|A|$ the cardinality of a set A. To avoid possible confusion, we denote the topological realization of a simplicial set X by ${}^\sharp X$ rather than $|X|$, as is usually done in mathematics. $\mathrm{Obj}_{\mathcal{C}}$ and $\mathrm{Hom}_{\mathcal{C}}$ denote the object class and homomorphism set of a category $\mathcal{C}$. Composition $\circ$ of maps, when occurring, is usually left understood. Finally, we adopt Dirac's bra-ket notation throughout this paper.

2. Simplicial sets

In this section, we shall review the main aspects of simplicial set theory. This topic has an elegant formulation in the framework of category theory. In what follows, however, we shall pursue a more direct approach which by its combinatorial nature is especially suitable for the algorithmic methods of computational topology. The presentation is kept as concrete as possible. Only a basic knowledge of category theory is assumed. Introductory accounts of simplicial set theory are provided by Refs. [50–53]. Standard references on the subject are [54,55].

In intuitive terms, a simplicial set X is a collection of sets X_n with $n \in \mathbb{N}$, whose elements are to be thought of as n-simplices, equipped with mappings that establish:

i) Which $n-1$-simplices are faces of which n-simplices.

ii) Which $n+1$-simplices are degeneracies of which n-simplices.

A simplicial set is an abstract combinatorial blueprint for constructing a topological space. Indeed, every simplicial set X has a topological realization (more commonly but less precisely called geometric realization) ${}^\sharp X$, a topological space of a special kind called a CW complex. $^\sharp X$ is built by associating with any n-simplexσ_n of X a copy of the standard topological n-simplex

$$^\sharp\Delta^n = \left\{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} | 0 \le t_i \le 1, \textstyle\sum_i t_i = 1\right\} \tag{1}$$

and then gluing together all the topological simplices generated in this fashion along their boundaries in a way that precisely correlates to how the simplices of X that they are associated with are related in consequence of the simplicial set's face and degeneracy maps.

It is precisely because a topological space can be encoded in the simplicial set of which it is the topological realization that many notions of ordinary algebraic topology have an analog in simplicial set theory. In particular, the homotopy and homology of a topological space are modeled by the homotopy and homology of the underlying simplicial set. In this paper, we concentrate on homology for its interest and greater simplicity.

2.1. Simplicial sets

In this subsection, we review the main notions of simplicial set theory.

Definition 1. *A simplicial set X is a collection of sets X_n, $n \in \mathbb{N}$, and mappings $d_{ni} : X_n \to X_{n-1}$, $n \ge 1$, $i = 0, \dots, n$, and $s_{ni} : X_n \to X_{n+1}$, $n \ge 0$, $i = 0, \dots, n$, obeying the simplicial relations*

$$d_{n-1i} d_{nj} = d_{n-1j-1} d_{ni} \qquad \text{if } 0 \le i, j \le n,\ i < j \tag{2a}$$

$$d_{n+1i} s_{nj} = s_{n-1j-1} d_{ni} \qquad \text{if } 0 \le i, j \le n,\ i < j \tag{2b}$$

$$d_{n+1i} s_{nj} = \mathrm{id}_n \qquad \text{if } 0 \le j \le n,\ i = j,\ j+1 \tag{2c}$$

$$d_{n+1i} s_{nj} = s_{n-1j} d_{ni-1} \qquad \text{if } 0 \le i, j \le n+1,\ i > j+1 \tag{2d}$$

$$s_{n+1i} s_{nj} = s_{n+1j+1} s_{ni} \qquad \text{if } 0 \le i, j \le n,\ i \le j \tag{2e}$$

For each of these relations, there are restrictions on the range of allowed values of n which follow from d_{ni} and s_{ni} being defined for $n \ge 1$ and $n \ge 0$, respectively. These conditions are evident from inspection and will not be stated explicitly. By the third relation, the maps d_{ni} are surjective whilst the maps s_{ni} are injective.

The integer n is called simplicial degree. The set X_n comprises the n-simplices of X. The maps d_{ni}, s_{ni} are the face and degeneracy maps of X_n. In the mathematical literature, the dependence of d_{ni}, s_{ni} on n is usually left implicit. In the applications to quantum computation treated in this paper, this is not always possible without yielding ambiguous or incomplete expressions. We have therefore decided to indicate it explicitly at the cost of somewhat complicating the notation.

The simplicial set can be represented diagrammatically as

$$\cdots \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} X_2 \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} X_1 \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} X_0 \tag{3}$$

where the rightward/leftward arrows stand for the face/degeneracy maps.

Definition 2. *A morphism $\phi : X \to X'$ of simplicial sets X, X' consists of a collection* *of maps $\phi_n : X_n \to X'_n$ with $n \geq 0$ obeying*

$$\phi_{n-1} d_{ni} = d'_{ni} \phi_n \qquad \text{if } 0 \leq i \leq n \tag{4a}$$

$$\phi_{n+1} s_{ni} = s'_{ni} \phi_n \qquad \text{if } 0 \leq i \leq n \tag{4b}$$

The morphism fits in a commutative diagram of the form

$$\begin{array}{ccccccc} \cdots & \substack{\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow} & X_2 & \substack{\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow} & X_1 & \substack{\longleftarrow\\[-1ex]\longrightarrow} & X_0 \\ & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} \\ \cdots & \substack{\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow} & X'_2 & \substack{\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow} & X'_1 & \substack{\longleftarrow\\[-1ex]\longrightarrow} & X'_0 \end{array} \tag{5}$$

Simplicial sets can be formed using other simplicial sets as building blocks via certain elementary operations. Two such operations will be relevant in our analysis.

Let X, X' be simplicial sets.

Definition 3. *The Cartesian product $X \times X'$ of X, X' is the simplicial set defined as follows. The n-simplex set of $X \times X'$ is the Cartesian product $X \times X'_n = X_n \times X'_n$. The face and degeneracy maps of $X \times X'$ at degree n are the Cartesian product maps $d \times d'_{ni} = d_{ni} \times d'_{ni}$ and $s \times s'_{ni} = s_{ni} \times s'_{ni}$. The Cartesian product of two morphisms $\phi : X \to X''$, $\psi : X' \to X'''$, of simplicial sets is the simplicial set morphism $\phi \times \psi : X \times X' \to X'' \times X'''$ defined by $\phi \times \psi_n = \phi_n \times \psi_n$ at degree n.*

Above, $\times$ denotes the Cartesian multiplication monoidal product of the category Set of sets and functions.

Definition 4. *The disjoint union $X \sqcup X'$ of X, X' is the simplicial set defined as follows. The n-simplex set of $X \sqcup X'$ is the disjoint union $X \sqcup X'_n = X_n \sqcup X'_n$. The face and degeneracy maps of $X \sqcup X'$ at degree n are the disjoint union maps $d \sqcup d'_{ni} = d_{ni} \sqcup d'_{ni}$ and $s \sqcup s'_{ni} = s_{ni} \sqcup s'_{ni}$. The disjoint union of two morphisms $\phi : X \to X''$, $\psi : X' \to X'''$, of simplicial sets is the simplicial set morphism $\phi \sqcup \psi : X \sqcup X' \to X'' \sqcup X'''$ defined by $\phi \sqcup \psi_n = \phi_n \sqcup \psi_n$ at degree n.*

Here, $\sqcup$ denotes the disjoint union monoidal product of the category Set.

With the operations of Cartesian product and disjoint union simplicial sets and morphisms form a bimonoidal category sSet. We shall treat sSet as a strict bimonoidal category, not completely rigorously, neglecting the fact that the Cartesian product and disjoint union of sets and functions are associative and unital only up to natural isomorphism only.

Definition 5. *A simplicial set X is called parafinite if the n-simplex set X_n is finite for all n.*

In this paper, we shall deal mainly with such simplicial sets. They form a full bimonoidal subcategory pfsSet of sSet.

The following seemingly trivial instance of simplicial set together with some of its variants sometimes useful in some constructions.

Example 1. *The discrete simplicial set of an ordinary set.*

Any ordinary non empty set A can be identified with the simplicial set DA with n-simplex set $D_nA = A$ and face and degeneracy maps $d_{ni} = s_{ni} = \mathrm{id}_A$. Such a simplicial set is called discrete and sometimes also simplicially constant. If the set A

is finite, then DA is parafinite as a simplicial set.

While the constructions elaborated in this work apply to any parafinite simplicial set, there are specific simplicial sets for which they exhibit special properties. Nerves of categories and simplicial sets of ordered simplicial complexes are among these.

Example 2. *The nerve of a category.*

The nerve $N\mathcal{C}$ of a category $\mathcal{C}$ is defined as follows. The 0-simplex set of $N\mathcal{C}$ is just the set of objects of $\mathcal{C}$: $N_0\mathcal{C} = \mathrm{Obj}_{\mathcal{C}}$. Thus, a 0-simplex is just an object x of $\mathcal{C}$. For $n \geq 1$, the n-simplex set of $N\mathcal{C}$ consists of the ordered n element sequences of composable morphisms of $\mathcal{C}$: $N_n\mathcal{C} = \mathrm{Hom}_{\mathcal{C}} \times_{\mathrm{Obj}_{\mathcal{C}}} \cdots \times_{\mathrm{Obj}_{\mathcal{C}}} \mathrm{Hom}_{\mathcal{C}}$ (n factors). Therefore, an n-simplex $\sigma_n \in N_n\mathcal{C}$ is representable as

$$\sigma_n = (f_1, \ldots, f_n) \tag{6}$$

where $f_1, \ldots, f_n \in \mathrm{Hom}_{\mathcal{C}}$ are morphisms such that $t(f_k) = s(f_{k+1})$ for $1 \leq k \leq n-1$, s, t denoting the source and target maps of $\mathcal{C}$. The face maps $d_{ni} : N_n\mathcal{C} \to N_{n-1}\mathcal{C}$ read as follows:

$$d_{10}(f_1) = t(f_1) \quad d_{11}(f_1) = s(f_1) \tag{7}$$

for $n = 1$ and

$$\begin{aligned} &d_{n0}(f_1, \ldots, f_n) = (f_2, \ldots, f_n) \\ &d_{ni}(f_1, \ldots, f_n) = (f_1, \ldots, f_{i-1}, f_{i+1} \circ f_i, f_{i+2}, \ldots, f_n) \quad \text{for } 0 < i < n \\ &d_{nn}(f_1, \ldots, f_n) = (f_1, \ldots, f_{n-1}) \end{aligned} \tag{8}$$

for $n \geq 2$. The degeneracy maps $s_{ni} : N_n\mathcal{C} \to N_{n+1}\mathcal{C}$ take the form

$$s_{00}x = \mathrm{id}_x \tag{9}$$

for $n = 0$ and

$$s_{ni}(f_1, \ldots, f_n) = (f_1, \ldots, f_i, \mathrm{id}_{t(f_i)}, f_{i+1}, \ldots, f_n) \tag{10}$$

for $n \geq 1$. A category $\mathcal{C}$ is called finite if its object and morphism sets are finite. In that case, its nerve $N\mathcal{C}$ is a parafinite simplicial set.

A groupoid is a category $\mathcal{G}$ all of whose morphisms are invertible. The nerve $N\mathcal{G}$ of a groupoid $\mathcal{G}$ exhibits as a consequence special properties not found in generic categories.

Example 3. *The simplicial set of an ordered abstract simplicial complex.*

An abstract simplicial complex $\mathcal{S}$ consists of a set of vertices, $\mathrm{Vert}_{\mathcal{S}}$, and a set of simplices, $\mathrm{Simp}_{\mathcal{S}}$, constituted by finite non-empty subsets of $\mathrm{Vert}_{\mathcal{S}}$ satisfying the following requirements. (1) If $\sigma \in \mathrm{Simp}_{\mathcal{S}}$ and $\emptyset \neq \tau \subseteq \sigma$, then $\tau \in \mathrm{Simp}_{\mathcal{S}}$, that is any non empty subset of a simplex is a simplex. (2) If $v \in \mathrm{Vert}_{\mathcal{S}}$, then $\{v\} \in \mathrm{Simp}_{\mathcal{S}}$, so every singleton of a vertex is a simplex. An n-simplex is a simplex of $\mathrm{Simp}_{\mathcal{S}}$ formed by $n+1$ distinct vertices of $\mathrm{Vert}_{\mathcal{S}}$. $\mathcal{S}$ is said to be ordered if $\mathrm{Vert}_{\mathcal{S}}$

is endowed with a total ordering $\leq$. *An* n*-simplex* σ_n *is then representable as an increasing sequence of* $n+1$ *vertices:* $\sigma_n = (v_0, \ldots, v_n)$ *with* $v_0, \ldots, v_n \in \mathrm{Vert}_{\mathcal{S}}$ *and* $v_0 < \ldots < v_n$.

With any ordered abstract simplicial complex $\mathcal{S}$*, there is associated a simplicial set* $K\mathcal{S}$ *defined as follows. For* $n \geq 1$*, the* n*-simplex set consists of the non decreasing sequences of* $n+1$ *vertices of* $\mathcal{S}$ *whose underlying vertex set is a simplex of* $\mathcal{S}$*. Therefore, an* n*-simplex* $\sigma_n \in K_n\mathcal{S}$ *has a representation of the form*

$$\sigma_n = (v_0, \ldots, v_n) \tag{11}$$

where $v_0, \ldots, v_n \in \mathrm{Vert}_{\mathcal{S}}$ *are vertices with* $v_0 \leq \ldots \leq v_n$ *and* $|\sigma_n| \in \mathrm{Simp}_{\mathcal{S}}$*,* $|\sigma_n|$ *being the set constituted by the distinct vertices* v_k*. The face maps* $d_{ni} : K_n\mathcal{S} \to K_{n-1}\mathcal{S}$ *are given by the expression*

$$d_{ni}(v_0, \ldots, v_n) = (v_0, \ldots v_{i-1}, v_{i+1}, \ldots, v_n) \tag{12}$$

The degeneracy maps $s_{ni} : K_n\mathcal{S} \to K_{n+1}\mathcal{S}$ *read as follows:*

$$s_{ni}(v_0, \ldots, v_n) = (v_0, \ldots v_i, v_i, \ldots, v_n) \tag{13}$$

$\mathcal{S}$ *is called finite when* $\mathrm{Vert}_{\mathcal{S}}$ *is a finite set. In that case, its simplicial set* $K\mathcal{S}$ *is parafinite.*

A distinguishing feature of simplicial sets when compared to simplicial complexes is the appearance of degenerate simplices.

Let X be a simplicial set.

Definition 6. *An* n*-simplex* $\sigma_n \in X_n$ *is said to be degenerate if there is some* $n-1$*-simplex* $\tau_{n-1} \in X_{n-1}$ *and index* i *with* $0 \leq i \leq n-1$ *such that* $\sigma_n = s_{n-1i}\tau_{n-1}$.

0-simplices are regarded as non degenerate. We denote by ${}^sX_n$ the subset of the degenerate simplices of X_n.

Example 4. *The degenerate simplices of the discrete simplicial set of a set.*

In the discrete simplicial set DA *of a non empty set* A *(cf. Example 1)* ${}^sD_nA = D_nA$ *for* $n > 0$*: all positive degree simplices are degenerate.*

Example 5. *The degenerate simplices of the nerve of a category.*

For $n > 0$*, the degenerate* n*-simplex set* ${}^sN_n\mathcal{C}$ *of the nerve* $N\mathcal{C}$ *of a category* $\mathcal{C}$ *(cf. Example 2) consists of all simplices* $(f_1, \ldots, f_n)$ *at least one of whose components* f_i *is an identity morphisms.*

Example 6. *The degenerate simplices of the simplicial set of a simplicial complex.*

For $n > 0$*, the degenerate* n*-simplex set* ${}^sK_n\mathcal{S}$ *of the simplicial set* $K\mathcal{S}$ *of an ordered abstract simplicial complex* $\mathcal{S}$ *(cf. Example 3) consists of all simplices* $(v_0, \ldots, v_n)$*, at least two of whose components* v_i *are equal.*

Note that a degenerate simplex can have a non degenerate face and vice versa.

2.2. Simplicial objects

Simplicial objects in a general category $\mathcal{C}$ are defined in analogy to and generalize simplicial sets.

Definition 7. *A simplicial object X in $\mathcal{C}$ is a collection of non empty objects X_n, $n \in \mathbb{N}$, of $\mathcal{C}$ and morphisms $d_{ni} : X_n \to X_{n-1}$, $n \geq 1$, $i = 1, \ldots, n$, and $s_{ni} : X_n \to X_{n+1}$, $n \geq 0$, $i = 1, \ldots, n$, of $\mathcal{C}$ obeying the simplicial relations* (2).

The d_{ni}, s_{ni} are called face and degeneracy morphisms of X.

Instances of simplicial objects are encountered in many areas of mathematics.

Example 7. *A simplicial set.*

A simplicial set X is just a simplicial object in the category Set *of sets and functions.*

Example 8. *A simplicial group.*

A simplicial group X is a simplicial object in the category Grp *of groups and group homomorphisms for which the objects X_n are groups and the face and degeneracy morphisms are group morphisms between them.*

Example 9. *A simplicial manifold.*

A simplicial manifold X is a simplicial object in the category Mnfld *of smooth manifolds and manifold maps for which the objects X_n are smooth manifolds and the face and degeneracy morphisms are smooth maps between them.*

In this paper, we shall deal specifically with simplicial Hilbert spaces. This kind of simplicial object emerges quite naturally in the construction of Section 3.

Example 10. *A simplicial Hilbert space.*

A simplicial Hilbert space is a simplicial object in the category Hilb *of finite dimensional Hilbert spaces and linear maps.*

There exists an obvious notion of simplicial object morphism that generalizes that of simplicial set morphism of Definition 2.

Definition 8. *A morphism $\phi : X \to X'$ of simplicial objects X, X' of $\mathcal{C}$ is a collection of morphisms $\phi_n : X_n \to X'_n$ with $n \geq 0$ obeying the relations* (4).

Note that every simplicial object in a concrete category $\mathcal{C}$ is also a simplicial set. In this paper, we consider mainly simplicial objects of this kind.

2.3. Simplicial homology

Homology is a basic structure playing an important role in our analysis. We introduce the notion first from a purely algebraic point of view. Later, we concentrate on simplicial homology.

Definition 9. *An abstract chain complex (A, δ) is a sequence of Abelian groups and group morphisms of the form*

$$\cdots \xrightarrow{\delta_3} A_2 \xrightarrow{\delta_2} A_1 \xrightarrow{\delta_1} A_0 \tag{14}$$

such that the homological relations

$$\delta_n \delta_{n+1} = 0 \tag{15}$$

with $n \geq 1$ are satisfied.

The index n labelling the segments of the sequence (14) is called chain degree. A_n and δ_n are named respectively chain group and boundary morphism at degree n.

By virtue of Equation (15), we have that $\operatorname{ran} \delta_{n+1} \subseteq \ker \delta_n$. The sequence (14) would be exact if $\operatorname{ran}\delta_{n+1} = \ker\delta_n$, but this is not the case in general. The homology of (A, δ) measures the failure of (14) to be exact.

Definition 10. *For $n \geq 0$, the homology group of degree n of the chain complex (A, δ) is the quotient group*

$$\mathrm{H}_n(A, \delta) = \ker\delta_n / \operatorname{ran}\delta_{n+1} \tag{16}$$

Above, it is conventionally assumed that $\ker\delta_0 = A_0$. The homology groups $\mathrm{H}_n(A, \delta)$ constitute the homology $\mathrm{H}(A, \delta)$ of (A, δ).

There is a natural notion of morphism of chain complexes.

Definition 11. *A morphism $g : (A, \delta) \to (A', \delta')$ of chain complexes is a diagram of Abelian groups and group morphisms of the form*

$$\begin{array}{ccccccc} \cdots \xrightarrow{\delta_3} & A_2 & \xrightarrow{\delta_2} & A_1 & \xrightarrow{\delta_1} & A_0 \\ & \downarrow g_2 & & \downarrow g_1 & & \downarrow g_0 \\ \cdots \xrightarrow{\delta'_3} & A'_2 & \xrightarrow{\delta'_2} & A'_1 & \xrightarrow{\delta'_1} & A'_0 \end{array} \tag{17}$$

obeying the commutativity conditions

$$g_{n-1}\delta_n = \delta'_n g_n \tag{18}$$

with $n \geq 1$.

Chain complexes and chain complex morphisms form a category.

A chain complex morphism $g : (A, \delta) \to (A', \delta')$ induces a homology morphism $g_{*n} : \mathrm{H}_n(A, \delta) \to \mathrm{H}_n(A', \delta')$ for each n. Homology thus has functorial properties.

Cohomology is the dual notion of homology. All the basic definitions of cohomology can be obtained from those of homology roughly by inverting all the arrows. In particular, for any abstract cochain complex (M, σ) one can define the cohomology groups $\mathrm{H}^n(M, \sigma)$ for any degree $n \geq 0$. We leave to the reader the straightforward task of spelling this out in full detail.

The above homological algebraic framework can be employed in simplicial set theory leading to the formulation of simplicial homology.

Let G be an Abelian simplicial group (cf. Subsection 2.2). Using the face maps of G as constitutive elements, one can construct a series of Abelian group morphisms $d_n : G_n \to G_{n-1}$ for $n \geq 1$ viz

$$d_n = \sum_{0 \leq i \leq n} (-1)^i d_{ni} \tag{19}$$

These Abelian groups and group morphisms fit in the diagram

$$\cdots \xrightarrow{d_3} G_2 \xrightarrow{d_2} G_1 \xrightarrow{d_1} G_0 \tag{20}$$

By the simplicial relations (2a), the d_n obey further the homological relations

$$d_n d_{n+1} = 0 \tag{21}$$

By Equation (21), the diagram (20) constitutes a chain complex (G, d), the simplicial chain complex of G. For each n, G_n and d_n are respectively the group of simplicial chains and the simplicial boundary morphism at degree n.

The homology $\mathrm{H}(G) \equiv \mathrm{H}(G, d)$ is called the simplicial homology of G. $\mathrm{H}(G)$ characterizes G and provides valuable information on the data that underlie and specify it.

Denote by ${}^sG_n$ the subgroup of G_n generated by the degenerate n-simplices (cf. Subsection 2.1). The quotient group $\overline{G}_n = G_n/{}^sG_n$ is the normalized n–chain group. Owing to relations (2b)–(2d), $d_n{}^sG_n \subseteq {}^sG_{n-1}$ and therefore an induced map $\overline{d}_n : \overline{G}_n \to \overline{G}_{n-1}$ is defined. $\overline{d}_n$ obeys the homological relation (21). The normalized chain complex

$$\cdots \xrightarrow{\overline{d}_3} \overline{G}_2 \xrightarrow{\overline{d}_2} \overline{G}_1 \xrightarrow{\overline{d}_1} \overline{G}_0 \tag{22}$$

is in this way constructed. The homology $\mathrm{H}(\overline{G}) \equiv \mathrm{H}(\overline{G}, \overline{d})$ of $(\overline{G}, \overline{d})$ is the normalized simplicial homology of G. The following theorem establishes that normalized simplicial homology is just another incarnation of simplicial homology.

Theorem 1. *The simplicial homology* $\mathrm{H}(G)$ *and normalized simplicial homology* $\mathrm{H}(\overline{G})$ *of* G *are isomorphic (Normalization theorem* [56]).

This result reveals that the degenerate chains are homologically irrelevant in the computation of the simplicial homology $\mathrm{H}(G)$ and can be used in principle to simplify the computation of the latter.

Let X be a simplicial set and A an Abelian group. For $n \in \mathbb{N}$, the group of n-chains of X with coefficients in A is the Abelian group

$$C_n(X, \mathsf{A}) = \mathbb{Z}[X_n] \otimes \mathsf{A} \tag{23}$$

where $\mathbb{Z}[X_n]$ denotes the free Abelian group generated by the n-simplex set X_n. The face and degeneracy maps d_{ni}, s_{ni} of X extend uniquely to Abelian group morphisms $d_{ni} : C_n(X, \mathsf{A}) \to C_{n-1}(X, \mathsf{A})$, $s_{ni} : C_n(X, \mathsf{A}) \to C_{n+1}(X, \mathsf{A})$. These extensions obey the simplicial relations (2). So, the groups $C_n(X, \mathsf{A})$ and the morphisms d_{ni}, s_{ni} build up a simplicial Abelian group $C(X, \mathsf{A})$. By the general construction described above, it is then possible to construct via (19) boundary morphisms

$$\partial_n \sigma_n = \sum_{0 \le i \le n} (-1)^i d_{ni} \sigma_n \tag{24}$$

obeying the homological relations

$$\partial_n \partial_{n+1} = 0 \tag{25}$$

With X and A, so, there is associated a chain complex $(C(X, \mathsf{A}), \partial)$.

Definition 12. *The simplicial homology* $\mathrm{H}(X, \mathsf{A})$ *of* X *with coefficients in* A *is the simplicial homology* $\mathrm{H}(C(X, \mathsf{A}))$ *of the simplicial Abelian group* $C(X, \mathsf{A})$. *The normalized simplicial homology* $\overline{\mathrm{H}}(X, \mathsf{A})$ *of* X *with coefficients in* A *is the associated normalized simplicial homology* $\mathrm{H}(\overline{C}(X, \mathsf{A}))$.

By the normalization Theorem 1, the simplicial and normalized simplicial homologies $\mathrm{H}(X, \mathsf{A})$, $\overline{\mathrm{H}}(X, \mathsf{A})$ are isomorphic. The computation of the latter is however generally simpler than that of the former. For this reason, simplicial homology is sometimes defined directly as normalized simplicial homology in the mathematical literature.

Example 11. *The homology of the delooping of a group.*

A group G *can be regarded as a groupoid* BG *with a single object whose morphisms are in one-to-one correspondence with the elements of* G*, the so called delooping of* G*. The simplicial homology of the nerve* $N\mathsf{BG}$ *of* BG *with coefficients in* $\mathbb{Z}$*,* $\mathrm{H}(N\mathsf{BG}, \mathbb{Z})$ *can be shown to be isomorphic to the group homology* $\mathrm{H}(\mathsf{G})$*.*

Example 12. *The homology of the simplicial set of an ordered abstract simplicial complex.*

It is possible to define the simplicial homology of a simplicial complex with a given coefficient Abelian group on the same lines as that of a simplicial set. The simplices of an ordered simplicial complex $\mathcal{S}$ *are precisely the non degenerate simplices of the associated simplicial set* $K\mathcal{S}$ *studied in Example 3. By the normalization theorem 1, the simplicial homology* $\mathrm{H}(\mathcal{S}, \mathsf{A})$ *is then isomorphic to the simplicial homology* $\mathrm{H}(K\mathcal{S}, \mathsf{A})$*.*

The above set-up can be refined by working with vector spaces or modules instead of Abelian groups in an obvious fashion, resulting in (co)homology spaces or modules, respectively.

The Betti numbers with coefficients in a field $\mathbb{F}$ of a simplicial set X are

$$\beta_n(X, \mathbb{F}) = \dim \mathrm{H}_n(X, \mathbb{F}) \tag{26}$$

The Betti numbers are topological invariants of the topological space $^\sharp X$ realizing X and characterize it based on its connectivity. For $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and low degrees, the Betti numbers have simple intuitive topological interpretations: $\beta_0(X, \mathbb{F})$ is the number of connected components of $^\sharp X$, $\beta_1(X, \mathbb{F})$ is the number of holes of $^\sharp X$, $\beta_2(X, \mathbb{F})$ is the number of voids of $^\sharp X$, etc. If $^\sharp X$ is a d-dimensional topological manifold, $\beta_n(X, \mathbb{F}) = 0$ for $n > d$.

Much of computational topology aims at the computation of the Betti numbers for the important topological information they furnish about $^\sharp X$. For instance, finding out that the Betti numbers $\beta_n(X, \mathbb{R})$ vanish for $n > d$ for some d is an indication that $^\sharp X$ is a d-dimensional manifold.

3. Quantum simplicial set framework

In this section we shall work out and study in detail the quantum simplicial set theoretic set-up outlined in the introduction.

The quantum simplicial set framework furnishes a natural backdrop for the theoretical analysis and eventual implementation of simplicial quantum algorithms for computational topology. It is essentially an instance of quantum basis encoding of classical data (see e.g., [57,58] for background), where the latter are just basic simplicial data. Thanks to it, a given parafinite simplicial set is encoded into a finite dimensional simplicial Hilbert space much as a qubit register is into a finite dimensional Hilbert space. Correspondingly, the simplices and face and degeneracy maps of the simplicial set convert into the basis vectors and face and degeneracy operators of the simplicial

Hilbert space. The Hilbert dagger structure provides however also with the adjoints of these operators, making possible novel constructions.

Extending the quantum simplicial framework beyond the range of parafinite simplicial sets is not feasible because of the intrinsic limitations of implementable computation: any conceivable simplicial computer can operate only on a finite number of simplices of each given degree. Therefore, parafinite simplicial sets are the most general kind of simplicial sets that can be handled by such a device.

A criticism that can be leveled at the framework is that it still incorporates the simplex Hilbert spaces storing the simplicial data of all degrees, while a realistic computer can manage the simplicial data up to a finite maximum degree. In fact, doing so is only a convenient abstraction to simplify the analysis. In Section 4, we shall show how to deal with this problem by simplicial truncation or skeletonization of the underlying simplicial set.

The quantum simplicial framework will enable us to analyze the simplicial homology of the simplicial Hilbert space, show how it is controlled by appropriate simplicial Hilbert Laplacians and prove its isomorphism to the simplicial homology of the underlying simplicial set. It will also give us the means to construct the simplicial Hilbert space's appropriate form of normalized simplicial homology and show its isomorphism to the simplicial set's normalized simplicial homology. Last but not least, it will suggest to us a suitable notion of simplicial quantum circuits as the kind of quantum circuits capable of performing simplicial computations. Though we concentrate on homology, applications also to homotopy are conceivable.

3.1. Hilbert space encoding of a simplicial set

In this subsection, we shall show how a given parafinite simplicial set X (cf. Definitions 1 and 5) can be encoded in a simplicial finite dimensional Hilbert structure.

Definition 13. *For $n \in \mathbb{N}$, the n-simplex Hilbert space $\mathscr{H}_n$ is the Hilbert space generated by the n-simplex set X_n.*

$\mathscr{H}_n$ has thus a canonical orthonormal basis $|\sigma_n\rangle$ labelled by the n-simplices $\sigma_n \in X_n$. In the following, we shall refer to the basis $|\sigma_n\rangle$ as the n-simplex basis. It plays a role analogous to the computational basis in familiar quantum computing. Since X_n is a finite set, $\mathscr{H}_n$ is finite dimensional.

The face and degeneracy maps d_{ni} and s_{ni} induce face and degeneracy operators characterized by their action on the vectors of the n-simplex basis.

Definition 14. *The face operators $D_{ni} : \mathscr{H}_n \to \mathscr{H}_{n-1}$, $i = 0, \ldots, n$ and $n \geq 1$, and degeneracy operators $S_{ni} : \mathscr{H}_n \to \mathscr{H}_{n+1}$ and $i = 0, \ldots, n$ and $n \geq 0$. D_{ni}, S_{ni} are*

$$D_{ni} = \sum_{\sigma_n \in X_n} |d_{ni}\sigma_n\rangle \langle\sigma_n| \tag{27}$$

$$S_{ni} = \sum_{\sigma_n \in X_n} |s_{ni}\sigma_n\rangle \langle\sigma_n| \tag{28}$$

The Hilbert dagger structure of the $\mathscr{H}_n$ allows us to define the adjoint operators

${D_{ni}}^+ : \mathscr{H}_{n-1} \to \mathscr{H}_n$, ${S_{ni}}^+ : \mathscr{H}_{n+1} \to \mathscr{H}_n$ of D_{ni}, S_{ni}. They are given by

$$ {D_{ni}}^+ = \sum_{\sigma_{n-1} \in X_{n-1}} \sum_{\omega_n \in D_{ni}(\sigma_{n-1})} |\omega_n\rangle \langle \sigma_{n-1}| \tag{29} $$

$$ {S_{ni}}^+ = \sum_{\sigma_{n+1} \in X_{n+1}} \sum_{\omega_n \in S_{ni}(\sigma_{n+1})} |\omega_n\rangle \langle \sigma_{n+1}| \tag{30} $$

where $D_{ni}(\sigma_{n-1})$, $S_{ni}(\sigma_{n+1}) \subset X_n$ are the n-simplex subsets

$$ D_{ni}(\sigma_{n-1}) = \{\omega_n \in X_n | d_{ni}\omega_n = \sigma_{n-1}\} \tag{31} $$

$$ S_{ni}(\sigma_{n+1}) = \{\omega_n \in X_n | s_{ni}\omega_n = \sigma_{n+1}\} \tag{32} $$

An important problem of the theory is the determination of the content of $D_{ni}(\sigma_{n-1})$, $S_{ni}(\sigma_{n+1})$ for any assignment of σ_{n-1}, σ_{n+1}. Through the $D_{ni}(\sigma_{n-1})$, $S_{ni}(\sigma_{n+1})$, the adjoint operators ${D_{ni}}^+$, ${S_{ni}}^+$ encode basic features of and provide important information about the underlying simplicial set X. Not much can be said about $D_{ni}(\sigma_{n-1})$, $S_{ni}(\sigma_{n+1})$ in general. The following general properties hold anyway. Since d_{ni} is surjective while s_{ni} generally is not, $D_{ni}(\sigma_{n-1})$ is always non empty while $S_{ni}(\sigma_{n+1})$ may be empty. Further, as s_{ni} is injective while d_{ni} generally is not, $|S_{ni}(\sigma_{n+1})| \leq 1$ while $|D_{ni}(\sigma_{n-1})| \geq 1$.

The basic simplicial relations (2) obeyed by the face and degeneracy maps d_{ni}, s_{ni} imply that the face and degeneracy operators D_{ni}, S_{ni} and their adjoints ${D_{ni}}^+$, ${S_{ni}}^+$ satisfy a host of *exchange identities* relating products of pairs of these latter. These characterize to a considerable extent the simplicial Hilbert framework constructed in this paper and are therefore analyzed in detail in the rest of this subsection.

The exchange identities come in pairs related by adjunction. We shall show explicitly only one of the two relations of each pair leaving to the reader the rather straightforward task of writing down the other.

The exchange identities involving the D_{ni}, S_{ni} ensue directly from Equations (27), (28) and the simplicial relations (2).

$$ D_{n-1i}D_{nj} - D_{n-1j-1}D_{ni} = 0 \qquad \text{for } 0 \leq i, j \leq n, i < j \tag{33a} $$

$$ D_{n+1i}S_{nj} - S_{n-1j-1}D_{ni} = 0 \qquad \text{for } 0 \leq i, j \leq n, i < j \tag{33b} $$

$$ D_{n+1i}S_{nj} = 1_n \qquad \text{for } 0 \leq j \leq n, i = j, j+1 \tag{33c} $$

$$ D_{n+1i}S_{nj} - S_{n-1j}D_{ni-1} = 0 \qquad \text{for } 0 \leq i, j \leq n+1, i > j+1 \tag{33d} $$

$$ S_{n+1i}S_{nj} - S_{n+1j+1}S_{ni} = 0 \qquad \text{for } 0 \leq i, j \leq n, i \leq j \tag{33e} $$

where $1_n = 1_{\mathscr{H}_n}$. The Equation (33) are identical in form to the simplicial relations and for this reason are referred to as the simplicial Hilbert identities (2). The exchange relations involving the ${D_{ni}}^+$, ${S_{ni}}^+$ are obtained from Equation (33) by adjunction. They are given by the Equation (33) except for the order of the factors of each term, which is reversed. They are formally equal to the simplicial theoretic cosimplicial relations and so are called the cosimplicial Hilbert identities.

The exchange identities involving one operator of each of the operator sets D_{ni}, S_{ni} and ${D_{ni}}^+$, ${S_{ni}}^+$ are not so easily obtained and do not take a form analogous to that of the simplicial and cosimplicial Hilbert identities. These mixed identities exhibit however an analogous structure. Explicitly, they read as

$$D_{ni}{}^{+}D_{nj} - D_{n+1j+1}D_{n+1i}{}^{+} = \mathit{\Delta}^{DD}{}_{nij} \qquad \text{for } 0 \le i,j \le n,\, i \le j \tag{34a}$$

$$D_{n+2i}{}^{+}S_{nj} - S_{n+1j+1}D_{n+1i}{}^{+} = \mathit{\Delta}^{DS}{}_{nij} \qquad \text{for } 0 \le i,j \le n,\, i \le j \tag{34b}$$

$$S_{n-2i}{}^{+}D_{nj} - D_{n-1j-1}S_{n-1i}{}^{+} = \mathit{\Delta}^{SD}{}_{nij} \qquad \text{for } 0 \le i,j \le n,\, i+1 < j \tag{34c}$$

$$S_{ni}{}^{+}S_{nj} - S_{n-1j-1}S_{n-1i}{}^{+} = \mathit{\Delta}^{SS}{}_{nij} \qquad \text{for } 0 \le i,j \le n,\, i < j \tag{34d}$$

where

$$\mathit{\Delta}^{DD}{}_{nij} = -\sum_{\sigma_n \in X_n} \sum_{\omega_n \in D_{ni}(d_{nj}\sigma_n)} |\omega_n\rangle(|D_{n+1i}(\sigma_n) \cap D_{n+1j+1}(\omega_n)| - 1)\langle\sigma_n| \tag{35a}$$

$$\mathit{\Delta}^{DS}{}_{nij} = -\sum_{\sigma_n \in X_n} \sum_{\omega_{n+2} \in D_{n+2i}(s_{nj}\sigma_n)} |\omega_{n+2}\rangle(|D_{n+1i}(\sigma_n) \cap S_{n+1j+1}(\omega_{n+2})| - 1)\langle\sigma_n| \tag{35b}$$

$$\mathit{\Delta}^{SD}{}_{nij} = -\sum_{\sigma_n \in X_n} \sum_{\omega_{n-2} \in S_{n-2i}(d_{nj}\sigma_n)} |\omega_{n-2}\rangle(|S_{n-1i}(\sigma_n) \cap D_{n-1j-1}(\omega_{n-2})| - 1)\langle\sigma_n| \tag{35c}$$

$$\mathit{\Delta}^{SS}{}_{nij} = -\sum_{\sigma_n \in X_n} \sum_{\omega_n \in S_{ni}(s_{nj}\sigma_n)} |\omega_n\rangle(|S_{n-1i}(\sigma_n) \cap S_{n-1j-1}(\omega_n)| - 1)\langle\sigma_n| \tag{35d}$$

Recall that $|I|$ denotes the cardinality of a set I. The expressions shown result from straightforward computations relying solely on the expressions (27)–(30) of the operators D_{ni}, S_{ni} and $D_{ni}{}^{+}$, $S_{ni}{}^{+}$ and the basic simplicial relations (2).

Definition 15. *The defects of the simplicial Hilbert structure are the four operators* $\mathit{\Delta}^{DD}{}_{nij} : \mathscr{H}_n \to \mathscr{H}_n$, $0 \le i,j \le n$, $i \le j$ $\mathit{\Delta}^{DS}{}_{nij} : \mathscr{H}_n \to \mathscr{H}_{n+2}$, $0 \le i,j \le n$, $i \le j$ $\mathit{\Delta}^{SD}{}_{nij} : \mathscr{H}_n \to \mathscr{H}_{n-2}$, $0 \le i,j \le n$, $i+1 < j$ *and* $\mathit{\Delta}^{SS}{}_{nij} : \mathscr{H}_n \to \mathscr{H}_n$, $0 \le i,j \le n$, $i < j$ *given by the Equation* (35).

A basic problem of the theory is determining under which conditions some or all defects vanish and identifying the simplicial sets for which such conditions are fulfilled. For the SS defects $\mathit{\Delta}^{SS}{}_{nij}$, the problem however does not arise by the following theorem.

Theorem 2. *(No degeneracy defect theorem) We have*

$$\mathit{\Delta}^{SS}{}_{nij} = 0 \qquad \textit{for } 0 \le i,j \le n,\, i < j \tag{36}$$

Proof. See Appendix I for the proof. □

The defects $\mathit{\Delta}^{DD}{}_{nij}$, $\mathit{\Delta}^{DS}{}_{nij}$, $\mathit{\Delta}^{SD}{}_{nij}$ conversely are non zero in general. The exchange identities of the operators S_{ni}, $S_{ni}{}^{+}$ have in this way a simple universal form akin to that of simplicial and cosimplicial Hilbert identities. The exchange identities of the operators D_{ni}, $D_{ni}{}^{+}$, S_{ni}, $D_{ni}{}^{+}$ and D_{ni}, $S_{ni}{}^{+}$ instead do not.

The concrete form the $\mathit{\Delta}^{DD}{}_{nij}$, $\mathit{\Delta}^{DS}{}_{nij}$, $\mathit{\Delta}^{SD}{}_{nij}$ take depends on the underlying simplicial set X. In fact, they encode features of X not deducible from the simplicial relations and hence specific to X. For distinguished instances of simplicial sets, there exist perfectness results establishing the vanishing of some of these defects. In this regard, the following definition is apposite.

Definition 16. *The simplicial set X is said to be semiperfect if*

$$\varDelta^{DS}{}_{nij} = 0 \qquad \text{for } 0 \le i, j \le n,\ i \le j \tag{37}$$
$$\varDelta^{SD}{}_{nij} = 0 \qquad \text{for } 0 \le i, j \le n,\ i + 1 < j \tag{38}$$

X is said to be quasi perfect if (37), (38) *hold and further*

$$\varDelta^{DD}{}_{nij} = 0 \qquad \text{for } 0 \le i, j \le n,\ i < j \tag{39}$$

X is said to be perfect if (37), (38) *hold and further*

$$\varDelta^{DD}{}_{nij} = 0 \qquad \text{for } 0 \le i, j \le n,\ i \le j \tag{40}$$

Example 13. *The nerve of a category.*

Nerves of categories are a special type of simplicial sets introduced in Example 2.

Proposition 1. *(Perfectness proposition for nerves of categories) The nerve $N\mathcal{C}$ of a finite category $\mathcal{C}$ is quasi perfect. If $\mathcal{C}$ is also a groupoid then $N\mathcal{C}$ is perfect.*

Proof. See Appendix II for the proof. ☐

Example 14. *The simplicial set of an ordered abstract simplicial complex.*

Simplicial sets of an ordered abstract simplicial complex are another distinctive type of simplicial set introduced in Example 3.

Proposition 2. *(Perfectness proposition for simplicial sets of simplicial complexes) The simplicial set $K\mathcal{S}$ of an ordered finite abstract simplicial complex $\mathcal{S}$ is semiperfect.*

Proof. See again Appendix II for the proof. ☐

The mixed exchange identities (34) do not cover all possible products of one of the operators D_{ni}, S_{ni} and one of the adjoint operators $D_{ni}{}^{+}$, $S_{ni}{}^{+}$. The missing products are $D_{n+1i}D_{n+1i}{}^{+}$, $0 \le i \le n+1$, $D_{n+2i+1}{}^{+}S_{ni}$, $0 \le i \le n$, $S_{n+1i}D_{n+1i}{}^{+}$, $0 \le i \le n+1$ and $S_{ni}{}^{+}S_{ni}$, $0 \le i \le n$. Some of these products will reemerge as elemental terms in the expression of the simplicial Hilbert Laplacians studied in Subsection 3.3.

Every morphism $\phi : X \to X'$ of parafinite simplicial sets X, X' (cf. Definition 2) also has a simplicial Hilbert encoding.

Definition 17. *The morphism operators $\varPhi_n : \mathscr{H}_n \to \mathscr{H}'_n$, $n \ge 0$, of ϕ are*

$$\varPhi_n = \sum_{\sigma_n \in X_n} |\phi_n \sigma_n\rangle\langle\sigma_n| \tag{41}$$

By the relations (4) and (27), (28), the $\varPhi_n$ satisfy

$$\varPhi_{n-1} D_{ni} - D'_{ni} \varPhi_n = 0 \qquad \text{if } 0 \le i \le n \tag{42a}$$
$$\varPhi_{n+1} S_{ni} - S'_{ni} \varPhi_n = 0 \qquad \text{if } 0 \le i \le n \tag{42b}$$

The Equation (42) are identical in form to the simplicial morphism relations and for this reason are referred to as the simplicial Hilbert morphism identities.

Again, the Hilbert dagger structure of the $\mathscr{H}_n$, $\mathscr{H}'_n$ allows us to define the adjoint operators ${\Phi_n}^+ : {\mathscr{H}'}_n \to \mathscr{H}_n$, $n \geq 0$, which in terms of the simplex basis read

$$ {\Phi_n}^+ = \sum_{{\sigma'}_n \in {X'}_n} \sum_{\omega_n \in X_n, \phi_n \omega_n = {\sigma'}_n} |\omega_n\rangle\langle{\sigma'}_n| \tag{43} $$

The ${\Phi_n}^+$ obey the identities following from Equation (42) by adjunction. They have the same form as the Equation (42) except for the reversed order of the factors and are therefore called cosimplicial Hilbert morphism relations.

3.2. The simplicial Hilbert space of a simplicial set

In this subsection, we shall show how the Hilbert space encoding of a simplicial set described in Subsection 3.1 can be naturally represented as a simplicial Hilbert space.

The category fdsHilb of finite dimensional simplicial Hilbert spaces and operators is described as follows. An object $\mathscr{H}$ of fdsHilb is a collection $\{\mathscr{H}_n, D_{ni}, S_{ni}\}$ consisting of finite dimensional Hilbert spaces $\mathscr{H}_n$ together with face and degeneracy operators $D_{ni} : \mathscr{H}_n \to \mathscr{H}_{n-1}$, $S_{ni} : \mathscr{H}_n \to \mathscr{H}_{n+1}$ obeying the simplicial Hilbert identities (33). A morphism $\Phi : \mathscr{H} \to \mathscr{H}'$ of fdsHilb is a collection of linear operators $\Phi_n : \mathscr{H}_n \to {\mathscr{H}'}_n$ satisfying the simplicial Hilbert morphism identities (42).

The category fdsHilb is bimonoidal, its two monoidal products being given by degreewise direct product and sum $\otimes$ and $\oplus$. Explicitly, $\otimes$ and $\oplus$ act as follows. Let $\mathscr{H}$, $\mathscr{H}'$ be finite dimensional simplicial Hilbert spaces. Then, $\mathscr{H} \otimes \mathscr{H}'$ is the simplicial Hilbert space with $\mathscr{H} \otimes {\mathscr{H}'}_n = \mathscr{H}_n \otimes {\mathscr{H}'}_n$, $D \otimes {D'}_{ni} = D_{ni} \otimes {D'}_{ni}$, $S \otimes {S'}_{ni} = S_{ni} \otimes {S'}_{ni}$. Similarly, $\mathscr{H} \oplus \mathscr{H}'$ is the simplicial Hilbert space with $\mathscr{H} \oplus \mathscr{H}_n = \mathscr{H}_n \oplus {\mathscr{H}'}_n$, $D \oplus {D'}_{ni} = D_{ni} \oplus {D'}_{ni}$, $S \oplus {S'}_{ni} = S_{ni} \oplus {S'}_{ni}$. Let $\Phi : \mathscr{H} \to \mathscr{H}''$, $\Phi' : \mathscr{H}' \to \mathscr{H}'''$ be morphisms of finite dimensional simplicial Hilbert spaces. Then, $\Phi \otimes \Phi' : \mathscr{H} \otimes \mathscr{H}' \to \mathscr{H}'' \otimes \mathscr{H}'''$ is the simplicial Hilbert space morphism given by $\Phi \otimes {\Phi'}_n = \Phi_n \otimes {\Phi'}_n$. Equally, $\Phi \oplus \Phi' : \mathscr{H} \oplus \mathscr{H}' \to \mathscr{H}'' \oplus \mathscr{H}'''$ is the simplicial Hilbert space morphism such that $\Phi \oplus {\Phi'}_n = \Phi_n \oplus {\Phi'}_n$.

We shall treat fdsHilb as a strict bimonoidal category, not completely rigorously, neglecting the fact that the direct multiplication and summation of Hilbert spaces and operators are associative and unital only up to natural isomorphisms only.

In Subsection 3.1, we have detailed a construction that associates with every parafinite simplicial set X the simplex Hilbert spaces $\mathscr{H}_n$ and the face and degeneracy operators D_{ni}, S_{ni} given by Equations (27) and (28). The simplicial Hilbert identities (33) obeyed by the D_{ni}, S_{ni} entail that the Hilbert data collection $\{\mathscr{H}_n, D_{ni}, S_{ni}\}$ constitutes a finite dimensional simplicial Hilbert space $\mathscr{H}$. We have also shown that with any morphism $\phi : X \to X'$ of parafinite simplicial sets there are associated linear operators Φ_n mapping $\mathscr{H}_n$ to ${\mathscr{H}'}_n$ given by Equation (41). By the simplicial Hilbert morphism relations (42) the Φ_n obey, the operator collection $\{\Phi_n\}$ forms a morphism $\Phi : \mathscr{H} \to \mathscr{H}'$ of the simplicial Hilbert spaces $\mathscr{H}$, $\mathscr{H}'$ of X, X'. The correspondence $X \mapsto \mathscr{H}$ and $\phi : X \to X' \mapsto \Phi : \mathscr{H} \to \mathscr{H}'$ is further compatible with morphism composition and identity assignment. We have thus constructed a functor from the category pfsSet of parafinite simplicial sets into the category of finite dimensional simplicial Hilbert spaces fdsHilb.

Definition 18. *The simplicial Hilbert functor is the functor* $\mathfrak{h}$: pfsSet $\to$ fdsHilb *described in the previous paragraph.*

The functor $\mathfrak{h}$ enjoys a nice property.

Theorem 3. *The simplicial Hilbert functor* $\mathfrak{h}$ *is bimonoidal.*

Essentially, this states that $\mathfrak{h}$ maps the bimonoidal product structure of pfsSet, consisting of Cartesian multiplication and disjoint union (cf. Subsection 2.1), into that of fdsHilb, comprising direct multiplication and summation. The most significant categorical features of parafinite simplicial sets are so reproduced in the appropriate form by finite dimensional simplicial Hilbert spaces.

Proof. We provide only a sketch of the proof. Let X, X' be simplicial sets and let $\mathscr{H} = \mathfrak{h}(X)$, $\mathscr{H}' = \mathfrak{h}(X')$ be their simplicial Hilbert spaces. Consider now the Cartesian product $X \times X'$ of X, X' (cf. Definition 3) and its associated simplicial Hilbert space $\mathscr{H} \times \mathscr{H}' := \mathfrak{h}(X \times X')$. There exists a simplicial Hilbert isomorphism $\Lambda : \mathscr{H} \times \mathscr{H}' \xrightarrow{\simeq} \mathscr{H} \otimes \mathscr{H}'$ of $\mathscr{H} \times \mathscr{H}'$ and the direct product $\mathscr{H} \otimes \mathscr{H}'$ of $\mathscr{H}$, $\mathscr{H}'$. At degree n, Λ is defined by $\Lambda_n|\sigma_n, \sigma'_n\rangle = |\sigma_n\rangle \otimes |\sigma'_n\rangle$ with $\sigma_n \in X_n$, $\sigma'_n \in X'_n$. This furnishes the identification $\mathscr{H} \times \mathscr{H}' \simeq \mathscr{H} \otimes \mathscr{H}'$ by which we conclude that $\mathfrak{h}(X \times X') \simeq \mathfrak{h}(X) \otimes \mathfrak{h}(X')$. Consider likewise the disjoint union $X'' = X \sqcup X'$ of X, X' (cf. Definition 4) and its associated simplicial Hilbert space $\mathscr{H} \sqcup \mathscr{H}' := \mathfrak{h}(X \sqcup X')$. There exists a simplicial Hilbert isomorphism $\Sigma : \mathscr{H} \sqcup \mathscr{H}' \xrightarrow{\simeq} \mathscr{H} \oplus \mathscr{H}'$ of $\mathscr{H} \sqcup \mathscr{H}'$ and the direct sum $\mathscr{H} \oplus \mathscr{H}'$ of $\mathscr{H}$, $\mathscr{H}'$. At degree n, Σ is defined by the expressions $\Sigma_n|\sigma_n\rangle = |\sigma_n\rangle \oplus 0$, $\Sigma_n|\sigma'_n\rangle = 0 \oplus |\sigma'_n\rangle$ for $\sigma_n \in X_n$, $\sigma'_n \in X'_n$ respectively. This leads to the identification $\mathscr{H} \sqcup \mathscr{H}' \simeq \mathscr{H} \oplus \mathscr{H}'$ from which it is concluded that $\mathfrak{h}(X \sqcup X') \simeq \mathfrak{h}(X) \oplus \mathfrak{h}(X')$.

Let $\phi : X \to X''$, $\phi' : X' \to X'''$ be morphisms of simplicial sets and let $\Phi = \mathfrak{h}(\phi)$, $\Phi' = \mathfrak{h}(\phi')$ be the associated simplicial Hilbert space morphisms, so that, setting $\mathscr{H} = \mathfrak{h}(X)$, $\mathscr{H}' = \mathfrak{h}(X')$, $\mathscr{H}'' = \mathfrak{h}(X'')$, $\mathscr{H}''' = \mathfrak{h}(X''')$, we have $\Phi : \mathscr{H} \to \mathscr{H}''$, $\Phi' : \mathscr{H}' \to \mathscr{H}'''$. Consider now the Cartesian product $\varphi \times \varphi' : X \times X' \to \varphi' : X'' \times X'''$ and disjoint union of $\varphi \sqcup \varphi' : X \sqcup X' \to \varphi' : X'' \sqcup X'''$ of φ, φ' (cf. Definitions 3 and 4) and their associated simplicial Hilbert space morphisms $\Phi \times \Phi' := \mathfrak{h}(\varphi \times \varphi')$ and $\Phi \sqcup \Phi' := \mathfrak{h}(\varphi \sqcup \varphi')$. Then, we have $\Phi \times \Phi' : \mathscr{H} \times \mathscr{H}' \to \mathscr{H}'' \times \mathscr{H}'''$ and $\Phi \sqcup \Phi' : \mathscr{H} \sqcup \mathscr{H}' \to \mathscr{H}'' \sqcup \mathscr{H}'''$. Using the isomorphisms we introduced in the previous paragraph, $\Lambda : \mathscr{H} \times \mathscr{H}' \xrightarrow{\simeq} \mathscr{H} \otimes \mathscr{H}'$, $\Lambda' : \mathscr{H}'' \times \mathscr{H}''' \xrightarrow{\simeq} \mathscr{H}'' \otimes \mathscr{H}'''$ and $\Sigma : \mathscr{H} \sqcup \mathscr{H}' \xrightarrow{\simeq} \mathscr{H} \oplus \mathscr{H}'$, $\Sigma : \mathscr{H}'' \sqcup \mathscr{H}''' \xrightarrow{\simeq} \mathscr{H}'' \oplus \mathscr{H}'''$, we find then that $\Phi \times \Phi' \simeq \Phi \otimes \Phi'$ and $\Phi \sqcup \Phi' \simeq \Phi \oplus \Phi'$. It follows that $\mathfrak{h}(\varphi \times \varphi') \simeq \mathfrak{h}(\varphi) \otimes \mathfrak{h}\varphi')$ and $\mathfrak{h}(\varphi \sqcup \varphi') \simeq \mathfrak{h}(\varphi) \oplus \mathfrak{h}(\varphi')$ as required.

The category fdcsHilb of finite dimensional cosimplicial Hilbert spaces and operators is defined analogously to its simplicial counterpart. An object $\mathscr{H}$ of fdsHilb is a collection $\{\mathscr{H}_n, D_{cni}, S_{cni}\}$ consisting of finite dimensional Hilbert spaces $\mathscr{H}_n$ and coface and codegeneracy operators $D_{cni} : \mathscr{H}_{n-1} \to \mathscr{H}_n$, $S_{cni} : \mathscr{H}_{n+1} \to \mathscr{H}_n$ obeying the cosimplicial Hilbert identities, relations of the same form as Equation (33) except for the order of the factors which is inverted. A morphism $\Phi : \mathscr{H}' \to \mathscr{H}$ of fdsHilb is a collection of linear operators $\Phi_{cn} : \mathscr{H}'_n \to \mathscr{H}_n$ satisfying the cosimplicial Hilbert morphism identities, relations of the same form as Equation (42) except again for the reversed order of the factors.

Just as fdsHilb, the category fdcsHilb is bimonoidal, its two monoidal products being given again by degreewise direct product and sum $\otimes$ and $\oplus$. The explicit expressions $\otimes$ and $\oplus$ take in fdcsHilb and are essentially the same as they do in fdsHilb, which was detailed above.

It is easy to see that the dagger autofunctor $^+$ of the finite dimensional Hilbert space category fdHilb, which implements operator adjunction, induces a dagger isofunctor $^+$: fdsHilb $\to$ fdcsHilb$^{\text{op}}$, where the superscript $^{\text{op}}$ denotes opposite of a category. $^+$ associates with every simplicial Hilbert space $\mathscr{H} = \{\mathscr{H}_n, D_{ni}, S_{ni}\}$ its adjoint cosimplicial Hilbert space $\mathscr{H}^+ = \{\mathscr{H}_n, D_{ni}{}^+, S_{ni}{}^+\}$ and with every simplicial Hilbert operator $\Phi = \{\Phi_n\}$ its adjoint cosimplicial Hilbert operator $\Phi^+ = \{\Phi_n{}^+\}$ with $\Phi^+ : \mathscr{H}'^+ \to \mathscr{H}^+$ if $\Phi : \mathscr{H} \to \mathscr{H}'$. The functor $^+$ preserves direct multiplication and summation and is therefore bimonoidal. The categories fdsHilb, fdcsHilb$^{\text{op}}$ can therefore be identified.

We can compose the simplicial Hilbert functor $\mathfrak{h}$ and the dagger isofunctor $^+$ just introduced to obtain a functor $\mathfrak{h}_c$: pfsSet $\to$ fdcsHilb$^{\text{op}}$ from the parafinite simplicial set category pfsSet into the opposite finite dimensional cosimplicial Hilbert space category fdcsHilb$^{\text{op}}$. By means of $\mathfrak{h}_c$, we can associate with any parafinite simplicial set X a finite dimensional cosimplicial Hilbert space $\mathscr{H}^+$, the adjoint of the simplicial Hilbert space $\mathscr{H}$ assigned to X by $\mathfrak{h}$. Similarly, we can associate with any morphism $\phi : X \to X'$ of parafinite simplicial sets a cosimplicial Hilbert operator $\Phi^+ : \mathscr{H}'^+ \to \mathscr{H}^+$, the adjoint of the simplicial Hilbert operator $\Phi : \mathscr{H} \to \mathscr{H}'$ assigned to ϕ by $\mathfrak{h}$. We shall call $\mathfrak{h}_c$ the cosimplicial Hilbert functor.

In this way, parafinite simplicial sets have both a simplicial and a cosimplicial encoding related by the dagger isofunctor. This dagger structure provides the formal framework for the analysis of the unitarity of the simplicial Hilbert operators associated with the simplicial set morphisms arising in reversible simplicial computation (cf. Subsection 3.6 below).

We conclude this subsection with the following remark. Let X be a parafinite simplicial set with associated simplicial Hilbert space $\mathscr{H}$. Suppose that the Hilbert structure of the spaces $\mathscr{H}_n$ is forgotten so that the $\mathscr{H}_n$ are regarded just as sets and the operators D_{ni}, S_{ni} as maps. Then, the simplicial Hilbert encoding maps $\varkappa_n : X_n \to \mathscr{H}_n$ given by $\varkappa_n(\sigma_n) = |\sigma_n\rangle$ are the components of a simplicial set monomorphism $\varkappa : X \to \mathscr{H}$ as follows from Equations (27) and (28). The morphism $\varkappa$ will enter the discussion of truncation and skeletonization of simplicial sets in Section 4.

3.3. The simplicial Hilbert Hodge Laplacians and their properties

In this subsection, we shall study in some depth the simplicial Hilbert Hodge Laplacians and their properties having in mind the problem of the computation of the simplicial homology spaces of a simplicial set analyzed later in Subsection 3.4. We shall do so from a perspective more general than that strictly required by such a problem, considering three kinds of Laplacians. The reason for proceeding like so is twofold. First, as a way of achieving a broader and more complete understanding of the quantum simplicial operator framework developed in the preceding subsections. Second, for its potential relevance in a reinterpretation of the simplicial Hilbert structure as an instance of $N = 4$ supersymmetric quantum mechanics on the lines of the analogous formulation of the quantum simplicial complex framework of Ref. [17] as an instance of $N = 2$ supersymmetric quantum mechanics proposed and studied in Refs. [34], though we shall delve into this matter in the present work.

We begin by introducing simplicial Hilbert homological operators which will be key in the study of simplicial Hilbert homology later in Subsection 3.4.

Definition 19. *The simplicial Hilbert face boundary operators* $Q_{Dn} : \mathscr{H}_n \to \mathscr{H}_{n-1}$*,* $n \geq 1$*, and degeneracy coboundary operators* $Q_{Sn} : \mathscr{H}_n \to \mathscr{H}_{n+1}$*,* $n \geq 0$ *are given by*

$$Q_{Dn} = \sum_{0 \leq i \leq n} (-1)^i D_{ni} \tag{44}$$

$$Q_{Sn} = \sum_{0 \leq i \leq n} (-1)^i S_{ni} \tag{45}$$

The Q_{Dn}, Q_{Sn} are the building blocks of the simplicial Hilbert Hodge Laplacians.

Definition 20. *The face, mixed and degeneracy simplicial Hilbert Hodge Laplacians are the operators* $H_{DDn} : \mathscr{H}_n \to \mathscr{H}_n$*,* $n \geq 0$*,* $H_{SDn} : \mathscr{H}_n \to \mathscr{H}_{n-2}$*,* $n \geq 2$*, and* $H_{SSn} : \mathscr{H}_n \to \mathscr{H}_n$*,* $n \geq 0$ *given by*

$$H_{DDn} = {Q_{Dn}}^+ Q_{Dn} + Q_{Dn+1} {Q_{Dn+1}}^+ \tag{46}$$

$$H_{SDn} = {Q_{Sn-2}}^+ Q_{Dn} + Q_{Dn-1} {Q_{Sn-1}}^+ \tag{47}$$

$$H_{SSn} = {Q_{Sn}}^+ Q_{Sn} + Q_{Sn-1} {Q_{Sn-1}}^+ \tag{48}$$

Above, it is tacitly understood that the first term on the right hand side of Equation (46) and the second term on the right hand side of Equation (48) are absent when $n = 0$. We observe that the H_{DDn}, H_{SSn} are Hermitian while the H_{SDn} are not.

The H_{DDn}, H_{DSn}, H_{SSn} can be expressed through the basic face and degeneracy operators D_{ni}, S_{ni} and their adjoints ${D_{ni}}^+$, ${S_{ni}}^+$ on account of Equations (44) and (45). The resulting expressions of H_{DDn}, H_{DSn}, H_{SSn} exhibit a similar structure:

$$H_{DDn} = \varUpsilon_{DDn} + {\varUpsilon_{DDn}}^+ + {H^0}_{DDn} \tag{49}$$

$$H_{SDn} = \varUpsilon_{SDn} + {\varUpsilon_{DSn-2}}^+ + {H^0}_{SDn} \tag{50}$$

$$H_{SSn} = {H^0}_{SSn} \tag{51}$$

Here, $\varUpsilon_{DDn}$, $\varUpsilon_{DSn}$, $\varUpsilon_{SDn}$, are operators directly related to the defects ${\varDelta^{DD}}_{nij}$, ${\varDelta^{DS}}_{nij}$, ${\varDelta^{SD}}_{nij}$ introduced in Subsection 3.1 and given by Equations (35a)–(35c),

$$\varUpsilon_{DDn} = \sum_{0 \leq i,j \leq n, i<j} (-1)^{i+j} {\varDelta^{DD}}_{nij} \tag{52a}$$

$$\varUpsilon_{DSn} = \sum_{0 \leq i,j \leq n, i \leq j} (-1)^{i+j} {\varDelta^{DS}}_{nij} \tag{52b}$$

$$\varUpsilon_{SDn} = \sum_{0 \leq i,j \leq n, i+1<j} (-1)^{i+j} {\varDelta^{SD}}_{nij} \tag{52c}$$

Above, $\varUpsilon_{DD0} = 0$ by convention. We note here that $\varUpsilon_{DSn}$, $\varUpsilon_{SDn}$ vanish when X is a semiperfect simplicial set and that $\varUpsilon_{DDn}$ also vanishes when X is a quasi perfect simplicial set (cf. Definition 16). A term of the form $\varUpsilon_{SSn} + {\varUpsilon_{SSn}}^+$ depending in an analogous manner on the defects ${\varDelta^{SS}}_{nij}$, ${{\varDelta^{SS}}_{nij}}^+$ does not appear in the expression of H_{SSn} in Equation (51), because the ${\varDelta^{SS}}_{nij}$ always vanish by the no degeneracy defect Theorem 2. The operators ${H^0}_{DDn}, {H^0}_{SDn}, {H^0}_{SSn}$ instead are not reducible to the defects and are genuinely new. They provide additional structure to our quantum simplicial framework. In particular, they involve all the missing products that are $D_{n+1i}{D_{n+1i}}^+, 0 \leq i \leq n+1, {D_{n+2i+1}}^+S_{ni}, 0 \leq i \leq n, S_{n+1i}{D_{n+1i}}^+, 0 \leq i \leq n+1$ and ${S_{ni}}^+S_{ni}, 0 \leq i \leq n$ not covered by the mixed exchange identities (34).

The operator ${H^0}_{DDn}$ is Hermitian. ${H^0}_{DDn}$ can be expressed through two sets of elementary operators. The first set consists of the operators

$$\varOmega_{ni} = D_{n+1i}{D_{n+1i}}^+ \tag{53}$$

with $0 \leq i \leq n+1$. The $\varOmega_{ni}$ are clearly Hermitian. A simple application of the basic expressions (27), (29) shows that the $\varOmega_{ni}$ are diagonal in the simplex basis $|\sigma_n\rangle$,

$$\varOmega_{ni} = \sum_{\sigma_n \in X_n} |\sigma_n\rangle |D_{n+1i}(\sigma_n)| \langle\sigma_n| \tag{54}$$

So, for every i and n-simplex σ_n, $\varOmega_{ni}$ counts the number of $n+1$-simplices ω_{n+1} whose i-face is σ_n. The second set is constituted by the operators

$$\varTheta_{ni} = {D_{ni}}^+D_{ni} \tag{55}$$

$$\varGamma_{ni} = D_{n+1i+1}{D_{n+1i}}^+ \tag{56}$$

with $0 \leq i \leq n$. Above, we conventionally set $\varTheta_{00} = 0$. The $\varTheta_{ni}$ are Hermitian, whilst the $\varGamma_{ni}$ are not. Another straightforward application of (27), (29) furnishes the following formulae:

$$\begin{aligned} \varTheta_{ni} &= \sum_{\sigma_n \in X_n} \sum_{\omega_n \in D_{ni}(d_{ni}\sigma_n)} |\omega_n\rangle\langle\sigma_n| \\ \varGamma_{ni} &= \sum_{\sigma_n, \omega_n \in X_n} |\omega_n\rangle |D_{n+1i}(\sigma_n) \cap D_{n+1i+1}(\omega_n)| \langle\sigma_n| \end{aligned} \tag{57}$$

Hence, for each i, $\varTheta_{ni}$ detects all the pairs σ_n, ω_n of n-simplices sharing the i–face whilst $\varGamma_{ni}$ provides information about the number of $n+1$-simplices ω_{n+1} having σ_n, ω_n as their i, $i+1$-th faces. We note that for $n \geq 1$, owing to the simplicial relation (2a), the effective summation range of $\varGamma_{ni}$ consists of pairs $\sigma_n, \omega_n \in X_n$ such that $d_{ni}\omega_n = d_{ni}\sigma_n$ and thus it is contained in the summation range of $\varTheta_{ni}$.

The operator ${H^0}_{DDn}$ is given by a sum of operator products of the kind appearing on the right hand side of Equations (53), (55) and (56). ${H^0}_{DDn}$ is in this way expressible in terms of the operators $\varOmega_{ni}$, $\varTheta_{ni}$,

$${H^0}_{DDn} = \sum_{0 \leq i \leq n+1} \varOmega_{ni} + \sum_{0 \leq i \leq n} (\varTheta_{ni} - \varGamma_{ni} - {\varGamma_{ni}}^+) \tag{58}$$

${H^0}_{DDn}$ so encodes all the information about face relations in the underlying simplicial set X that the $\varOmega_{ni}$, $\varTheta_{ni}$ and $\varGamma_{ni}$ do.

The operators ${H^0}_{SDn}$ and the Hermitian operators ${H^0}_{SSn}$ have a more elementary structure. They are reducible to a common set of elementary orthogonal projectors as we now show.

For $0 \leq i \leq n$, the operator S_{ni} is an isometry of $\mathscr{H}_n$ into $\mathscr{H}_{n+1}$,

$$S_{ni}{}^{+} S_{ni} = 1_n \tag{59}$$

This follows immediately from the expressions (28), (30) of S_{ni}, $S_{ni}{}^{+}$ and the fact that the sets $S_{ni}(\sigma_{n+1})$ defined in Equation (32) contain at most one element. Consequently, $S_{ni} S_{ni}{}^{+}$ is the orthogonal projector on the range ran S_{ni} of S_{ni}.

In general, the Hermitian operators

$$\mathit{\Pi}_{ni} = S_{n-1i} S_{n-1i}{}^{+} = S_{ni+1}{}^{+} S_{ni} = S_{ni}{}^{+} S_{ni+1} \tag{60}$$

where $0 \leq i \leq n-1$, are orthogonal projectors in $\mathscr{H}_n$. The identity of the three expressions of $\mathit{\Pi}_{ni}$ follows from the S–S^{+} exchange identities (34d). It is simple to verify using (28), (30) that the $\mathit{\Pi}_{ni}$ are diagonal in the simplex basis $|\sigma_n\rangle$,

$$\mathit{\Pi}_{ni} = \sum_{\sigma_n \in X_n} |\sigma_n\rangle |S_{n-1i}(\sigma_n)| \langle \sigma_n| \tag{61}$$

Since $|S_{n-1i}(\sigma_n)| \leq 1$, for fixed i $\mathit{\Pi}_{ni}$ detects whether a given n-simplex σ_n lies in the range of s_{n-1i} or not. The $\mathit{\Pi}_{ni}$ evidently commute pairwise. The $\mathit{\Pi}_{ni}$ do not furnish however a resolution of the identity of $\mathscr{H}_n$ because $\mathit{\Pi}_{ni} \mathit{\Pi}_{nj} \neq 0$ in general for $i \neq j$. Via the projectors $\mathit{\Pi}_{ni}$, the adjoint degeneracy operators $S_{ni}{}^{+}$ are reducible to the face operators D_{ni}, since

$$S_{ni}{}^{+} = D_{n+1i} \mathit{\Pi}_{n+1i} = D_{n+1i+1} \mathit{\Pi}_{n+1i} \tag{62}$$

as follows immediately from Equation (33c).

The operators $H^0{}_{SDn}$, $H^0{}_{SSn}$ are expressible in a simple manner in terms of the projectors $\mathit{\Pi}_{ni}$,

$$H^0{}_{DSn} = \sum_{0 \leq i \leq n-1} D_{n-1i} D_{ni+1} \mathit{\Pi}_{ni} - \sum_{0 \leq i \leq n-2} D_{n-1i} \mathit{\Pi}_{n-1i} D_{ni+1} \tag{63}$$

$$H^0{}_{SSn} = (n+1) 1_n - \sum_{0 \leq i \leq n-1} \mathit{\Pi}_{ni} \tag{64}$$

The second term on the right hand side of Equation (64) is conventionally set to 0 for $n = 0$. The verification of these identities is straightforward enough from Equations (59) and (62).

3.4. Simplicial Hilbert homology

In Subsection 2.3, we showed that a chain complex can be associated with any simplicial group. This scheme can be applied in particular to the simplicial Hilbert space of a parafinite simplicial set introduced in Subsections 3.1 and 3.2, adding new elements to our analysis. In fact, the richness of the operator structure of the quantum simplicial framework enables one to introduce several types of simplicial and cosimplicial Hilbert homology and cohomology. Since simplicial Hilbert theory is just a special codification of standard simplicial set theory, we expect that eventually we shall recover the ordinary simplicial homology of the underlying simplicial set, if we succeed, and no more. This is indeed the case: all the homologies and cohomologies that can be constructed turn out to be either trivial or isomorphic to simplicial homology. This subsection is devoted to the illustration of such construction.

Let X be a parafinite simplicial set and let $\mathscr{H}$ be its associated simplicial Hilbert space. The application of the homological set-up of Subsection 2.3 to $\mathscr{H}$ seen as a simplicial Abelian group yields a simplicial chain complex, the simplicial Hilbert face chain complex $(\mathscr{H}, Q_D)$. Its chain spaces are the Hilbert spaces $\mathscr{H}_n$; its boundary operators are the simplicial Hilbert face boundary operators Q_{Dn}, $n \geq 1$, defined in Equation (44). As is readily verified also using the simplicial identities (33a), the Q_{Dn} do indeed obey the basic homological relations (15) reading currently as

$$Q_{Dn} Q_{Dn+1} = 0 \tag{65}$$

Associated with $(\mathscr{H}, Q_D)$ there are then the simplicial Hilbert face homology spaces $\mathrm{H}_{Dn}(\mathscr{H}) = \ker Q_{Dn} / \operatorname{ran} Q_{Dn+1}$ with $n \geq 0$, where $\ker Q_{D0} = \mathscr{H}_0$ by convention.

Similarly, the application of the homological set-up to the cosimplicial Hilbert space $\mathscr{H}^+$ seen as a cosimplicial Abelian group yields a cosimplicial cochain complex, the cosimplicial Hilbert face cochain complex $(\mathscr{H}^+, Q_D{}^+)$. Its cochain spaces are the Hilbert spaces $\mathscr{H}_n$; its coboundary operators are the adjoints $Q_{Dn}{}^+$ of the simplicial Hilbert face boundary operators Q_{Dn}. The $Q_{Dn}{}^+$ obey indeed cohomological relations following from the Equation (65) by adjunction and identical to these but for the order of the factors of the operator products. Associated with $(\mathscr{H}^+, Q_D{}^+)$ there are then the cosimplicial Hilbert face cohomology spaces $\mathrm{H}_D{}^n(\mathscr{H}^+) = \ker Q_{Dn+1}{}^+ / \operatorname{ran} Q_{Dn}{}^+$ with $n \geq 0$, where $\operatorname{ran} Q_{D0}{}^+ = 0$ by convention.

Consider a morphism $\phi : X \to X'$ of the parafinite simplicial sets X, X'. As we have seen in Subsection 3.2, with ϕ there is associated a morphism $\varPhi : \mathscr{H} \to \mathscr{H}'$ of the simplicial Hilbert spaces $\mathscr{H}$, $\mathscr{H}'$ of X, X'. In the spirit of the homological theory of Subsection 2.3, this can be regarded as a morphism of simplicial Abelian groups. A morphism of chain complexes is then yielded, the associated morphism of simplicial Hilbert chain complexes $\varPhi : (\mathscr{H}, Q_D) \to (\mathscr{H}', Q'_D)$. Its components are the operators $\varPhi_n$ given by Equation (41), By virtue of Equation (44), the $\varPhi_n$ do indeed obey the identities (18), reading here as

$$\varPhi_{n-1} Q_{Dn} = Q'_{Dn} \varPhi_n \tag{66}$$

Similarly regarding the morphism $\varPhi^+ : \mathscr{H}'^+ \to \mathscr{H}^+$ of the cosimplicial Hilbert spaces $\mathscr{H}'^+$, $\mathscr{H}^+$ of X', X as a morphism of cosimplicial Abelian groups, we find a morphism of cochain complexes, the associated morphism of cosimplicial Hilbert cochain complexes $\varPhi^+ : (\mathscr{H}'^+, Q'_D{}^+) \to (\mathscr{H}^+, Q_D{}^+)$. Its components $\varPhi_n{}^+$, given by Equation (43), satisfy indeed the adjoints of relations (66). $\varPhi$, $\varPhi^+$ hereby give rise to morphisms $\varPhi_{*n} : \mathrm{H}_{Dn}(\mathscr{H}) \to \mathrm{H}_{Dn}(\mathscr{H}')$ and $\varPhi^{+*n} : \mathrm{H}_D{}^n(\mathscr{H}'^+) \to \mathrm{H}_D{}^n(\mathscr{H}^+)$ of the associated Hilbert face homology and cohomology spaces.

The computation of the homology/cohomology spaces $\mathrm{H}_{Dn}(\mathscr{H})$, $\mathrm{H}_D{}^n(\mathscr{H}^+)$

can be carried out by mimicking that of the de Rham cohomology spaces of closed Riemannian manifolds in Hodge theory: it reduces to the determination of the kernels of appropriate simplicial Hilbert Hodge Laplacians [17].

Theorem 4. *(Simplicial Hilbert face Hodge theorem) For $n \geq 0$, the isomorphism*

$$\mathrm{H}_{Dn}(\mathscr{H}) \simeq \mathrm{H}_D{}^n(\mathscr{H}^+) \simeq \ker H_{DDn} \quad \textit{with } n \geq 0 \tag{67}$$

holds, where H_{DDn} is the face simplicial Hilbert Hodge Laplacian (cf. Equation (46)*).*

Proof. The proof of Theorem 4 is based on the finite dimensional Hodge theorem. Although this theorem is well-known, we provide a simple proof of it in Appendix III for the reader's benefit. □

On account of Equations (24), (27) and (44), the face chain complex $(\mathscr{H}, Q_D)$ is manifestly isomorphic to the simplicial chain complex $(C(X,\mathbb{C}), \partial)$ with complex coefficients of the simplicial set X. At degree n, the isomorphism is given by the chain encoding map $\varkappa_n : C_n(X,\mathbb{C}) \rightarrow \mathscr{H}_n$ engendered by the simplicial encoding map introduced at the end of Subsection 3.2. Such isomorphism translates into one of the corresponding homology spaces leading to the following.

Theorem 5. *The simplicial Hilbert face homology and the cosimplicial face cohomology are isomorphic to the simplicial homology of the underlying simplicial set X with complex coefficients: for $n \geq 0$*

$$\mathrm{H}_{Dn}(\mathscr{H}) \simeq \mathrm{H}_D{}^n(\mathscr{H}^+) \simeq \mathrm{H}_n(X,\mathbb{C}) \tag{68}$$

Consequently, one has

$$\mathrm{H}_n(X,\mathbb{C}) \simeq \ker H_{DDn} \tag{69}$$

Consider again a morphism $\phi : X \rightarrow X'$ of the parafinite simplicial sets X, X'. ϕ gives rise to a morphism $\phi : (C(X,\mathbb{C}), \partial) \rightarrow (C(X',\mathbb{C}), \partial')$ of the complex simplicial chain complexes of X, X' (cf. Subsection 2.3, Definition 11) and by virtue of this a morphism $\phi_{*n} : \mathrm{H}_n(X,\mathbb{C}) \rightarrow \mathrm{H}_n(X',\mathbb{C})$ of the associated complex simplicial homology spaces for each n. Evidently, the simplicial Hilbert homology and cohomology space morphism $\Phi_{*n} : \mathrm{H}_{Dn}(\mathscr{H}) \rightarrow \mathrm{H}_{Dn}(\mathscr{H}')$ and $\Phi^{+*n} : \mathrm{H}_D{}^n(\mathscr{H}'^+) \rightarrow \mathrm{H}_D{}^n(\mathscr{H}^+)$ we have constructed earlier are the simplicial Hilbert encoding of the simplicial homology morphisms ϕ_{*n}.

As anticipated at the beginning of this subsection, the quantum simplicial framework is characterized by further homology and cohomology spaces, which we briefly illustrate next. Such spaces can be shown to be trivial, as expected also on general grounds. The uninterested reader can skip this discussion and move directly to the last paragraph of this subsection, if he/she wishes so.

In Subsection 3.3, we have also introduced the simplicial Hilbert degeneracy coboundary operators Q_{Sn}. Using the simplicial identities (33e), it is not difficult to show that the operators Q_{Sn} obey the basic cohomological relations

$$Q_{Sn+1} Q_{Sn} = 0 \tag{70}$$

Exploiting the simplicial identities (33b)–(33d), one finds in addition that the Q_{Sn} satisfy a further relation,

$$Q_{Sn-1}Q_{Dn} + Q_{Dn+1}Q_{Sn} = 0 \tag{71}$$

involving the simplicial Hilbert face boundary operators Q_{Dn} considered earlier. In this wise, the Q_{Dn} are part of a broader homological structure including the Q_{Sn}.

By virtue of relations (70), the simplicial Hilbert space $\mathscr{H}$ of X underlies the simplicial Hilbert degeneracy cochain complex $(\mathscr{H}, Q_S)$ with cochain spaces $\mathscr{H}_n$ and coboundary operators Q_{Sn}. Associated with this there are the simplicial Hilbert degeneracy cohomology spaces $\mathrm{H}_S{}^n(\mathscr{H}) = \ker Q_{Sn}/\operatorname{ran} Q_{Sn-1}$ for any $n \geq 0$, where $\operatorname{ran} Q_{S-1} = 0$ conventionally.

Similarly, by the adjoint of relations (70), the cosimplicial Hilbert space $\mathscr{H}^+$ of X supports the cosimplicial Hilbert degeneracy chain complex $(\mathscr{H}^+, Q_S{}^+)$ with chain spaces $\mathscr{H}_n$ and boundary operators $Q_{Sn}{}^+$. Associated with this there are the cosimplicial Hilbert degeneracy homology spaces $\mathrm{H}_{Sn}(\mathscr{H}^+) = \ker Q_{Sn-1}{}^+/\operatorname{ran} Q_{Sn}{}^+$ for any $n \geq 0$ with $\ker Q_{S-1}{}^+ = \mathscr{H}_0$.

Let $\phi : X \to X'$ be a morphism of the parafinite simplicial sets X, X'. The components $\varPhi_n$ of the attached simplicial Hilbert space morphism $\varPhi : \mathscr{H} \to \mathscr{H}'$ obey relations (42b). Consequently, by Equation (45), the $\varPhi_n$ also obey the identities

$$\varPhi_{n+1}Q_{Sn} = Q'_{Sn}\varPhi_n \tag{72}$$

Owing to Equation (18), the $\varPhi_n$ define then a morphism $\varPhi : (\mathscr{H}, Q_S) \to (\mathscr{H}', Q'_S)$ of cochain complexes. In the same way, by virtue of the Hilbert dagger structure, the components $\varPhi_n{}^+$ of the adjoint morphism $\varPhi^+ : \mathscr{H}'^+ \to \mathscr{H}^+$ give rise to a morphism $\varPhi^+ : (\mathscr{H}'^+, Q'_S{}^+) \to (\mathscr{H}^+, Q_S{}^+)$ of chain complexes. One has in this wise morphisms $\varPhi_*{}^n : \mathrm{H}_S{}^n(\mathscr{H}) \to \mathrm{H}_S{}^n(\mathscr{H}')$ and $\varPhi^{+*}{}_n : \mathrm{H}_{Sn}(\mathscr{H}'^+) \to \mathrm{H}_{Sn}(\mathscr{H}^+)$ of the associated simplicial and cosimplicial Hilbert degeneracy cohomology and homology spaces.

In spite of the formal similarities of the homology/cohomology spaces $\mathrm{H}_{Dn}(\mathscr{H})$, $\mathrm{H}_D{}^n(\mathscr{H}^+)$ and the cohomology/homology spaces $\mathrm{H}_S{}^n(\mathscr{H})$, $\mathrm{H}_{Sn}(\mathscr{H}^+)$, while the former are generally non trivial, the latter always are, in accordance with our expectations, as we show next.

The computation of the spaces $\mathrm{H}_S{}^n(\mathscr{H})$, $\mathrm{H}_{Sn}(\mathscr{H}^+)$ is again reduced to the determination of the kernels of appropriate simplicial Hilbert Hodge Laplacians. The triviality of the $\mathrm{H}_S{}^n(\mathscr{H})$, $\mathrm{H}_{Sn}(\mathscr{H}^+)$ is an immediate consequence of that of such kernels.

Theorem 6. *(Hilbert degeneracy cohomology and homology triviality theorem) The simplicial Hilbert degeneracy cohomology and homology spaces are trivial,*

$$\mathrm{H}_S{}^n(\mathscr{H}) \simeq \mathrm{H}_{Sn}(\mathscr{H}^+) \simeq 0 \tag{73}$$

for $n \geq 0$.

Proof. By the finite dimensional Hodge theorem, proven in Appendix III, we have

$$\mathrm{H}_S{}^n(\mathscr{H}) \simeq \mathrm{H}_{Sn}(\mathscr{H}^+) \simeq \ker H_{SSn} \tag{74}$$

for $n \geq 0$, where H_{SSn} is the degeneracy simplicial Hilbert Hodge Laplacian (cf. Equation (48)). The result follows by showing that $\ker H_{SSn} = 0$. See Appendix IV. □

A simplicial degeneracy cochain complex $(C(X,\mathbb{C}),\mathfrak{b})$ can be built also in standard simplicial theory with no reference to its eventual Hilbert space encoding alongside the face chain complex $(C(X,\mathbb{C}),\partial)$. While the complex $(C(X,\mathbb{C}),\partial)$ is contemplated and analyzed in simplicial theory, to our knowledge the complex $(C(X,\mathbb{C}),\mathfrak{b})$ has not appeared and found any application so far. The reason for this is presumably that the homology of $(C(X,\mathbb{C}),\partial)$, which is just the complex simplicial homology $\mathrm{H}(X,\mathbb{C})$ of X, is significant and generally non trivial, while the cohomology of $(C(X,\mathbb{C}),\mathfrak{b})$, as we have shown, vanishes. Interestingly, we have been able to provide a completely quantum Hilbert space proof of this fact.

From now on, for the reasons explained above, we concentrate on the simplicial Hilbert face homology, which we shall call simply simplicial Hilbert homology.

3.5. Normalized simplicial Hilbert homology

The determination of the simplicial homology $\mathrm{H}(X,\mathbb{C})$ of a parafinite simplicial set X via that of the isomorphic simplicial Hilbert homology $\mathrm{H}_D(\mathscr{H})$ is computationally more costly than necessary, as it involves also the subspaces of the simplex Hilbert spaces spanned by the degenerate simplices (cf. Subsection 2.1, Definition 6), which are homologically irrelevant by the normalization Theorem 1. In this subsection, we shall explain how this redundant degenerate structure can be disposed of in our formulation opening a cheaper route to the homology computation.

Definition 21. *For $n \geq 0$, the degenerate n-simplex space is*

$${}^s\mathscr{H}_n = \textstyle\sum_{i=0}^{n-1} \mathrm{ran}\, S_{n-1i} \tag{75}$$

where ${}^s\mathscr{H}_0 = 0$ by convention.

The expression on the right hand side of Equation (75) denotes the linear span of the ranges of the operators S_{n-1i}. ${}^s\mathscr{H}_n$ is therefore the subspace of $\mathscr{H}_n$ spanned by the degenerate n-simplex vectors as alluded to by its name.

By the isomorphism of the chain complexes $(C(X,\mathbb{C}),\partial)$ and $(\mathscr{H}, Q_D)$ disclosed in Subsection 3.4, the fact that the simplicial boundary operators of X preserve the subspaces of complex degenerate chains of X recalled in Subsection 2.3 translates into the property that the face boundary operators of $\mathscr{H}$ preserve the degenerate simplex subspaces of $\mathscr{H}$. So, for each $n \geq 1$ $Q_{Dn}{}^s\mathscr{H}_n \subseteq {}^s\mathscr{H}_{n-1}$. Let $\overline{\mathscr{H}}_n = \mathscr{H}_n/{}^s\mathscr{H}_n$. An operator $\overline{Q}_{Dn} : \overline{\mathscr{H}}_n \to \overline{\mathscr{H}}_{n-1}$ is then induced by Q_{Dn}, which obeys the homological relation $\overline{Q}_{Dn-1}\overline{Q}_{Dn} = 0$ (cf. Equation (65)). We have in this way a chain complex $(\overline{\mathscr{H}}, \overline{Q}_D)$ called the abstract normalized simplicial Hilbert face complex of X below. The associated abstract normalized simplicial Hilbert homology spaces are $\mathrm{H}_{Dn}(\overline{\mathscr{H}}) = \ker \overline{Q}_{Dn}/\,\mathrm{ran}\,\overline{Q}_{Dn+1}$ with $n \geq 0$ (with $\ker \overline{Q}_{D0} = \overline{\mathscr{H}}_0$).

Theorem 7. *For every $n \geq 0$, one has*

$$\mathrm{H}_{Dn}(\overline{\mathscr{H}}) \simeq \mathrm{H}_n(X, \mathbb{C}) \tag{76}$$

Proof. The chain complexes $(C(X, \mathbb{C}), \partial)$ and $(\mathscr{H}, Q_D)$ are isomorphic. Further, by Equation (75), for each n we have ${}^sC_n(X, \mathbb{C}) \simeq {}^s\mathscr{H}_n$. The normalized chain complexes $(\overline{C}(X, \mathbb{C}), \overline{\partial})$ and $(\overline{\mathscr{H}}, \overline{Q}_D)$ are also isomorphic and so $\overline{\mathrm{H}}_n(X, \mathbb{C}) \simeq \mathrm{H}_{Dn}(\overline{\mathscr{H}})$. By the normalization Theorem 1, $\mathrm{H}_n(X, \mathbb{C}) \simeq \overline{\mathrm{H}}_n(X, \mathbb{C})$. (76) then follows. □

The isomorphism (76) offers an alternative way of computing the simplicial homology of X, which is less expensive in that it does away with degenerate simplices. The modding out of these latter is however hardly implementable algorithmically by its abstract form. A full operator formulation is necessary for that purpose.

With the above in mind, we introduce the orthogonal projector Π_n on the degenerate n-simplex space ${}^s\mathscr{H}_n$. An expression of Π_n can be obtained in terms of the orthogonal projectors Π_{ni} introduced in Equation (60): one has

$$\Pi_n = 1_n - \prod_{0 \leq i \leq n-1} (1_n - \Pi_{ni}) \tag{77}$$

for $n \geq 0$ with $\Pi_0 = 0$ by convention. The reader is referred to Appendix V for some details about the derivation of this formula.

We now introduce another Hilbert complex closely related to the normalized Hilbert complex $(\overline{\mathscr{H}}, \overline{Q}_D)$. For $n \geq 1$, let ${}^c\mathscr{H}_n = {}^s\mathscr{H}_n{}^{\perp}$, where $^{\perp}$ denotes orthogonal complement, and let ${}^cQ_{Dn} : {}^c\mathscr{H}_n \to {}^c\mathscr{H}_{n-1}$ be defined by

$${}^cQ_{Dn} = (1_{n-1} - \Pi_{n-1}) Q_{Dn}\big|_{{}^c\mathscr{H}_n} \tag{78}$$

The ${}^cQ_{Dn}$ satisfy the homological relation ${}^cQ_{Dn-1}{}^cQ_{Dn} = 0$. The reader is referred again to Appendix V for some details about the derivation of this identity. We have consequently a chain complex $({}^c\mathscr{H}, {}^cQ_D)$ which we shall denominate the concrete normalized simplicial Hilbert face complex of X in the following. The associated concrete normalized simplicial Hilbert homology spaces with $n \geq 0$ are defined as $\mathrm{H}_{Dn}({}^c\mathscr{H}) = \ker {}^cQ_{Dn} / \operatorname{ran} {}^cQ_{Dn+1}$ (with $\ker {}^cQ_{D0} = {}^c\mathscr{H}_0$).

The following result shows the isomorphism of the two homologies we have introduced above.

Proposition 3. *The abstract and concrete simplicial Hilbert face homologies are isomorphic: for every $n \geq 0$, it holds that*

$$\mathrm{H}_{Dn}(\overline{\mathscr{H}}) \simeq \mathrm{H}_{Dn}({}^c\mathscr{H}) \tag{79}$$

Proof. The proof of the isomorphism (79) can be achieved by constructing a chain equivalence of the abstract and concrete Hilbert complexes $(\overline{\mathscr{H}}, Q_D)$, $({}^c\mathscr{H}, {}^cQ_D)$. The chain equivalence consists of a sequence of chain operators $I_n : \overline{\mathscr{H}}_n \to {}^c\mathscr{H}_n$, $J_n : {}^c\mathscr{H}_n \to \overline{\mathscr{H}}_n$, $n \geq 0$, such that the composite operators J_nI_n, I_nJ_n are chain homotopic to $\overline{1}_n$, ${}^c1_n$, respectively. The property of I_n, J_n being chain operators is just I_n, J_n

satisfying the relations

$$I_{n-1}\overline{Q}_{Dn} = {}^cQ_{Dn}I_n \tag{80a}$$

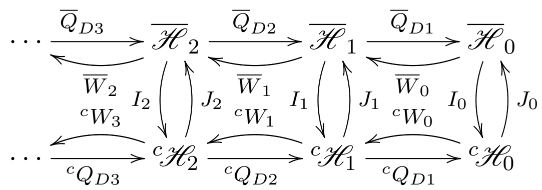
$$J_{n-1}{}^cQ_{Dn} = \overline{Q}_{Dn}J_n \tag{80b}$$

for $n \geq 1$. The chain homotopy of J_nI_n, I_nJ_n and 1_n, ${}^c1_n$ descends from the existence of operators $\overline{W}_n : \overline{\mathscr{H}}_n \to \overline{\mathscr{H}}_{n+1}$, ${}^cW_n : {}^c\mathscr{H}_n \to {}^c\mathscr{H}_{n+1}$ such that

$$J_nI_n - \overline{1}_n = \overline{Q}_{Dn+1}\overline{W}_n + \overline{W}_{n-1}\overline{Q}_{Dn} \tag{81a}$$

$$I_nJ_n - {}^c1_n = {}^cQ_{Dn+1}{}^cW_n + {}^cW_{n-1}{}^cQ_{Dn} \tag{81b}$$

for all $n \geq 0$, where the second term on the right hand side of both relations is absent when $n = 0$. The diagram



(82)

represents graphically the operator structure described above.

The chain equivalence I_n, J_n has the following explicit form. I_n is the operator from $\overline{\mathscr{H}}_n$ to ${}^c\mathscr{H}_n$ induced by the orthogonal projector $1_n - \mathit{\Pi}_n$ by virtue of the fact that ${}^s\mathscr{H}_n = \ker(1_n - \mathit{\Pi}_n)$. J_n is the canonical projection of ${}^c\mathscr{H}_n$ onto $\overline{\mathscr{H}}_n$. In Appendix V it is shown that I_n, J_n are both chain operators and are chain homotopic to $\overline{1}_n$, ${}^c1_n$, as required. □

The isomorphism (79) presumably reflects an equivalence of categories of finite dimensional simplicial Hilbert spaces, fdsHilb, and the category of chain complexes of finite dimensional Hilbert spaces, ChfdHilb, as a version of the Dold-Kan correspondence [59–61][1].

Because of the isomorphism (79), we shall no longer distinguish the abstract and concrete simplicial Hilbert homologies.

The following theorem is the main result of this subsection.

Theorem 8. *(Normalized simplicial Hilbert homology theorem) For all $n \geq 0$,*

$$\mathrm{H}_{Dn}({}^c\mathscr{H}) \simeq \mathrm{H}_n(X, \mathbb{C}) \tag{83}$$

Proof. The homology isomorphism (83) follows readily from the isomorphisms (76) and (79). □

The isomorphism (83) provides an alternative pathway to the determination of the complex simplicial homology of the simplicial set X grounded on normalized simplicial Hilbert homology. As for the non normalized homology studied in Subsection 3.4, the normalized homology can be computed via finite dimensional Hodge theory.

Definition 22. *The normalized simplicial Hilbert Laplacians are the operators*

${}^{c}H_{DDn} : {}^{c}\mathscr{H}_n \to {}^{c}\mathscr{H}_n$, $n \geq 0$, *given by*

$$ {}^{c}H_{DDn} = {}^{c}Q_{Dn}{}^{+c}Q_{Dn} + {}^{c}Q_{Dn+1}{}^{c}Q_{Dn+1}{}^{+} \tag{84} $$

Above, it is tacitly understood that the first term on the right hand side of Equation (84) is absent when $n = 0$.

The following theorem, like Theorem 4, relates the normalized simplicial Hilbert homology spaces to the kernels of the normalized simplicial Hilbert Laplacians.

Theorem 9. *(Normalized simplicial Hilbert Hodge theorem) The isomorphism*

$$ \mathrm{H}_{Dn}({}^{c}\mathscr{H}) \simeq \ker {}^{c}H_{DDn} \tag{85} $$

holds for every $n \geq 0$. *Consequently*

$$ \mathrm{H}_n(X, \mathbb{C}) \simeq \ker {}^{c}H_{DDn} \tag{86} $$

Proof. The proof of Theorem 9 is based again on the finite dimensional Hodge theorem reviewed in Appendix III. □

The isomorphism (85) provides a potentially more efficient way of computing the simplicial homology $\mathrm{H}(X, \mathbb{C})$ of X with complex coefficients than the isomorphism (69), as by virtue of it non degenerate simplices have been effectively excised.

We conclude this subsection by presenting explicit expressions of some of the operators introduced above for their relevance and later usefulness. In what follows, ${}^{c}X_n = X_n \setminus {}^{s}X_n$ denotes the set of non degenerate n-simplices of X. The normalized simplicial Hilbert face boundary operator ${}^{c}Q_{Dn}$ reads as

$$ {}^{c}Q_{Dn} = \sum_{0 \leq i \leq n} (-1)^i \sum_{\sigma_n \in {}^{c}X_n,\, d_{ni}\sigma_n \in {}^{c}X_{n-1}} |d_{ni}\sigma_n\rangle\langle\sigma_n|\big|_{{}^{c}\mathscr{H}_n} \tag{87} $$

The adjoint ${}^{c}Q_{Dn}{}^{+}$ of ${}^{c}Q_{Dn}$ is similarly given by

$$ {}^{c}Q_{Dn}{}^{+} = \sum_{0 \leq i \leq n} (-1)^i \sum_{\sigma_{n-1} \in {}^{c}X_{n-1}} \sum_{\omega_n \in D_{ni}(\sigma_{n-1}) \cap {}^{c}X_n} |\omega_n\rangle\langle\sigma_{n-1}|\big|_{{}^{c}\mathscr{H}_{n-1}} \tag{88} $$

Finally, the normalized simplicial Hilbert Laplacian ${}^{c}H_{DDn}$ takes the form

$$ \begin{aligned} {}^{c}H_{DDn} = & \sum_{0 \leq i,j \leq n} (-1)^{i+j} \sum_{\sigma_n \in {}^{c}X_n,\, d_{nj}\sigma_n \in {}^{c}X_{n-1}} \sum_{\omega_n \in D_{ni}(d_{nj}\sigma_n) \cap {}^{c}X_n} |\omega_n\rangle\langle\sigma_n|\big|_{{}^{c}\mathscr{H}_n} \\ & + \sum_{0 \leq i,j \leq n+1} (-1)^{i+j} \sum_{\sigma_n \in {}^{c}X_n} \sum_{\omega_{n+1} \in D_{n+1j}(\sigma_n) \cap {}^{c}X_n,\, d_{n+1i}\omega_{n+1} \in {}^{c}X_n} |d_{n+1i}\omega_{n+1}\rangle\langle\sigma_n|\big|_{{}^{c}\mathscr{H}_n} \end{aligned} \tag{89} $$

These expressions follow by straightforward calculations from relations (27), (29) and (44) and the identity $\mathit{\Pi}_n = \sum_{\sigma_n \in {}^{c}X_n} |\sigma_n\rangle\langle\sigma_n|$.

3.6. Simplicial quantum circuits

In this section, we introduce the notion of simplicial quantum circuit, a special kind of quantum circuit naturally emerging in the quantum simplicial set-up and capable in principle of performing meaningful simplicial computations.

Our treatment will be admittedly idealized. We shall assume, in fact, not very realistically, that no ancilla registers and no intermediate measurements are involved.

We consider again a parafinite simplicial set X and the associated simplicial Hilbert space $\mathscr{H}$.

The simplicial quantum register of X is a pre-Hilbert space $\mathscr{H}^{(\infty)}$ that stores all the simplicial data of X in the same way as a quantum register is a Hilbert space $\mathbb{C}^{2\otimes n}$ that stores all the configurations of a classical n bit string. Mathematically, $\mathscr{H}^{(\infty)}$ is the infinite dimensional pre–Hilbert space

$$\mathscr{H}^{(\infty)} = \bigoplus_{0\leq n<\infty} \mathscr{H}_n \tag{90}$$

where the direct summation is algebraic.

Before proceeding further, it is important to emphasize the purely algebraic nature of the simplicial quantum register. $\mathscr{H}^{(\infty)}$ is indeed a formal device we introduce in order to study simplicial quantum circuits without having to bother about the fact that the storage capability of a computer is large but finite. In fact any finite computation is carried out within a finite dimensional subspace $\mathscr{H}^{(N)}$ of $\mathscr{H}^{(\infty)}$ for some large $N \in \mathbb{N}$ (cf. Equation (111)). Using $\mathscr{H}^{(\infty)}$ avoids putting an upper bound on the size of the computation, that is the values of N. This simplifies the analysis to a considerable extent. In Subsections 4.1 and 4.2 we shall tackle the problem of the finite storage capability of a simplicial quantum computer by resorting to the skeletonization/truncation of the underlying simplicial Hilbert space $\mathscr{H}$ to simplicial degree N. This results in a corresponding reduction of the pre Hilbert space $\mathscr{H}^{(\infty)}$ to the finite dimensional subspace $\mathscr{H}^{(N)}$.

The completion $\overline{\mathscr{H}}^{(\infty)}$ of $\mathscr{H}^{(\infty)}$ could be considered. $\overline{\mathscr{H}}^{(\infty)}$ is an infinite dimensional Hilbert space. It is separable since it is the completed direct sum of the denumerable family of the finite dimensional Hilbert spaces $\mathscr{H}_n$. An orthonormal basis is obtained by taking the union of the simplex bases $|\sigma_n\rangle$ of the $\mathscr{H}_n$ for all n. Note however that switching from $\mathscr{H}^{(\infty)}$ to $\overline{\mathscr{H}}^{(\infty)}$ is neither necessary nor useful as finite computations take place in subspaces $\mathscr{H}^{(N)}$ of $\mathscr{H}^{(\infty)}$ of arbitrarily large but finite dimension. Functional analytic issues such as convergence of infinite series and domain problems of unbounded operators indeed never arise. Everything can be described in terms of finite dimensional linear algebra, though in varying dimensions.

A simplicial quantum circuit is a quantum circuit based on the register $\mathscr{H}^{(\infty)}$, whose functioning is compatible with the structure of the underlying simplicial set X. Mathematically, so, a simplicial quantum circuit is a unitary operator $U \in \mathsf{U}(\mathscr{H}^{(\infty)})$ that satisfies certain simplicial conditions.

Definition 23. *A simple simplicial quantum circuit is a collection of unitary operators $U_n \in \mathsf{U}(\mathscr{H}_n)$ with $n \in \mathbb{N}$ such that*

$$U_{n-1}D_{ni} = D_{ni}U_n \qquad \textit{for } 0 \leq i \leq n \tag{91}$$

$$U_{n+1}S_{ni} = S_{ni}U_n \qquad \textit{for } 0 \leq i \leq n \tag{92}$$

In the language of simplicial Hilbert theory, a simple simplicial quantum circuit

$\{U_n\}$ is therefore a simplicial unitary operator of the simplicial Hilbert space $\mathscr{H}$ (cf. Equations (42)). Intuitively, for each n the operator U_n embodies a quantum circuit implementing a reversible computation involving the simplices of X_n. For the above notion to be really meaningful, the computations performed for the various values of n should have the same simplicial nature and be compatible with the simplicial face and degeneracy relations occurring between the underlying simplicial data. These properties are precisely codified by relations (91) and (92).

A simple quantum circuit $\{U_n\}$ can be thought of as a whole collection of simplicial quantum gates of the form

$$U^{(n)} = U_n \oplus \bigoplus_{0 \le n' < \infty, n' \neq n} 1_{n'} \tag{93}$$

The unitary operator $U \in \mathsf{U}(\mathscr{H}^{(\infty)})$ corresponding to the circuit is

$$U = \prod_{0 \le n < \infty} U^{(n)} = \bigoplus_{0 \le n < \infty} U_n \tag{94}$$

We notice that simple simplicial quantum circuits form a group $\mathsf{U}(\mathscr{H})$ under simplicial degreewise multiplication and inversion.

Simple simplicial quantum circuits can perform only computations at fixed simplicial degree, an important limitation. We need more general circuits for more general computations. The simplicial conditions that a general simplicial quantum circuit obeys should be an appropriate generalization of those obeyed by simple circuits. To formulate it, we need to introduce an appropriate notation.

For a finite subset $A \subset \mathbb{N}$ with $A \neq \emptyset$, the simplicial A-subregister is the finite dimensional Hilbert space

$$\mathscr{H}_A = \bigoplus_{n \in A} \mathscr{H}_n \subset \mathscr{H}^{(\infty)} \tag{95}$$

For a finite subset $A \subset \mathbb{N}$ with $A \neq \emptyset$, we let F_A be the set of all mappings $\alpha : A \to \Sigma$ with the property that $\alpha_0 = +1$ when $0 \in A$, where $\Sigma = \{-1, +1\}$ is the sign alphabet. We also set $\mathbb{N}_n = \{n' | n' \in \mathbb{N}, 0 \le n' \le n\}$, where $n \in \mathbb{N}$. For $\alpha \in F_A$ and $i \in \prod_{n \in A} \mathbb{N}_n$, we define the operator ${X^{(\alpha)}}_{Ai} : \mathscr{H}_A \to \mathscr{H}_{A+\alpha}$ by

$${X^{(\alpha)}}_{Ai} = \bigoplus_{n \in A} {X^{(\alpha_n)}}_{ni_n} \tag{96}$$

where ${X^{(-1)}}_{ni} = D_{ni}$, ${X^{(+1)}}_{ni} = S_{ni}$ and $A + \alpha = \{n + \alpha_n | n \in A\} \subset \mathbb{N}$.

Definition 24. *Let $p \in \mathbb{N}$, $p > 0$. A p-ary simplicial quantum circuit consists of a collection of unitary operators $U_A \in \mathsf{U}(\mathscr{H}_A)$ with $A \subset \mathbb{N}$ and $|A| = p$ such that for all $\alpha \in F_A$ and $i \in \prod_{n \in A} \mathbb{N}_n$*

$${X^{(\alpha)}}_{Ai} U_A = U_{A+\alpha} {X^{(\alpha)}}_{Ai} \tag{97}$$

The simple simplicial quantum circuits introduced in Definition 23 are just 1-ary simplicial quantum circuits.

A p-ary quantum circuit $\{U_A\}$ encodes a family of simplicial quantum gates,

$$U^{(A)} = U_A \oplus \bigoplus_{n \notin A} 1_n \tag{98}$$

Unlike in the simple case, these gates generally do not commute since the subspaces $\mathscr{H}_A$ may have non trivial intersections. The unitary operator $U \in \mathsf{U}(\mathscr{H}^{(\infty)})$ of the circuit is gotten by multiplying some subset of simplicial gates in a prescribed order.

If $\{U_n\}$ is a simple simplicial quantum circuit, the operators $U_A = \bigoplus_{n\in A} U_n$, $A \subset \mathbb{N}$ and $|A| = p$, constitute a p-ary simplicial quantum circuit. More generally, given a collection $\{U_{\alpha A_\alpha}\}$ of p_α-ary simplicial quantum circuits, $\alpha = 1, \ldots, a$, one can construct a p-ary simplicial quantum circuit $\{U_A\}$ with $p = \sum_\alpha p_\alpha$ as follows. Every subset $A \subset \mathbb{N}$ with $|A| = p$ has a unique partition $A = \bigcup_\alpha A_\alpha$ such that $A_\alpha \subset \mathbb{N}$ with $|A_\alpha| = p_\alpha$ and that for every $\alpha < \beta$, $m \in A_\alpha$, $n \in A_\beta$ one has $m < n$. Then, $U_A = \bigoplus_\alpha U_{\alpha A_\alpha}$.

Definition 25. *A p-ary simplicial quantum circuit* $\{U_A\}$ *of the kind constructed above is called reducible.*

They are so because they have a fixed non trivial block diagonal structure on each simplicial A-subregister.

We present now a template for generating interesting examples of simple simplicial quantum circuits. The data of the construction are the following:

1) a pair of parafinite simplicial sets X, X';
2) a simplicial morphism $\phi : X \to X'$;
3) a structure of simplicial group on X'.

The role played by the simplicial group structure of X' (cf. Example 8) is essential.

We now turn to the Cartesian product $X \times X'$ of X, X' (cf. Definition 3). By means of the components ϕ_n of ϕ, we define maps $\hat{\phi}_n : X \times X'_n \to X \times X'_n$ by setting

$$\hat{\phi}_n(\sigma_n, \sigma'_n) = (\sigma_n, \sigma'_n \phi_n(\sigma_n)) \tag{99}$$

The second component of the pair on the right hand side exhibits the product of the simplices σ'_n, $\phi_n(\sigma_n) \in X'_n$ in the group X'_n. Unlike the ϕ_n, the $\hat{\phi}_n$ are always invertible, as they are injective and the sets $X \times X'_n$ are finite. The $\hat{\phi}_n$ constitute indeed a reversible form of the ϕ_n with a structure analogous to that of similar maps employed in reversible computation.

Since X' is a simplicial group, its face and degeneracy maps d'_{ni}, s'_{ni} are group morphisms. Furthermore, as ϕ is a simplicial morphism, its components ϕ_n satisfy the simplicial relations (4). Exploiting these properties, it is straightforward to check that the maps $\hat{\phi}_n$ obey (4) as well and so are the components of a simplicial morphism $\hat{\phi} : X \times X' \to X \times X'$. Being the $\hat{\phi}_n$ invertible, $\hat{\phi}$ is in fact an isomorphism.

We define next operators $\hat{U}_{\phi n} : \mathscr{H} \otimes \mathscr{H}'_n \to \mathscr{H} \otimes \mathscr{H}'_n$ by

$$\hat{U}_{\phi n} = \sum_{(\sigma_n, \sigma'_n) \in X_n \times X'_n} |\hat{\phi}_n(\sigma_n, \sigma'_n)\rangle\langle(\sigma_n, \sigma'_n)| \tag{100}$$

where the kets $|(\sigma_n, \sigma'_n)\rangle = |\sigma_n\rangle \otimes |\sigma'_n\rangle$ are the n-simplex basis of $\mathscr{H} \otimes \mathscr{H}'_n$.

Proposition 4. *The operators $\hat{U}_{\phi n}$, $n \in \mathbb{N}$, constitute a simple simplicial quantum circuit $\hat{U}_\phi$ of $X \times X'$.*

Proof. The $\hat{U}_{\phi n}$ are unitary operators since the maps $\hat{\phi}_n$ are invertible. Using that $\hat{\phi}$ is a simplicial morphism, it is now straightforward to verify that the $\hat{U}_{\phi n}$ obey relations (91), (92) and are therefore the component of a simplicial quantum circuit $\hat{U}_\phi$ of $X \times X'$. □

Indeed, the $\hat{U}_{\phi n}$ are just the components of the simplicial Hilbert automorphism $\hat{U}_\phi : \mathscr{H} \otimes \mathscr{H}' \to \mathscr{H} \otimes \mathscr{H}'$ associated with the simplicial isomorphism $\hat{\phi}$.

The eventual relevance of the construction just outlined for useful applications is still to be clarified because of its very special nature. The example derived by it which we illustrated next is anyway worthy of mentioning.

Example 15. *Simplicial quantum circuits of a simplicial group.*

Let X be a simplicial group. Since the simplex sets X_n are groups, multiplication and inversion maps $\mu_n : X_n \times X_n \to X_n$ and $\iota_n : X_n \to X_n$ are defined at each degree n. The property of the face and degeneracy maps d_{ni}, s_{ni} as group morphisms entails that the μ_n and ι_n are the components of simplicial morphisms $\mu : X \times X \to X$, and $\iota : X \to X$. Application of the construction scheme illustrated above shows that with these there are associated simplicial quantum circuits U_μ and U_ι of $X \times X \times X$ and $X \times X$, respectively.

The challenge facing one presently is constructing irreducible simplicial quantum circuits beyond the simple ones and more broadly to devise a classification scheme of such circuits reflecting basic properties of simplicial set theory.

3.7. Simplicial quantum circuits and homology

In this final subsection, we shall examine the interplay of simplicial quantum circuits studied in Section 3.6 and simplicial Hilbert homology as analyzed in Subsections 3.4 and 3.5. Our discussion will be limited to simple simplicial quantum circuits (cf. Definition 23), its extension to p-ary simplicial quantum circuits (cf. Definition 24) being straightforward.

Owing to Equations (44) and (91), the components U_n of a simple simplicial quantum circuit also intertwine the simplicial Hilbert face coboundary operators Q_{Dn},

$$U_{n-1} Q_{Dn} = Q_{Dn} U_n \tag{101}$$

and hence define by Equation (66) a unitary chain endomorphism of the simplicial Hilbert face chain complex $(\mathscr{H}, Q_D)$ studied in Subsection 3.4. For each degree n, so, there exists an automorphism of the homology space $\mathrm{H}_{Dn}(\mathscr{H})$ associated with U_n and hence, by the isomorphism (68), also one of the simplicial homology $\mathrm{H}_n(X, \mathbb{C})$. Such automorphism can be concretely described as follows. Relations (46) and (101) entail that simplicial Hilbert Laplacian H_{DDn} commutes with the U_n,

$$H_{DDn} U_n = U_n H_{DDn} \tag{102}$$

By virtue of this relation it holds that $U_n \ker H_{DDn} \subseteq \ker H_{DDn}$. Under the isomorphisms $\mathrm{H}_{Dn}(\mathscr{H}) \simeq \ker H_{DDn}$ of Equation (67), the resulting action of U_n on $\ker H_{DDn}$ represents the aforementioned homology automorphism.

Simple simplicial quantum circuits can be analyzed in a similar fashion also in the normalized simplicial Hilbert homological framework elaborated in Subsection 3.5, which as we have seen is the one best suited for homology computation. Let us see this in some detail. A straightforward verification using (60) and (92) shows that for each n the operator U_n commutes with the projectors Π_{ni}. By Equation (77), U_n commutes then also with the projector Π_n,

$$U_n \Pi_n = \Pi_n U_n \tag{103}$$

The degenerate n-simplex space ${}^s\mathscr{H}_n$ in Equation (75) is so invariant under U_n, as Π_n projects on ${}^s\mathscr{H}_n$. Its orthogonal complement ${}^c\mathscr{H}_n$ is then invariant too, by the unitarity of U_n, and the restriction ${}^cU_n$ of U_n to ${}^c\mathscr{H}_n$ is a unitary operator of ${}^c\mathscr{H}_n$. By Equation (101), the normalized boundary operator ${}^cQ_{Dn}$ defined in Equation (78) satisfies then

$${}^cU_{n-1}{}^cQ_{Dn} = {}^cQ_{Dn}{}^cU_n \tag{104}$$

Relation (104), analogously to Equation (101), shows that the circuit defines a unitary chain endomorphism of the normalized simplicial Hilbert face chain complex $({}^c\mathscr{H}, {}^cQ_D)$ and so each circuit component U_n yields an automorphism of the normalized homology space $\mathrm{H}_{Dn}({}^c\mathscr{H})$ and consequently, by virtue of the isomorphism (76) again, one of the simplicial homology $\mathrm{H}_n(X, \mathbb{C})$. The automorphism can be described similarly to the unnormalized case. From Equations (84) and (104), the normalized simplicial Hilbert Laplacian ${}^cH_{DDn}$ obeys the relations

$${}^cH_{DDn}{}^cU_n = {}^cU_n{}^cH_{DDn} \tag{105}$$

analogous to Equation (102). Proceeding as done earlier, one finds that the homology automorphism is realized as an action of ${}^cU_n$ on $\ker {}^cH_{DDn}$.

3.8. Expected extensions of the quantum simplicial framework

In this concluding subsection, we discuss a few possible ramifications of the quantum simplicial framework we have developed above.

In computational topology, one aims at the computation of interesting topological invariants of a relevant topological space, in particular this latter's homology. This requires a simplicial model of the topological space. By its nature the model is not unique. While the results the model provides cannot depend on its choice, the computational effort required to obtain those results does. This brings to the forefront the problem of reduction: given a simplicial model of a topological space, derive from it an equivalent simpler model that yields the same results at a lower computational cost. The natural question arises about whether reduction can be implemented in the quantum simplicial framework elaborated in this paper.

Our simplicial set-up involves a fixed parafinite simplicial set as an input datum, that is a collection of simplices sorted according to their degrees together with a set of face and degeneracy maps. By contrast, a reduction algorithm changes this datum. A possible way of circumventing the problem arising here would be working within a large 'ambient' simplicial set, in which the given simplicial set and its reductions are contained as subsets. It remains to be seen whether this is indeed feasible and the resource cost of a set-up of this kind does not offset the computational advantage of reduction.

We have characterized a simplicial quantum circuit as a quantum circuit based on the simplicial quantum register whose operation is compatible with the structure of the underlying simplicial set. Mathematically, a simplicial quantum circuit is a collection of unitary operators satisfying the simplicial conditions stated in Definition 23 in the simple case and in Definition 24 in the general case. Any simplicial quantum circuit performing an ordinary simplicial computation should satisfy the conditions listed in those definitions. However, we ask, is any simplicial quantum circuit associated with one such calculation?

The problem posed in the previous paragraph is in a sense one of 'reverse engineering'. We cannot provide a solution presently. We speculate however that a negative answer would show the existence of new 'quantum' computations in algebraic topology. Let us clarify this point. A classical computation in algebraic topology is any computation that can be performed by a simplicial classical computer at least in principle. All standard computations in algebraic topology, such as homology and homotopy computations, are classical in this sense. A quantum computation in algebraic topology is any computation that can be carried out by a simplicial quantum computer but not by a simplicial classical computer. Therefore, a quantum computation, if such a thing exists at all, should be something substantially novel in algebraic topology, whose import for this discipline would have to be explored.

4. Implementation of the quantum simplicial set framework

In this section, relying on the quantum simplicial framework worked out in Section 3, we examine the problems that may arise in the implementation of simplicial set theoretic topological algorithms in a quantum computer taking into account its finite storage capabilities. The results found are only preliminary and the issue will require further closer examination in future work. We present no new algorithms and limit ourselves to formulating some necessary conditions for their working.

The issues analyzed below range from truncation and skeletonization of a parafinite simplicial set to its digital encoding by simplex counting and parametrizing. The way such operations are implemented in the quantum simplicial framework is then investigated. We also outline mainly for illustrative purposes an algorithmic scheme combining a number of basic quantum algorithms capable of computing the complex simplicial homology spaces and Betti numbers of the simplicial set along the lines of that of Ref. [17].

4.1. Truncating and skeletonizing simplicial sets and Hilbert spaces

Finite simplicial complexes have a finite total number of simplices. By contrast, parafinite simplicial sets always have an infinite total number of simplices. Even the non degenerate simplices may be infinitely many.

Algorithms implemented on computers can process only a finite amount of input data. Therefore, any algorithms of computational topology cannot be grounded on a simplicial set containing infinitely many simplices but only on an approximation of it including only finitely many of them. This is achieved by setting a cut-off on simplicial degree so that only simplices of degree not exceeding the cut-off are kept. This approach goes under the name of simplicial truncation. Next, we describe this approach in more formal terms.

Fix a cut-off integer $N \in \mathbb{N}$.

Definition 26. *An N-truncated simplicial set X consists of a collection of sets X_n, $0 \leq n \leq N$, and mappings $d_{ni} : X_n \to X_{n-1}$, $1 \leq n \leq N$, $i = 1, \ldots, n$, and $s_{ni} : X_n \to X_{n+1}$, $0 \leq n \leq N-1$, $i = 1, \ldots, n$, obeying the simplicial relations* (2) *when defined.*

Comparing Definitions 1 and 26, we realize that an N-truncated simplicial set is much like a simplicial set except for the existence of an upper bound N to simplicial degree.

Definition 27. *A morphism $\phi : X \to X'$ of N-truncated simplicial sets consists in a collection of maps $\phi_n : X_n \to X'_n$ with $0 \leq n \leq N$ obeying the simplicial morphism relations* (4) *when defined.*

Again, inspection of Definitions 2 and 27 reveals that the simplicial morphisms of N-truncated simplicial sets are the truncated analog of the simplicial morphisms of simplicial sets. The operations of Cartesian product and disjoint union extend also to N-truncated simplicial sets. N-truncated simplicial sets form a bimonoidal category $\underline{\mathrm{sSet}}_N$ analogous to the bimonoidal category $\underline{\mathrm{sSet}}$ of simplicial sets. The homology of N-truncated simplicial sets is defined in the usual manner though it is limited to degree $n \leq N-1$.

There exists an obvious truncation functor $\mathrm{tr}_N : \underline{\mathrm{sSet}} \to \underline{\mathrm{sSet}}_N$ that discards all the simplices of degree $n > N$ of the simplicial sets on which it acts. It can be shown that tr_N admits a left adjoint functor $\mathrm{lk}_N : \underline{\mathrm{sSet}}_N \to \underline{\mathrm{sSet}}$, its so-called left Kan extension [62]. The resulting composite functor $\mathrm{sk}_N = \mathrm{lk}_N \circ \mathrm{tr}_N : \underline{\mathrm{sSet}} \to \underline{\mathrm{sSet}}$ goes under the name of N-skeleton functor and can be characterized as follows. If $X \in \underline{\mathrm{sSet}}$ is a simplicial set, then $\mathrm{sk}_N X$ is the smallest simplicial subset of X such that $\mathrm{sk}_N X_n = X_n$ for $n \leq N$ and $\mathrm{sk}_N X_n \subseteq {}^s X_n$ for $n > N$, where ${}^s X_n \subseteq X_n$ is the subset of degenerate n-simplices (cf. Subsection 2.1, Definition 6). In this way, $\mathrm{sk}_N X$ reproduces X in degree $n \leq N$ whilst in degree $n > N$ it keeps only certain degenerate simplices of X.

Definition 28. *A simplicial set X is called N-skeletal if $X = \mathrm{sk}_N X^*$ for some simplicial set X^*.*

The following property of simplicial homology is now fairly evident.

Proposition 5. *Let G be an Abelian group. Then, it holds that in degree $n \leq N-1$* $\mathrm{H}_n(X, \mathsf{G}) \simeq \mathrm{H}_n(\mathrm{tr}_N X, \mathsf{G}) \simeq \mathrm{H}_n(\mathrm{sk}_N X, \mathsf{G})$.

Therefore, working with the N-truncation or N-skeletonization of a simplicial set X rather than with X itself still allows one to recover the homology of X up to degree $N-1$ inclusive.

The above discussion can be extended in an evident fashion to simplicial objects, in particular to simplicial Hilbert spaces, which are our main focus.

An N-truncated simplicial Hilbert space $\mathscr{H}$ is a collection of Hilbert spaces $\mathscr{H}_n$, $0 \leq n \leq N$, and operators $D_{ni} : \mathscr{H}_n \to \mathscr{H}_{n-1}$, $1 \leq n \leq N$, $i = 1, \ldots, n$, and $S_{ni} : \mathscr{H}_n \to \mathscr{H}_{n+1}$, $0 \leq n \leq N-1$, $i = 1, \ldots, n$, which constitute an N-truncated simplicial set when these are regarded as sets and maps of sets.

A morphism $\Phi : \mathscr{H} \to \mathscr{H}'$ of N-truncated simplicial Hilbert spaces consists of a collection of linear operators $\Phi_n : \mathscr{H}_n \to \mathscr{H}'_n$ with $0 \leq n \leq N$, which constitute a map of N-truncated simplicial sets when they are regarded as maps of sets.

The N-truncation and N-skeletonization functors can be built also for simplicial objects and in particular for simplicial Hilbert spaces. Hence, with any simplicial Hilbert space $\mathscr{H}$ we can associate its truncation $\mathrm{tr}_N\,\mathscr{H}$ and N-skeleton $\mathrm{sk}_N\,\mathscr{H}$, which have the property that $\mathrm{tr}_N\,\mathscr{H}_n = \mathrm{sk}_N\,\mathscr{H}_n = \mathscr{H}_n$ for $n \leq N$ and $\mathrm{sk}_N\,\mathscr{H}_n \subseteq {}^s\mathscr{H}_n$ for $n > N$, ${}^s\mathscr{H}_n$ being the degenerate n-simplex subspace (cf. Definition 21). A simplicial Hilbert space is N-skeletal when $\mathscr{H} = \mathrm{sk}_N\,\mathscr{H}^*$ for some simplicial Hilbert space $\mathscr{H}^*$.

In computational topology, setting a cut-off N on the simplicial degree of the relevant parafinite simplicial set X is tantamount to replacing X by its N-truncation $\mathrm{tr}_N\,X$. $\mathrm{tr}_N\,X$, however, belongs to the category of N-truncated simplicial sets, which is related to but distinct from the category of simplicial sets. To remain within this latter while essentially keeping the essence of the truncation operation, one needs to consider, instead that $\mathrm{tr}_N\,X$, the N-skeleton $\mathrm{sk}\,X_N$ of X. Both the truncation $\mathrm{tr}_N\,X$ and the skeleton $\mathrm{sk}\,X_N$ may be viewed as an approximation of X in the appropriate sense. In practice, one works with $\mathrm{tr}_N\,X$. In more formal considerations, dealing with $\mathrm{sk}\,X_N$ may be more natural, since it allows using the analysis carried out in Section 3 simply by restricting to N-skeletal simplicial sets.

In the quantum simplicial set framework of Subsections 3.1 and 3.2, to each parafinite simplicial set X there corresponds a simplicial Hilbert space $\mathscr{H}$. The simplicial Hilbert encoding map, which defines the simplex basis, is a simplicial set morphism $\varkappa : X \to \mathscr{H}$. The N-truncation functor so yields a map $\mathrm{tr}_N\,\varkappa : \mathrm{tr}_N\,X \to \mathrm{tr}_N\,\mathscr{H}$ of N-truncated simplicial sets with components $\mathrm{tr}_N\,\varkappa_n = \varkappa_n$ for $0 \leq n \leq N$. Similarly, the N-skeletonization functor yields a map $\mathrm{sk}_N\,\varkappa : \mathrm{sk}_N\,X \to \mathrm{sk}_N\,\mathscr{H}$ of N-skeletal simplicial sets with components $\mathrm{sk}_N\,\varkappa_n = \varkappa_n$ for $0 \leq n \leq N$. Therefore, the operations of N-truncation and N-skeletonization of parafinite simplicial sets turn under simplicial Hilbert encoding into the corresponding operations of the associated simplicial Hilbert spaces.

4.2. Simplicial digital encoding and quantum simplicial set framework

The digital encoding of the simplices of a given parafinite simplicial set is a precondition for the implementation of simplicial set based algorithms of computational topology in a quantum computer. This matter is analyzed in detail in the present subsection.

Consider a parafinite simplicial set X and a truncation $\mathrm{tr}_N\,X$ of it. The full simplex set of the truncation is

$$X^{(N)} = \bigsqcup_{0 \leq n \leq N} X_n \tag{106}$$

To encode the simplices of $X^{(N)}$, one needs a k-bit register with $k \geq \kappa_{XN}$, where

$$\kappa_{XN} = \min \left\{ l \,\middle|\, l \in \mathbb{N}, |X^{(N)}| \leq 2^l \right\} \tag{107}$$

We shall show next how the encoding creates a digitized image of the whole simplicial structure of $\mathrm{tr}_N X$ in the register.

Let $B_2 = \{0, 1\}$ be the digital Boolean domain.

Definition 29. *A digital encoding of* $\mathrm{tr}_N X$ *in a length* k *register is a bijective map* $\chi : X^{(N)} \to X_\chi{}^{(N)}$*, where* $X_\chi{}^{(N)} \subseteq B_2{}^k$ *is a k-bit string set with* $|X_\chi{}^{(N)}| = |X^{(N)}|$.

The images via χ of the simplex sets X_n are the subsets $X_{\chi n} := \chi(X_n) \subseteq X_\chi{}^{(N)}$. They constitute a partition of $X_\chi{}^{(N)}$, so that

$$X_\chi{}^{(N)} = \bigsqcup_{0 \leq n \leq N} X_{\chi n} \tag{108}$$

From here, it is promptly verified that the register contains a full digital image of $\mathrm{tr}_N X$. The restrictions $\chi\big|_{X_n}$ of χ to the X_n induce bijective maps $\chi_n : X_n \to X_{\chi n}$, through which further maps $d_{\chi ni} : X_{\chi n} \to X_{\chi n-1}$, $1 \leq n \leq N$, $i = 1, \ldots, n$, and $s_{\chi ni} : X_{\chi n} \to X_{\chi n+1}$, $0 \leq n \leq N-1$, $i = 1, \ldots, n$, can be defined by

$$d_{\chi ni} = \chi_{n-1} d_{ni} {\chi_n}^{-1} \tag{109}$$

$$s_{\chi ni} = \chi_{n+1} s_{ni} {\chi_n}^{-1} \tag{110}$$

These $d_{\chi ni}$, $s_{\chi ni}$ obey the simplicial relations (2) as a consequence of d_{ni}, s_{ni} doing so. The sets $X_{\chi n}$ and the maps d_{ni}, s_{ni} are so the simplex sets and the face and degeneracy maps of an N-truncated simplicial set X_χ. Moreover, the maps χ_n are the components of an N-truncated simplicial set isomorphism $\chi : \mathrm{tr}_N X \to X_\chi$.

Let $X_0 \subseteq B_2{}^k$ be a k-bit string set such that $|X_0| = |X^{(N)}|$. If χ_0 is a reference encoding of $\mathrm{tr}_N X$ with $X_{\chi_0}{}^{(N)} = X_0$ and π is any permutation of X_0, then $\chi = \pi\chi_0$ is an encoding of $\mathrm{tr}_N X$ with $X_\chi{}^{(N)} = X_0$ too. Furthermore, each encoding χ with $X_\chi{}^{(N)} = X_0$ is of this form for precisely one permutation π, viz $\pi = \chi{\chi_0}^{-1}$. Therefore, there are altogether $|X^{(N)}|!$ encodings with a given range X_0.

We now shall examine in detail how a digital encoding χ of the N-truncation of a parafinite simplicial set X in a k-bit register is implemented in the quantum simplicial set framework of Subsections 3.1 and 3.2. In Subsection 4.1, we saw that when X is replaced by its N-truncation $\mathrm{tr}_N X$, the simplicial Hilbert space of $\mathscr{H}$ of X gets replaced by its N-truncation $\mathrm{tr}_N \mathscr{H}$. The simplicial quantum register of $X^{(N)}$ is

$$\mathscr{H}^{(N)} = \bigoplus_{0 \leq n \leq N} \mathscr{H}_n \tag{111}$$

By Equation (106), $\dim \mathscr{H}^{(N)} = |X^{(N)}|$, because $\dim \mathscr{H}_n = |X_n|$. In the Hilbert set-up, the k-bit register turns into a k-qubit register with quantum Hilbert space $\mathscr{Q}^{\otimes k}$, where $\mathscr{Q} = \mathbb{C}^2$. The encoding χ yields a linear operator $U_\chi : \mathscr{H}^{(N)} \to \mathscr{H}_\chi{}^{(N)}$, where $\mathscr{H}_\chi{}^{(N)} \subseteq \mathscr{Q}^{\otimes k}$ is a k-bit string space of dimension $\dim \mathscr{H}_\chi{}^{(N)} = |X^{(N)}|$. Explicitly

$$U_\chi = \sum_{0 \leq n \leq N} \sum_{\sigma_n \in X_n} |\chi\sigma_n\rangle_k \langle \sigma_n| \tag{112}$$

where the kets $|\xi\rangle_k$, $\xi \in B_2{}^k$, are those of the computational basis of $\mathscr{Q}^{\otimes k}$. U_χ is evidently unitary. U_χ generates an image of the direct sum structure of $\mathscr{H}^{(N)}$ in $\mathscr{H}_\chi{}^{(N)}$: setting $\mathscr{H}_{\chi n} = U_\chi \mathscr{H}_n$, one has indeed

$$\mathscr{H}_\chi{}^{(N)} = \bigoplus_{0 \le n \le N} \mathscr{H}_{\chi n} \tag{113}$$

The restrictions $U_\chi\big|_{\mathscr{H}_n}$ of U_χ to the $\mathscr{H}_n$ induce unitary operators $U_{\chi n} : \mathscr{H}_n \to \mathscr{H}_{\chi n}$ through which further operators $D_{\chi ni} : \mathscr{H}_{\chi n} \to \mathscr{H}_{\chi n-1}$, $1 \le n \le N$, $i = 1, \ldots, n$, and $S_{\chi ni} : \mathscr{H}_{\chi n} \to \mathscr{H}_{\chi n+1}$, $0 \le n \le N-1$, $i = 1, \ldots, n$, can be constructed by

$$D_{\chi ni} = U_{\chi n-1} D_{ni} U_{\chi n}{}^{-1} \tag{114}$$

$$S_{\chi ni} = U_{\chi n+1} S_{ni} U_{\chi n}{}^{-1} \tag{115}$$

$D_{\chi ni}$, $S_{\chi ni}$ obey the simplicial identities (33) as a consequence of D_{ni}, S_{ni} doing so. In fact, in the computational basis of $\mathscr{Q}^{\otimes k}$ these operators are given by expressions analogous to Equations (27) and (28), viz

$$D_{\chi ni} = \sum_{\xi_n \in X_{\chi n}} |d_{\chi ni}\xi_n\rangle_{k\,k}\langle \xi_n| \tag{116}$$

$$S_{\chi ni} = \sum_{\xi_n \in X_{\chi n}} |s_{\chi ni}\xi_n\rangle_{k\,k}\langle \xi_n| \tag{117}$$

The Hilbert spaces $\mathscr{H}_{\chi n}$ and the operators $D_{\chi ni}$, $S_{\chi ni}$ are thus the simplex spaces and the face and degeneracy operators of an N-truncated simplicial Hilbert space $\mathscr{H}_\chi$. Moreover, the maps $U_{\chi n}$ are the components of an N-truncated simplicial Hilbert space unitary operator $U_\chi : \mathrm{tr}_N \mathscr{H} \to \mathscr{H}_\chi$.

The above analysis aims only to show the possibility of creating through a digital encoding χ a digitized image of the truncation $\mathrm{tr}_N X$ of a parafinite simplicial set X and providing a precise formal characterization of such an image and similarly for the corresponding truncation $\mathrm{tr}_N \mathscr{H}$ of the associated simplicial Hilbert space $\mathscr{H}$. Depending on the specific features of X, there may be special instances of the encoding χ with high efficiency and distinctive formal properties. In particular, χ should be selected judiciously in such a way as to render the face and degeneracy maps $d_{\chi ni}$, $s_{\chi ni}$ as simple as possible. There is no general prescription for doing that and χ must be chosen on a case-by-case basis. By contrast, in the simplicial complex framework of Ref. [17], there is a canonical encoding of the simplices of the relevant simplicial complex in terms of which the boundary maps have a simple form.

4.3. Counting and parametrizing simplices

Subsection 4.2 provides a theoretical analysis of the digital encoding of a truncation of a simplicial set. Explicit construction of an encoding requires however further scrutiny of this matter, which we do in this subsection.

Let X be a parafinite simplicial set. In Subsection 2.1, for each n we considered the subset ${}^sX_n \subseteq X_n$ of degenerate simplices of X_n. ${}^cX_n = X_n \setminus {}^sX_n \subseteq X_n$ is hence the subset of non degenerate simplices of X_n.

The following theorem is an important structural property of simplicial sets.

Theorem 10. *(Eilenberg-Zilber lemma [41]) For every n, each simplex $\sigma_n \in X_n$ has a unique representation $\sigma_n = s_{n-1j_{n-m-1}} \cdots s_{mj_0}\tau_m$, where $m \leq n$, $\tau_m \in {}^cX_n$ and $0 \leq j_0 < \ldots < j_{n-m-1} \leq n-1$.*

When σ_n is non degenerate the degeneracy map string $s_{n-1j_{n-m-1}} \cdots s_{mj_0}$ is empty.

By the Eilenberg-Zilber lemma, we have

$$X_n \simeq \bigcup_{0 \leq m \leq n} J^n{}_m \times {}^cX_m \tag{118}$$

where for $m \leq n$ $J^n{}_m = \{(j_{n-m-1}, \ldots, j_0) | 0 \leq j_0 < \ldots < j_{n-m-1} \leq n-1\}$ is the set of index strings of height $n-1$ and length $n-m$. Note that $J^n{}_n = \{\emptyset\}$, where $\emptyset$ denotes the empty index string. It is a simple combinatorial exercise to show that $|J^n{}_m| = \binom{n}{m}$. By Equation (118), so, the number $|X_n|$ of n-simplices can be expressed in terms of the numbers $|{}^cX_m|$ of non degenerate m-simplices with $m \leq n$ as

$$|X_n| = \sum_{0 \leq m \leq n} \binom{n}{m} |{}^cX_m| \tag{119}$$

The total to non degenerate n-simplex ratio

$$\varrho_{Xn} = |X_n| / |{}^cX_n| \tag{120}$$

is an important indicator of the incidence of degenerate n-simplices. While $\varrho_{X0} = 1$, ϱ_{Xn} as a rule grows very rapidly as n gets large. The total number of simplices of an N-truncation of X reads from Equation (119) as

$$|X^{(N)}| = \sum_{0 \leq n \leq N} |X_n| = \sum_{0 \leq m \leq N} \binom{N+1}{m+1} |{}^cX_m| \tag{121}$$

(cf. Equation (106)). The content of the non degenerate n-simplex sets ${}^cX_m$ depends on the underlying simplicial set X. The above numerical measures of the simplex distribution and the size of a truncation of a simplicial set can consequently be evaluated only on a case-by-case basis.

The following simple examples serve as an illustration of the general techniques we described above.

Example 16. *The nerve of the delooping of a finite group.*

Recall that a group G can be viewed as a one-object groupoid BG, the delooping of G.

Consider the nerve $N\mathsf{BG}$ of the delooping BG of a finite group G (cf. Example 2). Then, $N_n\mathsf{BG} = \mathsf{G}^n$. The non degenerate n-simplices of $N\mathsf{BG}$ are precisely the n-tuples of G^n that do not contain the identity of G. Therefore,

$$|{}^cN_n\mathsf{BG}| = (|\mathsf{G}| - 1)^n \tag{122}$$

The number of non degenerate n-simplices such that $n \leq N$ is consequently

$$\sum_{0 \leq n \leq N} |^c N_n \mathrm{BG}| = \frac{(|\mathsf{G}| - 1)^{N+1} - 1}{|\mathsf{G}| - 2} \tag{123}$$

(when $|\mathsf{G}| = 2$, this takes the value $N + 1$). Inserting (122) *in the general Equation* (119)*, we recover the known number of n-simplices of $N\mathrm{BG}$*

$$|N_n \mathrm{BG}| = \sum_{0 \leq m \leq n} \binom{n}{m} |^c N_m \mathrm{BG}| = |\mathsf{G}|^n \tag{124}$$

The total to non degenerate n-simplex ratio of $N\mathrm{BG}$ is thus

$$\varrho_{N\mathrm{BG}n} = |N_n \mathrm{BG}| / |^c N_n \mathrm{BG}| = \left(\frac{|\mathsf{G}|}{|\mathsf{G}| - 1} \right)^n \tag{125}$$

This grows exponentially with n, but the larger $|\mathsf{G}|$ is the slower this growth is.

The total number of simplices of the N-truncation $\mathrm{tr}_N\, N\mathrm{BG}$ of $N\mathrm{BG}$ reads as

$$|N^{(N)} \mathrm{BG}| = \frac{|\mathsf{G}|^{N+1} - 1}{|\mathsf{G}| - 1} \tag{126}$$

(when $|\mathsf{G}| = 1$, this takes the value $N+1$). The encoding of $\mathrm{tr}_N\, N\mathrm{BG}$ requires therefore a k-bit register with $k \geq \varkappa_{N\mathrm{BG}N}$, where $\varkappa_{N\mathrm{BG}N} = \log_2 |N^{(N)}\mathrm{BG}|$. We note that $\varkappa_{N\mathrm{BG}N} = N \log_2 |\mathsf{G}| + O(1/|\mathsf{G}|)$ when $|\mathsf{G}|$ is large.

Example 17. *The simplicial set of a finite ordered discrete simplicial complex.*

Let $V = \{v_0, \ldots, v_d\}$ be a finite non empty set. Let $\mathscr{P}_V$ be the discrete simplicial complex of V, that is the simplicial complex whose vertex set $\mathrm{Vert}_{\mathscr{P}_V}$ is V and whose simplex set $\mathrm{Simp}_{\mathscr{P}_V}$ is the power set of V. We assume that V is endowed with a total ordering so that $v_a < v_b$ for $a < b$. $\mathscr{P}_V$ is so an ordered simplicial complex.

Consider the simplicial set $K\mathscr{P}_V$ associated with the complex $\mathscr{P}_V$ (cf. Example 3). The n-simplices of $\mathscr{P}_V$ constitute precisely the non degenerate n-simplices of $K\mathscr{P}_V$. The number of n-simplices of $\mathscr{P}_V$ is $\binom{d+1}{n+1}$ for $n \leq d$. This indicates us also the number of the non degenerate n-simplices of $K\mathscr{P}_V$

$$|^c K_n \mathscr{P}_V| = \binom{d+1}{n+1} \quad \textit{for } n \leq d, \qquad 0 \quad \textit{for } n > d \tag{127}$$

The number of non degenerate n-simplices such that $n \leq N$ with $N \leq d$ is found from here to be given by the expression

$$\sum_{0 \leq n \leq N} |^c K_n \mathscr{P}_V| = 2^{d+1} - 1 - \binom{d+1}{N+2} {}_2F_1(1, -d + N + 1; N + 3; -1) \tag{128}$$

The total number of non degenerate simplices is $2^{d+1} - 1$. Inserting (127) *in the general Equation* (119)*, we obtain the number of n-simplices of $K\mathscr{P}_V$ for $n \leq d$*

$$|K_n \mathscr{P}_V| = \sum_{0 \leq m \leq n} \binom{n}{m} |^c K_m \mathscr{P}_V| = \binom{d+n+1}{n+1} \tag{129}$$

The total to non degenerate n-simplex ratio of $K\mathscr{P}_V$ for $n \leq d$ is thus

$$\varrho_{K\mathscr{P}_V n} = |K_n\mathscr{P}_V|/|^{c}K_n\mathscr{P}_V| = \binom{d+n+1}{n+1}\Big/\binom{d+1}{n+1} \tag{130}$$

$\varrho_{K\mathscr{P}_V n}$ has the following expansions:

$$\begin{aligned}\varrho_{K\mathscr{P}_V n} &= 1 + O(n^2/d) \quad \textit{for } 1 \ll n \ll d^{1/2}, \\ &\frac{2^{2d+1}}{(\pi d)^{1/2}}[1 + O(d^{-1}, (n-d)\log_2 d)] \quad \textit{for } 1 \ll n \to d\end{aligned} \tag{131}$$

So, while for $n \ll d^{1/2}$ the numbers of degenerate and non degenerate n-simplices are comparable, for $n \sim d$ the number of degenerate simplices is exponentially greater than that of the non degenerate ones.

The total number of simplices of the N-truncation of $\mathrm{tr}_N\, K\mathscr{P}_V$ of $K\mathscr{P}_V$ is

$$|K^{(N)}\mathscr{P}_V| = \binom{d+N+2}{d+1} - 1 \tag{132}$$

provided $N \leq d$. The encoding of $\mathrm{tr}_N\, K\mathscr{P}_V$ requires therefore a k-bit register with $k \geq \varkappa_{K\mathscr{P}_V N}$, where $\varkappa_{K\mathscr{P}_V N} = \log_2 |K^{(N)}\mathscr{P}_V|$. $\varkappa_{K\mathscr{P}_V N}$ has the expansion

$$\begin{aligned}\varkappa_{K\mathscr{P}_V N} =& \log_2\left[\left(\frac{\mathrm{e}d}{N}\right)^N \frac{d}{(2\pi)^{1/2}N^{3/2}}\right] + O(1/N, N^2/d) \quad \textit{for } 1 \ll N \ll d^{1/2}, \\ & 2d + 2 - \tfrac{1}{2}\log_2(\pi d) + O(d^{-1}, (N-d)\log_2 d) \quad \textit{for } 1 \ll N \to d\end{aligned} \tag{133}$$

So, one needs a $2(d+1)$-bit register to encode all the simplex data in degree less than d of the simplicial set $K\mathscr{P}_V$. This is to be compared with the $(d+1)$-bit register required to encode the simplex data of the underlying simplicial complex $\mathscr{P}_V$ [17].

If $\mathcal{S}$ is a finite simplicial complex, then $\mathcal{S}$ is a subcomplex of the discrete simplicial complex $\mathscr{P}_V$ with $V = \mathrm{Vert}_{\mathcal{S}}$. The values of such quantities as $|^{c}K_n\mathscr{P}_V|$, $|K_n\mathscr{P}_V|$, ... computed above for $K\mathscr{P}_V$ are then upper bounds for the values of the corresponding quantities of $|^{c}K_n\mathcal{S}|$, $|K_n\mathcal{S}|$, ... of $K\mathcal{S}$.

Once the number of simplices of the chosen truncation $\mathrm{tr}_N\, X$ of a simplicial set X is ascertained, it becomes possible to construct a digital encoding of it (cf. Definition 29) by allocating a suitably sized register. The choice of the encoding is not unique. The following examples of encoding are presented here as an illustration of the theory. No claim is made that they constitute the optimal choice with regard to efficiency.

Example 18. *The nerve of the delooping of a finite group.*

Consider again the nerve $N\mathsf{BG}$ of the delooping BG of a finite group G (cf. Example 16). The simplices of a truncation $\mathrm{tr}_N\, N\mathsf{BG}$ of $N\mathsf{BG}$ can be digitally encoded in an $Nq+r$-bit register with q, r integers such that $q \geq \log_2 |\mathsf{G}|$ and $r \geq \log_2(N+1)$ as follows. We represent an $Nq+r$-bit string as $(x_1, \ldots, x_N; y)$, where the x_a are q-bit strings and y is an r-bit string. For convenience, we use the enumerative indexation of such bit strings, in terms of which the x_a are represented by integers in the range 0 to $2^q - 1$ and y as an integer in the range 0 to $2^r - 1$. Further, we select a digital encoding of G, i.e., a bijective map $\varphi : \mathsf{G} \to P_\varphi$ with $P_\varphi \subseteq B_2{}^q$, which we normalize *by requiring that* $\varphi(e) = 0$*, where* e *denotes the neutral element of* G*. Using these elements, we can construct a digital encoding* χ *of* $\mathrm{tr}_N\, \mathsf{N}\mathrm{B}\mathsf{G}$*. This is the bijective map* $\chi : \mathsf{N}^{(N)}\mathrm{B}\mathsf{G} \to \mathsf{N}_{\chi}{}^{(N)}\mathrm{B}\mathsf{G}$ *defined as follows. The encoding range is*

$$\mathsf{N}_{\chi}{}^{(N)}\mathrm{B}\mathsf{G} = \bigsqcup_{0\le n\le N} P_{\varphi}{}^{n} \times \{0\}^{N-n} \times \{n\} \subseteq B_2{}^{Nq+r} \tag{134}$$

Further, for $\sigma_n = (g_1, \ldots, g_n) \in \mathsf{N}_n\mathrm{B}\mathsf{G}$ *with* $n \le N$*,*

$$\chi(\sigma_n) = (0, \ldots, 0; 0) \quad \textit{for } n = 0 \tag{135}$$

$$\chi(\sigma_n) = (\varphi(g_1), \ldots, \varphi(g_n), 0, \ldots, 0; n) \quad \textit{for } 0 < n \le N \tag{136}$$

The encoding's face and degeneracy maps $d_{\chi n i}$*,* $s_{\chi n i}$ *(cf. Equations* (109) *and* (110)*) take the following form. We notice preliminarily that* $\mathsf{N}_{\chi n}\mathrm{B}\mathsf{G} = P_{\varphi}{}^{n} \times \{0\}^{N-n} \times \{n\}$*. Let* $(x_1, \ldots, x_n, 0, \ldots, 0; n) \in \mathsf{N}_{\chi n}\mathrm{B}\mathsf{G}$ *with* $n \le N$*. Then, if* $1 \le n$

$$d_{\chi n0}(x_1, \ldots, x_n, 0, \ldots, 0; n) = (x_2, \ldots, x_n, 0, \ldots, 0; n-1) \tag{137a}$$

$$\begin{aligned} &d_{\chi ni}(x_1, \ldots, x_n, 0, \ldots, 0; n) \\ &\quad = (x_1, \ldots, x_{i-1}, \varphi(\varphi^{-1}(x_i)\varphi^{-1}(x_{i+1})), x_{i+2}, \ldots, x_n, 0, \ldots, 0; n-1) \quad \textit{if } 0 < i < n, \end{aligned} \tag{137b}$$

$$d_{\chi nn}(x_1, \ldots, x_n, 0, \ldots, 0; n) = (x_1, \ldots, x_{n-1}, 0, \ldots, 0; n-1) \tag{137c}$$

Similarly, if $n \le N - 1$

$$s_{\chi ni}(x_1, \ldots, x_n, 0, \ldots, 0; n) = (x_1, \ldots, x_i, 0, x_{i+1}, \ldots, x_n, 0, \ldots, 0; n+1) \tag{138}$$

Example 19. *The simplicial set of a finite ordered discrete simplicial complex.*

Consider the simplicial set $\mathsf{K}\mathscr{P}_V$ *associated with the ordered simplicial complex* $\mathscr{P}_V$ *studied earlier (cf. Example 17). The simplices of a truncation* $\mathrm{tr}_N\, \mathsf{K}\mathscr{P}_V$ *of* $\mathsf{K}\mathscr{P}_V$ *can be digitally encoded in a* $(d+1)r$*-bit register with* r *being an integer such that* $r \ge \log_2(N+2)$ *as follows. We represent a* $(d+1)r$*-bit string as* $(x_0, \ldots, x_d)$*, where the* x_a *are* r*-bit strings, and employ again the enumerative indexation of the* r*-bit strings representing them as integers in the range* 0 *to* $2^r - 1$*. For each index* $0 \le a \le d$*, let* $\varphi_a : \mathsf{K}^{(\infty)}\mathscr{P}_V \to \mathbb{N}$*, where* $\mathsf{K}^{(\infty)}\mathscr{P}_V = \bigsqcup_{0\le n} \mathsf{K}_n\mathscr{P}_V$*, be the* a*-th vertex counting map: if* $\sigma_n \in \mathsf{K}_n\mathscr{P}_V$*, then* $\varphi_a(\sigma_n)$ *is the number of occurrences of the vertex* v_a *in* σ_n*. We note that for* $\sigma_n \in \mathsf{K}_n\mathscr{P}_V$ *the integers* $\varphi_a(\sigma_n)$ *must obey the sum rule* $\sum_{0\le a\le d} \varphi_a(\sigma_n) = n+1$*. Employing these elements, we can construct a digital encoding* χ *of* $\mathrm{tr}_N\, \mathsf{K}\mathscr{P}_V$*. This is the bijective map* $\chi : \mathsf{K}^{(N)}\mathscr{P}_V \to \mathsf{K}_{\chi}{}^{(N)}\mathscr{P}_V$ *defined as follows. The range of the encoding is*

$$\mathsf{K}_{\chi}{}^{(N)}\mathscr{P}_V = \left\{ (x_0, \ldots, x_d) \middle| 0 \le x_a \le N+1, 0 < \sum_{0\le a\le d} x_a \le N+1 \right\} \tag{139}$$

Further, for $\sigma_n \in \mathsf{K}_n\mathscr{P}_V$ *with* $n \le N$*, we have*

$$\chi(\sigma_n) = (\varphi_0(\sigma_n), \ldots, \varphi_d(\sigma_n)) \tag{140}$$

The face and degeneracy maps $d_{\chi ni}$*,* $s_{\chi ni}$ *of the encoding (cf. Equations* (109) *and*

(110)) *read as follows. We notice that* $K_{\chi n}\mathscr{P}_V = \{(x_0,\dots,x_d) | \sum_{0\le a\le d} x_a = n+1\}$. *For* $(x_0,\dots,x_d) \in \mathbb{N}^{d+1}$, *set*

$$\vartheta_{ai}(x_0,\dots,x_d) = \begin{cases} 1 & \text{if } \sum_{0\le b<a} x_b \le i < \sum_{0\le b\le a} x_b \\ 0 & \text{else.} \end{cases} \tag{141}$$

Let $(x_0,\dots,x_d) \in K_{\chi n}\mathscr{P}_V$ *with* $n \le N$. *Then, for* $1 \le n$

$$d_{\chi ni}(x_0,\dots,x_d) = (x_0 - \vartheta_{0i}(x_0,\dots,x_d),\dots,x_d - \vartheta_{di}(x_0,\dots,x_d)) \tag{142}$$

while for $n \le N-1$

$$s_{\chi ni}(x_0,\dots,x_d) = (x_0 + \vartheta_{0i}(x_0,\dots,x_d),\dots,x_d + \vartheta_{di}(x_0,\dots,x_d)) \tag{143}$$

4.4. Homology computation via quantum algorithms

In Subsection 3.5, we showed that the complex coefficient homology spaces of a parafinite simplicial set X, $\mathrm{H}_n(X,\mathbb{C})$, can be computed based on the isomorphism of these latter and the normalized simplicial Hilbert homology spaces of the associated simplicial Hilbert space $\mathscr{H}$, $\mathrm{H}_{Dn}({}^c\mathscr{H})$ (cf. Theorem 8). In turn, such spaces are given by $\mathrm{H}_{Dn}({}^c\mathscr{H}) \simeq \ker {}^cH_{DDn}$, where ${}^cH_{DDn}$ is the normalized simplicial Hilbert Laplacian expressible through the simplicial Hilbert boundary operators ${}^cQ_{Dn}$, ${}^cQ_{Dn+1}$ and their adjoints (cf. Definition 22 and Theorem 9). The problem we have to tackle next is the determination of $\ker {}^cH_{DDn}$ in an N-truncated setting as required by the implementation of simplicial set based quantum algorithms. This will be done along the lines of Ref. [17] by exploiting at two distinct stages two well-known quantum algorithms:

1) Grover multisolution quantum search algorithm [63];
2) Abrams and Lloyd's quantum algorithm for the solution of the general eigenvalue problem [64], a refinement of Kitaev's basic quantum phase estimation algorithm [65].

The implementation of such algorithms relies on a number of other algorithms, as will be explained in due course. The total complexity of the whole algorithm is determined by the combined complexity of all the algorithms that compose it.

To study normalized Hilbert homology in an N-truncated setting, we have to consider the $N+1$-tuple of equations ${}^cH_{DDNn}|\psi_n\rangle = 0$ with $|\psi_n\rangle \in {}^c\mathscr{H}_n$, where

$${}^cH_{DDNn} = {}^cH_{DDn} - \delta_{Nn}{}^cQ_{DN+1}{}^cQ_{DN+1}{}^+ \tag{144}$$

Note that ${}^cH_{DDNn} = {}^cH_{DDn}$ for $0 \le n < N$ only while ${}^cH_{DDNN} \ne {}^cH_{DDN}$. The operator ${}^cH_{DDNN}$ rather than ${}^cH_{DDN}$ appears here, as the operators ${}^cQ_{DN+1}$, ${}^cQ_{DN+1}{}^+$ are excluded by the truncation. Such operators enter only the expression of ${}^cH_{DDN}$ (cf. Equation (84) and their contribution is duly subtracted out.

We saw in Subsection 4.2 that the N-truncation of the simplicial Hilbert space $\mathscr{H}$ gives rise to the simplicial quantum register $\mathscr{H}^{(N)}$ shown in Equation (111). The normalized simplicial Hilbert homology rests on the subspace ${}^c\mathscr{H}^{(N)}$ of $\mathscr{H}^{(N)}$ spanned by the non degenerate n-simplex subspaces ${}^{c}\mathscr{H}_n$ of the $\mathscr{H}_n$ with $0 \leq n \leq N$,

$$ {}^{c}\mathscr{H}^{(N)} = \bigoplus_{0 \leq n \leq N} {}^{c}\mathscr{H}_n \tag{145} $$

By systematically exploiting the canonical projections $R_n : {}^{c}\mathscr{H}^{(N)} \to {}^{c}\mathscr{H}_n$ and injections $I_n : {}^{c}\mathscr{H}_n \to {}^{c}\mathscr{H}^{(N)}$, one can view the vector spaces ${}^{c}\mathscr{H}_n$ as subspaces of the space ${}^{c}\mathscr{H}^{(N)}$ and similarly the operator spaces $\mathrm{Hom}({}^{c}\mathscr{H}_n, {}^{c}\mathscr{H}_m)$ as subspaces of the space $\mathrm{End}({}^{c}\mathscr{H}^{(N)})$. In what follows we shall thoroughly adhere to this perspective. Accordingly, we shall use the same notation to denote vectors of ${}^{c}\mathscr{H}_n$ and operators of $\mathrm{Hom}({}^{c}\mathscr{H}_n, {}^{c}\mathscr{H}_m)$, e.g. $|{}^{c}\psi_n\rangle$, $|{}^{c}\phi_n\rangle$, …and ${}^{c}A_{n,m}$, ${}^{c}B_{n,m}$, …, irrespective of whether ${}^{c}\mathscr{H}_n$ and $\mathrm{Hom}({}^{c}\mathscr{H}_n, {}^{c}\mathscr{H}_m)$ are considered as vector spaces in their own or as subspaces of ${}^{c}\mathscr{H}^{(N)}$ and $\mathrm{End}({}^{c}\mathscr{H}^{(N)})$, respectively[2].

The quantum algorithm computing the normalized Hilbert homologies $\mathrm{H}_{Dn}({}^{c}\mathscr{H})$ requires as a preliminary step the implementation of the projection of $\mathscr{H}^{(N)}$ onto ${}^{c}\mathscr{H}^{(N)}$. This operation is done parallelly within each subspace $\mathscr{H}_n$ with $0 \leq n \leq N$ by the orthogonal projection operator $1_n - \mathit{\Pi}_n$, where $\mathit{\Pi}_n$ is the orthogonal projector of $\mathscr{H}_n$ onto ${}^{s}\mathscr{H}_n = {}^{c}\mathscr{H}_n{}^{\perp}$ given in Equation (77). That operator however cannot be a component of any quantum circuit as such, as it is not unitary. Luckily, the projection can be achieved in the appropriate form compatible with unitarity using Grover's quantum search algorithm [63] in the variant based on amplitude amplification [66]. We illustrate briefly how the algorithm works within $\mathscr{H}_n$. The quantum computer is initialized in a state $|\xi_{0n}\rangle \in \mathscr{H}_n$

$$ |\xi_{0n}\rangle = \sum_{\sigma_n \in X_n} |\sigma_n\rangle |X_n|^{-1/2} \tag{146} $$

that is a uniform superposition of all n-simplex states $|\sigma_n\rangle$. Through the algorithm, the state $|\xi_{0n}\rangle$ evolves unitarily toward the final state

$$ |{}^{c}\xi_{0n}\rangle = (1_n - \mathit{\Pi}_n)|\xi_{0n}\rangle(|X_n|/|{}^{c}X_n|)^{1/2} = \sum_{\sigma_n \in {}^{c}X_n} |\sigma_n\rangle |{}^{c}X_n|^{-1/2} \tag{147} $$

which constitutes a uniform superposition of all non degenerate n-simplex states $|\sigma_n\rangle$.

The algorithm comprises two stages: i) the preparation of the state $|\xi_{0n}\rangle$ and ii) the production of the state $|{}^{c}\xi_{0n}\rangle$ from $|\xi_{0n}\rangle$. These stages contribute additively to the algorithm's complexity. In stage i, the state $|\xi_{0n}\rangle$ is yielded by the action of some unitary operator W_n on a fiducial reference state $|o_n\rangle$, so that $|\xi_{0n}\rangle = W_n|o_n\rangle$. In stage ii, the algorithm is implemented through the action on the state $|\xi_{0n}\rangle$ of a unitary operator ${G_n}^{p_n}$ that is the p_n-th power of an elementary unitary operator G_n, showing the algorithm's iterative nature. The Grover operator G_n is of the form $G_n = -W_n D_{0n} {W_n}^{+} D_n$, where D_{0n}, D_n are unitary operators. $D_{0n} = 1_n - 2|o_n\rangle\langle o_n|$ is the conditional sign flip operator of the reference state $|o_n\rangle$. D_n is the conditional sign flip operator of the non degenerate simplex states $|\sigma_n\rangle$ and is oracular in nature. The Grover iteration number p_n reads as $p_n = \left[\frac{\pi}{4}(|\mathrm{x}_n|/|{}^{c}\mathrm{x}_n|)^{1/2}\right]$.

The overall complexity C_n of the above Grover type state preparation algorithm is given by $C_n = C(W_n) + p_n(2C(W_n) + C(D_{0n}) + C(D_n))$, where $C(W_n)$, $C(D_{0n})$,

$C(D_n)$ are the complexities of the unitaries W_n, D_{0n}, D_n. The values of $C(W_n)$, $C(D_n)$ depend on the underlying simplicial set X. Further, the values of $C(W_n)$, $C(D_{0n})$ depend on the reference state $|o_n\rangle$ chosen. $C(W_n)$, $C(D_{0n})$, $C(D_n)$ may also depend on the digital encoding of the truncation $\mathrm{tr}_N X$ used. The performance of the algorithm hinges in particular on that of the oracle D_n. Note that the iteration number p_n depends on the total to non degenerate n-simplex ratio $\varrho_{Xn} = |X_n|/|{}^cX_n|$ introduced in Subsection 4.3 (cf. Equation (120)). As ϱ_{Xn} typically grows very rapidly with n, the algorithm fails when n is large enough, if a maximal number of iterations is allowed. ϱ_{Xn} is unknown in general. In such a case, it must be determined previously using a quantum counting algorithm [67], which is an instance of a phase estimation algorithm [65] using the Grover operator G_n as the underlying unitary operator, since the eigenvalues $\mathrm{e}^{\pm i\theta_n}$ of G_n are given by $\sin(\theta_n/2) = \varrho_{Xn}{}^{-1/2}$.

The effective realization of the projection of $\mathscr{H}^{(N)}$ onto ${}^c\mathscr{H}^{(N)}$ renders possible the implementation of operators on ${}^c\mathscr{H}^{(N)}$ such as ${}^cQ_{DN+1}$, ${}^cQ_{DN+1}{}^+$ and ${}^cH_{DDNN}$ (cf. Equations (78) and (84)) and many more as constitutive elements of a quantum homological algorithm. Henceforth, so, we work on ${}^c\mathscr{H}^{(N)}$.

By the isomorphism (85) and (144), $\mathrm{H}_{Dn}({}^c\mathscr{H}) = \ker {}^cH_{DDNn}$ for $0 \leq n < N$. To compute the normalized simplicial Hilbert homology spaces $\mathrm{H}_{Dn}({}^c\mathscr{H})$, we thus have to solve the N equations ${}^cH_{DDNn}|{}^c\psi_n\rangle = 0$ with $|{}^c\psi_n\rangle \in {}^c\mathscr{H}_n$. To that end, let us introduce the following linear operator ${}^cH_{DD}{}^{(N)}$ of ${}^c\mathscr{H}^{(N)}$:

$$ {}^cH_{DD}{}^{(N)} = \sum_{0\leq n\leq N} {}^cH_{DDNn} \tag{148} $$

Then, the equations mentioned earlier are equivalent to a single equation, namely ${}^cH_{DD}{}^{(N)}|\psi^{(N)}\rangle = 0$, since ${}^cH_{DD}{}^{(N)}|\psi^{(N)}\rangle = \sum_{0\leq n\leq N} {}^cH_{DDNn}|\psi_n\rangle$. In this way, the calculation of the homology spaces $\mathrm{H}_{Dn}({}^c\mathscr{H})$ in degree $n < N$ is reduced to that of the kernel of $\ker {}^cH_{DD}{}^{(N)}$.

The determination of $\ker {}^cH_{DD}{}^{(N)}$ by the quantum phase estimation methods of Ref. [64] involves the unitary operators $\exp(i\tau {}^cH_{DD}{}^{(N)})$ for varying τ. The construction of this requires the use of a suitable Hamiltonian simulation algorithm [68]. For the sake of computational efficiency, it is convenient when possible to replace ${}^cH_{DD}{}^{(N)}$ with a Hermitian operator ${}^cB^{(N)}$ sparser than ${}^cH_{DD}{}^{(N)}$ such that $\ker {}^cB^{(N)} = \ker {}^cH_{DD}{}^{(N)}$ and construct $\exp(i\tau {}^cB^{(N)})$ instead, since the complexity of the algorithm depends inversely on the sparsity of the exponentiated operator [69,70]. A standard choice of ${}^cB^{(N)}$ is provided by the Dirac operator ${}^cB_D{}^{(N)}$ of ${}^cH_{DD}{}^{(N)}$, which is the operator of ${}^c\mathscr{H}^{(N)}$ given by

$$ {}^cB_D{}^{(N)} = \sum_{0\leq n\leq N-1} \left({}^cQ_{Dn+1} + {}^cQ_{Dn+1}{}^+\right) \tag{149} $$

${}^cB_D{}^{(N)}$ is evidently Hermitian and satisfies

$$ {}^cB_D{}^{(N)2} = {}^cH_{DD}{}^{(N)} \tag{150} $$

Consequently, $\ker {}^cH_{DD}{}^{(N)} = \ker {}^cB_D{}^{(N)}$ as required.

The Hamiltonian simulation algorithm constructing $\exp(i\tau {}^cB_D{}^{(N)})$ involves an oracle unitary operator O_{DN} providing the non zero matrix elements of ${}^{c}B_{D}{}^{(N)}$. The algorithm's total complexity is $C_{D}{}^{(N)}(\tau) = G(\tau) + Y_{D}{}^{(N)}(\tau)C(O_{DN})$, where $G(\tau)$, $Y_{D}{}^{(N)}(\tau)$ and $C(O_{DN})$ are the algorithm's gate and query complexity and the complexity of O_{DN}, respectively. $G(\tau)$, $Y_{D}{}^{(N)}(\tau)$ depend, besides the value of the parameter τ, on the precision desired and the specific algorithm employed [71–76]. $Y_{D}{}^{(N)}(\tau)$ depends further on the sparseness index of ${}^{c}B_{D}{}^{(N)}$, measured as the number of non zero matrix elements of ${}^{c}B_{D}{}^{(N)}$ per row or column, and the maximum magnitude of the matrix elements. Finally, $C(O_{DN})$ may depend through O_{DN} on the underlying simplicial set X and the digital encoding of the truncation $\mathrm{tr}_{N}\,X$ used.

We review briefly how the algorithm of Ref. [64] is implemented to find homology spaces $\mathrm{H}_{Dn}({}^{c}\mathscr{H})$ in our framework. The algorithm proceeds by determining the eigenvalues and eigenvectors of the unitary operators $\exp(i\tau{}^{c}B_{D}{}^{(N)})$ introduced above using quantum phase estimation. It operates specifically with the density operators. For each n with $0 \leq n \leq N$, let ${}^{c}\rho_{0n}$ be the uniform mixture of all non degenerate n-simplex states $|\sigma_{n}\rangle\langle\sigma_{n}|$. Explicitly, ${}^{c}\rho_{0n}$ reads as

$$ {}^{c}\rho_{0n} = |{}^{c}X_{n}|^{-1}{}^{c}1_{n} = \sum_{\sigma_{n}\in{}^{c}X_{n}} |\sigma_{n}\rangle|{}^{c}X_{n}|^{-1}\langle\sigma_{n}| \tag{151} $$

${}^{c}\rho_{0n}$ can be straightforwardly obtained from the state $|{}^{c}\xi_{0n}\rangle$ of Equation (147) constructed earlier by adding an ancilla, copying the simplex data into the ancilla to construct the state $\sum_{\sigma_{n}\in{}^{c}X_{n}} |\sigma_{n}\rangle \otimes |\sigma_{n}\rangle|{}^{c}X_{n}|^{-1/2}$ and then tracing the ancilla out [17]. One adjoins next to the 'vector' register ${}^{c}\mathscr{H}^{(N)}$ a b_{t}-bit 'clock' register $\mathscr{Q}^{\otimes b_{t}}$ with b_{t} suitably large, so that the total Hilbert space is $\mathscr{Q}^{\otimes b_{t}} \otimes {}^{c}\mathscr{H}^{(N)}$. The quantum computer is initialized in the mixed state

$$ {}^{tc}\rho_{0n} = |0\rangle_{t\,t}\langle 0| \otimes {}^{c}\rho_{0n} \tag{152} $$

where $|\lambda\rangle_{t}$, $0 \leq \lambda \leq 2^{b_{t}} - 1$, is the computational basis of $\mathscr{Q}^{\otimes b_{t}}$ (in the enumerative parametrization). The algorithm evolves unitarily the state ${}^{tc}\rho_{0n}$ and ends with a measurement of the clock register. Upon reiteration of the algorithm, the relevant clock value $\lambda = 0$ is found with probability $\dim\ker{}^{c}H_{DDNn}/|{}^{c}X_{n}|$. After $\lambda = 0$ is obtained, the computer is in the mixed state

$$ {}^{tc}\rho_{DDn} = |0\rangle_{t\,t}\langle 0| \otimes {}^{c}\rho_{DDn} \tag{153} $$

where ${}^{c}\rho_{DDn}$ is the density operator of ${}^{c}\mathscr{H}^{(N)}$ reading as

$$ {}^{c}\rho_{DDn} = \dim\ker{}^{c}H_{DDNn}{}^{-1c}P_{DDNn} \tag{154} $$

${}^{c}P_{DDNn}$ denoting the orthogonal projection operator of ${}^{c}\mathscr{H}_{n}$ onto $\ker{}^{c}H_{DDNn}$. By the isomorphism $\ker{}^{c}H_{DDn} \simeq \mathrm{H}_{Dn}({}^{c}\mathscr{H})$, the final state of the quantum computer for $\lambda = 0$ encodes the homology space $\mathrm{H}_{Dn}({}^{c}\mathscr{H})$. Further, the frequency with which $\lambda = 0$ occurs furnishes the Betti numbers $\beta_{n}(X, \mathbb{C}) = \dim\ker{}^{c}H_{DDNn}$ (cf. Equation (26)).

In the algorithm described in the previous paragraph, the value of b_{t} depends on the number of bits and the precision desired for the estimation of the eigenvalues of ${}^{c}B_D{}^{(N)}$. The algorithm involves the use of b_t-bit Welsh–Hadamard and quantum Fourier transforms with combined complexity $C_{WHT}(b_t) + C_{QFT}(b_t) = O(b_t{}^2)$ and one call of an oracle unitary operator U_{DNj} computing $\exp(i2^{j}{}^{c}B_D{}^{(N)})$ for each j with $0 \leq j \leq b_t - 1$. If $C(U_{DNj})$ is the complexity of U_{DNj}, the total complexity of the algorithm is $C_{QPE}{}^{(N)} = C_{WHT}(b_t) + C_{QFT}(b_t) + \sum_{0 \leq j \leq b_t - 1} C(U_{DNj})$. The values of the $C(U_{DNj})$ depend on the Hamiltonian simulation algorithm employed.

We conclude this subsection with the following remark. The complexity analysis of the quantum algorithmic scheme for the computation of the homology of a parafinite simplicial set that we have presented above is only preliminary. Many of the contributions to the scheme's complexity depend on the simplicial set considered and on the digital encoding of its truncation used in a way whose precise theoretical understanding and quantitative estimation definitely require further investigation.

5. Open problems and outlook

In this paper, we have attempted to construct a theoretical model of a simplicial quantum computer (SQC). It is an open question whether a computer of this kind can be eventually built and employed in practice in computational topology. In this final section, we briefly review a number of issues that may arise in this respect[3], without advancing any claim for a complete solution.

5.1. Structural issues of the SQC model

The following objections concerning the overall structure of the SQC model can be raised.

i) The applicability of the model is limited to parafinite simplicial sets, restricting its potential usefulness for a wider range of topological problems.

ii) Associating a finite dimensional simplicial Hilbert space with a simplicial set and implementing the simplicial operator set-up is arduous in a quantum computational setting.

The reply to the first objection is provided already in the inception of Section 3, which we quote here. Extending the quantum simplicial framework to non parafinite simplicial sets is not feasible because of the intrinsic limitations of implementable computation: by the way they are defined (cf. Definition 5), parafinite simplicial sets are the most general kind of simplicial sets that can be handled by an SQC because any such device, regardless of the way it is conceived and built, necessarily can operate only on a finite number of simplicial data of each given degree. Indeed, the simplicial complexes routinely used in topological data analysis are all instances of parafinite simplicial sets.

The reply to the second objection is that configuring the simplicial Hilbert space and arranging its operator scheme in a quantum computational setting is evidently impossible because of the overall infinite dimensional nature of these entities! The simplicial framework is in fact only a conceptual construct devised to simplify and be useful for the analysis. It is its truncated form, introduced in Subsection 4.1, that by virtue of its finite dimensionality can be assembled at least in principle on an SQC, with an effort comparable to that required by other quantum computational set-ups. The crucial problem to be addressed here, in our judgement, is rather the implementability of simplicial quantum circuits, perhaps in a form less idealized than that presented in Subsection 3.6.

5.2. Practical realization of the SQC model

The SQC model is admittedly abstract: its viability depends to a significant extent on theoretical assumptions that may not hold in practice. This raises a number of issues concerning generically the implementation and the practical realization of the SQC model, as we list next.

i) An investigation of noise and decoherence in an SQC and their effect on the reliability of the computations performed by it should be carried out.

ii) An analysis of error rates occurring in an SQC is also required; methods of error corrections specific for an SQC should be developed.

iii) A detailed study of the limitations and feasibility of the quantum hardware required by the effective functioning of an SQC should be undertaken.

iv) Resource requirements for running algorithms on an SQC also have to be taken into due consideration.

All these questions are undoubtedly important, but their resolution lies beyond the reach of the analysis carried out in the present work, which is theoretical in nature. Anyhow, the eventual validation of the effective functioning of an SQC can ultimately be only empirical and experimental.

5.3. Relevance of the SQC model for computational topology

The eventual relevance of the SQC model for computational topology should also be appraised. In particular, it is important to gauge the potential impact of the SQC model on the development of dedicated quantum algorithms for computational topology and more broadly its implications for the future development and advancement of algebraic topology. In this respect, the following issues are especially relevant:

i) The potential advantages and disadvantages of using algorithms designed for implementation on an SQC should be appraised in depth.

ii) The robustness of algorithms devised for an SQC requires verification.

iii) A comparative analysis of the effectiveness of such algorithms and their classical counterparts is required to evaluate the merits of the SQC model.

iv) A comparison of the effectiveness of new SQC algorithms and existing ones of quantum topological data analysis is also due.

Further points concern specifically the algorithm for the computation of simplicial homology we have outlined in Subsection 4.4.

v) The scalability of SQC quantum algorithmic schemes should be studied to assess its efficiency for large simplicial data sets.

vi) The computational complexity of SQC quantum algorithmic schemes is to be analyzed.

We have provided only an elementary treatment of these topics. A more refined analysis is definitely required. This can only be left to future work for reasons explained next.

Quantum circuit complexity is a fundamental issue of quantum computation whose significance extends to other fields such as holography [77], quantum field theory [78] and topological quantum field theory [79]. In quantum computation proper, quantum circuit complexity measures the computational resources required to run a given quantum algorithm using a quantum circuit. It evidently depends on the details of the quantum algorithm implemented (see Ref. [80] for a general review of this topic). In the SQC model, as we have explained in Section 1, the greater flexibility of simplicial set theory affords in general several simplicial models of a given topological space. Each of these, in turn, may have a variety of digital encodings. All these possibilities affect in different ways the size of the simplex sets at hand, impinging on the complexity of quantum search algorithms, and on the sparsity properties and matrix element size of the relevant Hamiltonians, having so a bearing on the complexity of quantum Hamiltonian simulation algorithms, just to mention a few of the many issues involved. Since our analysis concerns mostly the general architecture of an SQC, there is little specific we can say about the complexity issue. Definite statements are possible only on a case-by-case basis and as a result of extensive research work.

The halting problem and the closely related decidability problem [81, 82] are fundamental questions. Computational topology involves decision problems such as the contractibility and the transformation problems in simplicial complexes, which are known to be related to the word and conjugacy problems in computational group theory. These have been shown to be unsolvable in classical computer science: no algorithm can be found. It is conceivable but also debatable that these problems may be solvable in quantum computer science. We refer the reader to [83] for a general analysis of this matter. The SQC model is unlikely to add new elements toward an answer.

5.4. Extensions and generalizations of the SQC model

We conclude this section by assessing prospective enhancements of the SQC model. It would be useful.

i) to study whether the SQC framework is generalizable to understand its broader ramifications;

ii) to explore potential applications of the SQC model beyond simplicial sets.

We have no answer to these questions presently. We only observe here that simplicial sets as well as cubical sets are instances of a more general type of combinatorial structure: the so called simploidal sets [84, 85]. Simploidal sets have so far found useful applications mostly in computational geometry, to the best of our knowledge. In principle, the SQC model might be generalized to simploidal sets. In practice, this can be confirmed only by a detailed analysis.

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Data availability statement: The data that support the findings of this study are openly available on scientific journals or in the arxiv repository.

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Notes

1 The author thanks U. Schreiber for suggesting this to him.

2 The notation $|n\rangle \otimes |{}^c\psi_n\rangle$, $|n\rangle \otimes |{}^c\phi_n\rangle$,…and $|n\rangle\langle m| \otimes {}^cA_{n,m}$, $|n\rangle\langle m| \otimes {}^cB_{n,m}$, …is instead more commonly used for vectors of ${}^c\mathscr{H}_n$ and operators of $\mathrm{Hom}({}^c\mathscr{H}_n, {}^c\mathscr{H}_m)$, when these latter are regarded as subspaces of ${}^c\mathscr{H}^{(N)}$ and $\mathrm{End}({}^c\mathscr{H}^{(N)})$ to keep track of simplicial degree, where $|n\rangle$ is the canonical basis of an auxiliary Hilbert space $\mathscr{A}^{(N)} \simeq \mathbb{C}^{N+1}$.

3 The author thanks the referees of the paper for raising some of these points.

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Appendix

The following appendixes provide the more technical details of the proofs of several propositions and theorems stated in the main text of the paper. They are required for the completeness of the analysis presented and are illustrative of the kind of techniques used.

I. The no degeneracy defect theorem

In this appendix, we provide a proof of the no degeneracy defect Theorem 2.

On account of Equation (35d), to demonstrate Equation (36), one has just to show that for $0 \leq i, j \leq n$, $i < j$, if $\sigma_n, \omega_n \in X_n$ and $s_{ni}\omega_n = s_{nj}\sigma_n$, then there exists a unique $\zeta_{n-1} \in X_{n-1}$ with $s_{n-1i}\zeta_{n-1} = \sigma_n$, $s_{n-1j-1}\zeta_{n-1} = \omega_n$. We begin with recalling that for any $\rho_n \in X_n$ the set $S_{n-1i}(\rho_n)$ is either empty or contains precisely one element, which by Equation (2c) is then $\eta_{n-1} = d_{ni}\rho_n = d_{ni+1}\rho_n$. By the simplicial identities Equations (2b)–(2d), we have $s_{ni}\omega_n = s_{nj}\sigma_n \Rightarrow d_{ni}\sigma_n \in S_{n-1j-1}(\omega_n) = \{d_{nj}\omega_n\}$ and $s_{ni}\omega_n = s_{nj}\sigma_n \Rightarrow d_{nj}\omega_n \in S_{n-1i}(\sigma_n) = \{d_{ni}\sigma_n\}$. Then, $\zeta_{n-1} = d_{nj}\omega_n = d_{ni}\sigma_n \in X_n$ is the unique element such that $s_{n-1i}\zeta_{n-1} = \sigma_n$, $s_{n-1j-1}\zeta_{n-1} = \omega_n$.

II. The perfectness propositions

We provide below the proofs of the perfectness Propositions 1, 2.

We suppose first that the simplicial set X under consideration is the nerve of a finite category $\mathcal{C}$, $X = N\mathcal{C}$ (cf. Subsection 2.1). We have to verify whether Equations (37)–(40) are fulfilled.

Below, we shall adopt the following convention: any sequence $f_r, f_{r+1} \ldots, f_{s-1}, f_s$ of morphisms or identities between morphisms of $\mathcal{C}$ displayed in the course of the analysis is tacitly assumed to be absent whenever $r > s$.

Because of Equation (35b), to verify Equation (37) it is enough to prove that for $0 \leq i, j \leq n$, $i \leq j$, if $\sigma_n \in N_n\mathcal{C}$, $\omega_{n+2} \in N_{n+2}\mathcal{C}$ and $d_{n+2i}\omega_{n+2} = s_{nj}\sigma_n$, then there is a unique $\zeta_{n+1} \in N_{n+1}\mathcal{C}$ such that $d_{n+1i}\zeta_{n+1} = \sigma_n$, $s_{n+1j+1}\zeta_{n+1} = \omega_{n+2}$. Write

$$\sigma_n = (g_1, \ldots, g_n), \qquad \omega_{n+2} = (h_1, \ldots, h_{n+2}) \tag{A1}$$

in terms of morphisms g_k, h_l of $\mathcal{C}$. The condition $d_{n+2i}\omega_{n+2} = s_{nj}\sigma_n$ implies that

$$\begin{aligned} &h_1 = g_1,\ \ldots,\ h_{i-1} = g_{i-1},\ h_{i+1} \circ h_i = g_i,\ h_{i+2} = g_{i+1}, \\ &\qquad \ldots,\ h_{j+1} = g_j,\ h_{j+2} = \mathrm{id},\ h_{j+3} = g_{j+1},\ \ldots,\ h_{n+2} = g_n \end{aligned} \tag{A2}$$

Using relations (A2), it is straightforward to check that the morphism string

$$\zeta_{n+1} = (h_1, \ldots, h_{i+1}, g_{i+1}, \ldots, g_n) = (h_1, \ldots, h_{j+1}, g_{j+1}, \ldots, g_n) \tag{A3}$$

is composable and so that $\in N_{n+1}\mathcal{C}$ and that $d_{n+1i}\zeta_{n+1} = \sigma_n$, $s_{n+1j+1}\zeta_{n+1} = \omega_{n+2}$. The uniqueness of ζ_{n+1} is guaranteed by the injectivity of s_{n+1j+1}.

From Equation (35c), to demonstrate Equation (38) it suffices to prove that for $0 \leq i, j \leq n$, $i + 1 < j$, if $\sigma_n \in N_n\mathcal{C}$, $\omega_{n-2} \in N_{n-2}\mathcal{C}$ and $s_{n-2i}\omega_{n-2} = d_{nj}\sigma_n$, then there is a unique $\zeta_{n-1} \in N_{n-1}\mathcal{C}$ such that $s_{n-1i}\zeta_{n-1} = \sigma_n$, $d_{n-1j-1}\zeta_{n-1} = \omega_{n-2}$. Write

$$\sigma_n = (g_1, \ldots, g_n), \qquad \omega_{n-2} = (h_1, \ldots, h_{n-2}) \tag{A4}$$

in terms of morphisms g_k, h_l of $\mathcal{C}$. The condition $s_{n-2i}\omega_{n-2} = d_{nj}\sigma_n$ implies that

$$\begin{aligned} &h_1 = g_1, \dots, h_i = g_i, g_{i+1} = \mathrm{id}, h_{i+1} = g_{i+2}, \dots, \\ &h_{j-2} = g_{j-1}, h_{j-1} = g_{j+1} \circ g_j, h_j = g_{j+2}, \dots, h_{n-2} = g_n \end{aligned} \tag{A5}$$

Using Equation (A5), one checks readily that the morphism sequence

$$\zeta_{n-1} = (h_1, \dots, h_i, g_{i+2}, \dots, g_n) = (h_1, \dots, h_{j-2}, g_j, \dots, g_n) \tag{A6}$$

is composable so that $\in \mathsf{N}_{n-1}\mathcal{C}$ and that $s_{n-1i}\zeta_{n-1} = \sigma_n$, $d_{n-1j-1}\zeta_{n-1} = \omega_{n-2}$. The uniqueness of ζ_{n-1} is guaranteed by the injectivity of s_{n-1i}.

By virtue of Equation (35a), to show Equations (39) and (40), it is sufficient to show that for $0 \leq i, j \leq n$, $i \leq j$, if $\sigma_n, \omega_n \in \mathsf{N}_n\mathcal{C}$ and $d_{ni}\omega_n = d_{nj}\sigma_n$, then there is a unique $\zeta_{n+1} \in \mathsf{N}_{n+1}\mathcal{C}$ such that $d_{n+1i}\zeta_{n+1} = \sigma_n$, $d_{n+1j+1}\zeta_{n+1} = \omega_n$. Let

$$\sigma_n = (g_1, \dots, g_n), \qquad \omega_n = (h_1, \dots, h_n) \tag{A7}$$

in terms of morphisms g_k, h_l of $\mathcal{C}$. The condition $d_{ni}\omega_n = d_{nj}\sigma_n$ implies that

$$\begin{aligned} &h_1 = g_1, \dots, h_{i-1} = g_{i-1}, h_{i+1} \circ h_i = g_i, h_{i+2} = g_{i+1}, \dots, \\ &h_j = g_{j-1}, h_{j+1} = g_{j+1} \circ g_j, h_{j+2} = g_{j+2}, \dots, h_n = g_n \end{aligned} \tag{A8}$$

for $i < j$ and

$$h_1 = g_1, \dots, h_{i-1} = g_{i-1}, h_{i+1} \circ h_i = g_{i+1} \circ g_i, h_{i+2} = g_{i+2}, \dots, h_n = g_n \tag{A9}$$

for $i = j$. When $i < j$, it is simple to check exploiting Equation (A8) that

$$\zeta_{n+1} = (h_1, \dots, h_{i+1}, g_{i+1}, \dots, g_n) = (h_1, \dots, h_j, g_j, \dots, g_n) \tag{A10}$$

is composable and so $\zeta_{n+1} \in \mathsf{N}_{n+1}\mathcal{C}$ and that $d_{n+1i}\zeta_{n+1} = \sigma_n$, $d_{n+1j+1}\zeta_{n+1} = \omega_n$ as claimed. Letting $\zeta_{n+1} = (z_1, \dots, z_{n+1})$ for generic morphisms z_k and imposing that $d_{n+1i}\zeta_{n+1} = \sigma_n$, $d_{n+1j+1}\zeta_{n+1} = \omega_n$ we find further that ζ_{n+1} is precisely of the form Equation (A10), showing the uniqueness of ζ_{n+1}. When $i = j$, provided $\mathcal{C}$ is a groupoid, one similarly verifies from Equation (A9) that, setting $f_{i+1} = g_i \circ {h_i}^{-1} = {g_{i+1}}^{-1} \circ h_{i+1}$ and

$$\zeta_{n+1} = (h_1, \dots, h_i, f_{i+1}, g_{i+1}, \dots, g_n), \tag{A11}$$

$\zeta_{n+1} \in \mathsf{N}_{n+1}\mathcal{C}$ and that $d_{n+1i}\zeta_{n+1} = \sigma_n$, $d_{n+1j+1}\zeta_{n+1} = \omega_n$ again. When $i = 0$, only the second expression of f_{i+1} holds and when $i = n$ only the first. The equality of the two expressions of f_{i+1} for $0 < i < n$ is guaranteed by the ith Equation (A9). Writing $\zeta_{n+1} = (z_1, \dots, z_{n+1})$ for generic morphisms z_k and imposing that $d_{n+1i}\zeta_{n+1} = \sigma_n$, $d_{n+1j+1}\zeta_{n+1} = \omega_n$ we find further that ζ_{n+1} is precisely of the form of Equation (A11), showing once more the uniqueness of ζ_{n+1}.

We suppose next that the relevant simplicial set X considered is the simplicial set of an ordered finite simplicial complex $\mathcal{S}$, $X = K\mathcal{S}$ (cf. Subsection 2.1). We have to verify whether Equations (37)–(38) are fulfilled.

Below, analogously to what was done earlier, we shall adopt the convention that any sequence $v_r, v_{r+1} \dots, v_{s-1}, v_s$ of vertices or identities between vertices of $\mathcal{S}$ displayed in the analysis is tacitly assumed to be absent whenever $r > s$.

Because of Equation (35b) again, verifying Equation (37) reduces just to demonstrating that for $0 \leq i, j \leq n$, $i \leq j$, if $\sigma_n \in \mathsf{K}_n\mathcal{S}$, $\omega_{n+2} \in \mathsf{K}_{n+2}\mathcal{S}$ and $d_{n+2i}\omega_{n+2} = s_{nj}\sigma_n$, then there is a unique $\zeta_{n+1} \in \mathsf{K}_{n+1}\mathcal{S}$ such that $d_{n+1i}\zeta_{n+1} = \sigma_n$,

$s_{n+1j+1}\zeta_{n+1} = \omega_{n+2}$. Write

$$\sigma_n = (a_0, \dots, a_n), \qquad \omega_{n+2} = (b_0, \dots, b_{n+2}) \tag{A12}$$

in terms of vertices a_k, b_l of $\mathcal{S}$. The condition $d_{n+2i}\omega_{n+2} = s_{nj}\sigma_n$ implies that

$$b_0 = a_0, \dots, b_{i-1} = a_{i-1}, b_{i+1} = a_i, \dots, b_{j+1} = a_j, b_{j+2} = a_j, \dots, b_{n+2} = a_n \tag{A13}$$

Using Equation (A13), it is straightforward to check that

$$\zeta_{n+1} = (b_0, \dots, b_i, a_i, \dots, a_n) = (b_0, \dots, b_{j+1}, a_{j+1}, \dots, a_n) \tag{A14}$$

does the job. Since $\zeta_{n+1} = (b_0, \dots, b_{j+1}, b_{j+3}, \dots, b_{n+2})$, it holds that $\zeta_{n+1} \in K_{n+1}\mathcal{S}$. Further, $d_{n+1i}\zeta_{n+1} = \sigma_n$, $s_{n+1j+1}\zeta_{n+1} = \omega_{n+2}$ as is readily checked. The uniqueness of ζ_{n+1} is guaranteed by the injectivity of s_{n+1j+1}.

By Equation (35c), showing Equation (38) amounts to proving that for $0 \leq i, j \leq n$, $i + 1 < j$, if $\sigma_n \in K_n\mathcal{S}$, $\omega_{n-2} \in K_{n-2}\mathcal{S}$ and $s_{n-2i}\omega_{n-2} = d_{nj}\sigma_n$, then there exists a unique $\zeta_{n-1} \in K_{n-1}\mathcal{S}$ such that $s_{n-1i}\zeta_{n-1} = \sigma_n$, $d_{n-1j-1}\zeta_{n-1} = \omega_{n-2}$. Write

$$\sigma_n = (a_0, \dots, a_n), \qquad \omega_{n-2} = (b_0, \dots, b_{n-2}) \tag{A15}$$

in terms of vertices a_k, b_l of $\mathcal{S}$. The condition $s_{n-2i}\omega_{n-2} = d_{nj}\sigma_n$ implies that

$$b_0 = a_0, \dots, b_i = a_i, b_i = a_{i+1}, \dots, b_{j-2} = a_{j-1}, b_{j-1} = a_{j+1}, \dots, b_{n-2} = a_n \tag{A16}$$

Owing to the Equation (A16), it is straightforward to check that

$$\zeta_{n-1} = (b_0, \dots, b_i, a_{i+2}, \dots, a_n) = (b_0, \dots, b_{j-2}, a_j, \dots, a_n) \tag{A17}$$

has the required properties. As $\zeta_{n-1} = (a_0, \dots, a_i, a_{i+2}, \dots, a_n)$, $\zeta_{n-1} \in K_{n-1}\mathcal{S}$. Further, the identities $s_{n-1i}\zeta_{n-1} = \sigma_n$, $d_{n-1j-1}\zeta_{n-1} = \omega_{n-2}$ hold. The uniqueness of ζ_{n-1} is guaranteed by the injectivity of s_{n-1i}.

Though the above analysis completes the proof of the perfectness proposition for the simplicial set $K\mathcal{S}$, it may be interesting to get an intuition of the reason why $K\mathcal{S}$ fails to be quasi perfect or perfect. By Equation (35a), showing Equations (39) and (40) would require proving that for $0 \leq i, j \leq n$, $i \leq j$, if $\sigma_n, \omega_n \in K_n\mathcal{C}$ and $d_{ni}\omega_n = d_{nj}\sigma_n$, then there is a unique $\zeta_{n+1} \in K_{n+1}\mathcal{S}$ such that $d_{n+1i}\zeta_{n+1} = \sigma_n$, $d_{n+1j+1}\zeta_{n+1} = \omega_n$. Let

$$\sigma_n = (a_0, \dots, a_n), \qquad \omega_n = (b_0, \dots, b_n) \tag{A18}$$

Then, considerations analogous to those carried out above would yield for $i < j$

$$\zeta_{n+1} = (b_0, \dots, b_i, a_i, \dots, a_n) = (b_0, \dots, b_j, a_j, \dots, a_n) \tag{A19}$$

This simplex, however, cannot be written in terms of either the a_k or the b_k only, so there is no guarantee that $\zeta_{n+1} \in K_{n+1}\mathcal{S}$ by lack of increasing vertex ordering.

III. The simplicial Hilbert Hodge theorem

The proof of the simplicial Hilbert Hodge Theorem 4 follows immediately from the following proposition.

Proposition 6. *(Finite dimensional Hodge theorem) Let $\mathcal{K}_n$, $n \geq 0$, a sequence of finite dimensional Hilbert spaces. Let* *further*

$$\cdots \xleftarrow{Q_2} \mathscr{K}_2 \xleftarrow{Q_1} \mathscr{K}_1 \xleftarrow{Q_0} \mathscr{K}_0 \tag{A20}$$

be a Hilbert cochain complex and

$$\cdots \xrightarrow{{Q_2}^+} \mathscr{K}_2 \xrightarrow{{Q_1}^+} \mathscr{K}_1 \xrightarrow{{Q_0}^+} \mathscr{K}_0 \tag{A21}$$

be its adjoint chain complex. For $n \geq 0$*, denote by* $\mathrm{H}^n(\mathscr{K}, Q) = \ker Q_n / \operatorname{ran} Q_{n-1}$ *and* $\mathrm{H}_n(\mathscr{K}, Q^+) = \ker {Q_{n-1}}^+ / \operatorname{ran} {Q_n}^+$ *their (co)homology spaces, where by convention* $\operatorname{ran} Q_{-1} = 0$ *and* $\ker {Q_{-1}}^+ = \mathscr{K}_0$ *as usual. Then,*

$$\mathrm{H}^n(\mathscr{K}, Q) \simeq \ker H_n, \tag{A22}$$

$$\mathrm{H}_n(\mathscr{K}, Q^+) \simeq \ker H_n \tag{A23}$$

for any $n \geq 0$*, where* $H_n : \mathscr{K}_n \to \mathscr{K}_n$ *are the Laplacians*

$$H_n = {Q_n}^+ Q_n + Q_{n-1} {Q_{n-1}}^+, \tag{A24}$$

the second tern being missing when $n = 0$

We show the theorem. Let n be fixed. We notice that H_n is a Hermitian operator. So is then any real valued function of H_n. Two operators of this kind will be relevant for the proof. The first is the orthogonal projector P_n onto the kernel of H_n. The second is the generalized inverse ${H_n}^{-1}$ of H_n. They satisfy the following basic identities which will be employed repeatedly in the proof:

$${H_n}^{-1} H_n = 1_n - P_n, \tag{A25}$$

$$H_n P_n = 0, \qquad {H_n}^{-1} P_n = 0 \tag{A26}$$

where $1_n = 1_{\mathscr{K}_n}$.

The kernel of H_n is contained in that of Q_n and the range of Q_n is orthogonal to the kernel of H_{n+1}, since for $n \geq 0$

$$Q_n P_n = P_{n+1} Q_n = 0 \tag{A27}$$

Indeed, owing to the first Equation (A26),

$$0 = P_n H_n P_n \tag{A28}$$

$$= P_n({Q_n}^+ Q_n + Q_{n-1} {Q_{n-1}}^+) P_n \tag{A29}$$

$$= (Q_n P_n)^+ Q_n P_n + P_n Q_{n-1} (P_n Q_{n-1})^+ \tag{A30}$$

As $(Q_n P_n)^+ Q_n P_n$, $P_n Q_{n-1} (P_n Q_{n-1})^+ \geq 0$ as operators, we find that $Q_n P_n = 0$, $P_n Q_{n-1} = 0$, the second relation being missing for $n = 0$.

From Equation (A24), using that $Q_n Q_{n-1} = 0$, it is immediate to verify that

$$Q_n H_n = H_{n+1} Q_n \tag{A31}$$

for $n \geq 0$. From Equation (A31), it follows that

$$Q_n {H_n}^{-1} = {H_{n+1}}^{-1} Q_n \tag{A32}$$

Indeed, on account of Equations (A25), (A27),

$$\begin{aligned} {H_{n+1}}^{-1} Q_n - Q_n {H_n}^{-1} &= {H_{n+1}}^{-1} Q_n (1_n - P_n) - (1_{n+1} - P_{n+1}) Q_n {H_n}^{-1} \\ &= {H_{n+1}}^{-1} Q_n H_n {H_n}^{-1} - {H_{n+1}}^{-1} H_{n+1} Q_n {H_n}^{-1} \\ &= {H_{n+1}}^{-1} (Q_n H_n - H_{n+1} Q_n) {H_n}^{-1} = 0 \end{aligned} \tag{A33}$$

Using Equation (A32), we can show that for $n \geq 0$

$$1_n = P_n + {Q_n}^+ {H_{n+1}}^{-1} Q_n + Q_{n-1} {H_{n-1}}^{-1} {Q_{n-1}}^+ \tag{A34}$$

where for $n = 0$ the third term is missing. Indeed,

$$\begin{aligned} {Q_n}^+ {H_{n+1}}^{-1} Q_n + Q_{n-1} {H_{n-1}}^{-1} {Q_{n-1}}^+ &= ({Q_n}^+ Q_n + Q_{n-1} {Q_{n-1}}^+) {H_n}^{-1} \\ &= H_n {H_n}^{-1} = 1_n - P_n \end{aligned} \tag{A35}$$

whence Equation (A34) follows.

We have already observed that $\ker H_n \subseteq \ker Q_n$, since $Q_n P_n = 0$ by Equation (A27). As a consequence, a linear map $\mu_n : \ker H_n \to \mathrm{H}^n(\mathscr{K}, Q)$ is defined given by

$$\mu_n |\psi_n\rangle = |\psi_n\rangle + \operatorname{ran} Q_{n-1} \tag{A36}$$

with $|\psi_n\rangle \in \ker H_n$. We are going to show that μ_n is an isomorphism. We prove first that μ_n is injective. Suppose that $|\psi_n\rangle, |\phi_n\rangle \in \ker H_n$ and $|\psi_n\rangle - |\phi_n\rangle = Q_{n-1}|\chi_{n-1}\rangle$ for some $|\chi_{n-1}\rangle \in \mathscr{K}_{n-1}$, where $Q_{-1}|\chi_{-1}\rangle = 0$ by convention. Then, by the second Equation (A27),

$$|\psi_n\rangle - |\phi_n\rangle = P_n(|\psi_n\rangle - |\phi_n\rangle) = P_n Q_{n-1} |\chi_{n-1}\rangle = 0 \tag{A37}$$

proving the stated injectivity of μ_n. We demonstrate next that μ_n is a surjective map. If $Q_n|\psi_n\rangle = 0$, then by Equation (A34)

$$|\psi_n\rangle = P_n|\psi_n\rangle + Q_{n-1} {H_{n-1}}^{-1} {Q_{n-1}}^+ |\psi_n\rangle \tag{A38}$$

and consequently

$$|\psi_n\rangle + \operatorname{ran} Q_{n-1} = P_n|\psi_n\rangle + \operatorname{ran} Q_{n-1} = \mu_n P_n |\psi_n\rangle \tag{A39}$$

showing the stated surjectivity of μ_n. So, μ_n is an isomorphism as claimed. The isomorphism Equation (A22) follows.

The isomorphism Equation (A23) is shown similarly. We have that $\ker H_n \subseteq \ker {Q_{n-1}}^+$, by the relation ${Q_{n-1}}^+ P_n = 0$ from (A27) if $n > 0$ or trivially if $n = 0$. As a consequence, a linear map $\nu_n : \ker H_n \to \mathrm{H}_n(\mathscr{K}, Q^+)$

$$\nu_n |\psi_n\rangle = |\psi_n\rangle + \operatorname{ran} {Q_n}^+ \tag{A40}$$

with $|\psi_n\rangle \in \ker H_n$, is defined. ν_n can be shown to be an isomorphism by verifying its injectivity and surjectivity as done earlier for μ_n.

IV. Triviality of the kernel of the degeneracy simplicial Hodge Laplacian

Combining Equations (51), (64) and (61)

$$H_{SSn} = \sum_{\sigma_n \in X_n} |\sigma_n\rangle \Big(n + 1 - \sum_{0 \leq i \leq n-1} |S_{n-1i}(\sigma_n)| \Big) \langle \sigma_n| \tag{A41}$$

Since $|S_{n-1i}(\sigma_n)| \leq 1$, we have $H_{SSn} \geq 1_n$. Consequently, ker $H_{SSn} = 0$ as required.

V. The normalized Hilbert homology theorem

We begin with showing identity Equation (77) providing an expression of the orthogonal projector Π_n on the degenerate n-simplex space ${}^s\mathscr{H}_n \subseteq \mathscr{H}_n$ shown in Equation (75). Recall that the orthogonal projectors on the ranges ran S_{n-1i} of the degeneracy operators S_{n-1i} are the operators Π_{ni} given in Equation (60). Recall also that the Π_{ni} commute pairwise. Now, from Equation (75), the space ${}^s\mathscr{H}_n$ can be expressed as

$${}^s\mathscr{H}_n = \Big(\bigcap\nolimits_{i=0}^{n-1} \operatorname{ran} S_{n-1i}{}^{\perp} \Big)^{\perp} \tag{A42}$$

Now, the projectors onto the subspaces ran $S_{n-1i}{}^{\perp}$ are the operators $1_n - \Pi_{ni}$. By the commutativity of the Π_{ni}, the projector onto the subspace $\bigcap_{i=0}^{n-1} \operatorname{ran} S_{n-1i}{}^{\perp}$ is then $\prod_{i=0}^{n-1}(1_n - \Pi_{ni})$. Equation (A42) then implies that the projector Π_n is given by Equation (77) as claimed.

Next, we demonstrate the homological relation

$${}^cQ_{Dn-1}{}^cQ_{Dn} = 0 \tag{A43}$$

To that purpose, the relations

$$(1_{n-1} - \Pi_{n-1})Q_{Dn}\Pi_n = 0 \tag{A44}$$

valid for $n \geq 1$ are required. The proof of Equation (A44) runs as follows. Let $0 \leq k \leq n$. From Equation (60), using Equations (33b)–(33d) yields

$$\begin{aligned} Q_{Dn}\Pi_{nk} &= \sum_{0 \leq i \leq n} (-1)^i D_{ni} S_{n-1k} S_{n-1k}{}^{+} \\ &= \sum_{0 \leq i < k} (-1)^i S_{n-2k-1} D_{n-1i} S_{n-1k}{}^{+} + \sum_{k+1 < i \leq n} (-1)^i S_{n-2k} D_{n-1i-1} S_{n-1k}{}^{+} \end{aligned} \tag{A45}$$

where the first and second terms are 0 when respectively $k = 0$ and $k = n$. This expression shows that ran $Q_{Dn}\Pi_{nk} \subseteq$ ran S_{n-2k-1} + ran S_{n-2k}. Consequently,

$$(1_{n-1} - \Pi_{n-1k-1})(1_{n-1} - \Pi_{n-1k})Q_{Dn}\Pi_{nk} = 0 \tag{A46}$$

By Equation (77), we can replace the first two factors by $1_{n-1} - \Pi_{n-1}$. By Equation (77) again, Π_n is a polynomial in the Π_{ni} with vanishing degree 0 term. Hence, we can replace the last factor by Π_{n-1}, Equation (A44) follows. Using this latter, we can readily prove Equation (A43): from Equation (78),

$$\begin{aligned} {}^cQ_{Dn-1}{}^cQ_{Dn} &= (1_{n-2} - \Pi_{n-2})Q_{Dn-1}(1_{n-1} - \Pi_{n-1})Q_{Dn}\big|_{{}^c\mathscr{H}_n} \\ &= (1_{n-2} - \Pi_{n-2})Q_{Dn-1}Q_{Dn}\big|_{{}^c\mathscr{H}_n} - (1_{n-2} - \Pi_{n-2})Q_{Dn-1}\Pi_{n-1}Q_{Dn}\big|_{{}^c\mathscr{H}_n} = 0 \end{aligned} \tag{A47}$$

Next, we provide the details of the proof of Proposition 3. The design of the proof is already outlined in Subsection 3.5 and rests on the construction of a chain equivalence of the abstract and concrete Hilbert complexes $(\overline{\mathscr{H}}, \overline{Q}_D)$,

$({}^{c}\mathscr{H}, {}^{c}Q_D)$ (see Subsection 3.5 for their definition). The chain equivalence is given by a sequence of chain operators $I_n : \overline{\mathscr{H}}_n \to {}^{c}\mathscr{H}_n$, $J_n : {}^{c}\mathscr{H}_n \to \overline{\mathscr{H}}_n$, $n \geq 0$, with $J_n I_n$, $I_n J_n$ chain homotopic to $\overline{1}_n$, ${}^{c}1_n$ respectively (cf. Equations (80) and (81)). I_n is the operator from $\overline{\mathscr{H}}_n$ to ${}^{c}\mathscr{H}_n$ induced by the orthogonal projector $1_n - \Pi_n$ by virtue of the fact that ${}^{s}\mathscr{H}_n = \ker(1_n - \Pi_n)$. J_n is the canonical projection of ${}^{c}\mathscr{H}_n$ onto $\overline{\mathscr{H}}_n$.

We show first that the chain operator Equation (80) are fulfilled. By Equation (78), for $|\psi_n\rangle \in \mathscr{H}_n$, we have

$$\begin{aligned} I_{n-1}\overline{Q}_{Dn}(|\psi_n\rangle + {}^{s}\mathscr{H}_n) &= I_{n-1}(Q_{Dn}|\psi_n\rangle + {}^{s}\mathscr{H}_{n-1}) \\ &= (1_{n-1} - \Pi_{n-1})Q_{Dn}|\psi_n\rangle \\ &= (1_{n-1} - \Pi_{n-1})Q_{Dn}(1_n - \Pi_n)|\psi_n\rangle \\ &= {}^{c}Q_{Dn}(1_n - \Pi_n)|\psi_n\rangle \\ &= {}^{c}Q_{Dn}I_n(|\psi_n\rangle + {}^{s}\mathscr{H}_n) \end{aligned} \tag{A48}$$

showing Equation (80a). Similarly, by Equation (78) again, for $|\psi_n\rangle \in {}^{c}\mathscr{H}_n$

$$\begin{aligned} J_{n-1}{}^{c}Q_{Dn}|\psi_n\rangle &= J_{n-1}(1_{n-1} - \Pi_{n-1})Q_{Dn}|\psi_n\rangle \\ &= (1_{n-1} - \Pi_{n-1})Q_{Dn}|\psi_n\rangle + {}^{s}\mathscr{H}_{n-1} \\ &= Q_{Dn}|\psi_n\rangle + {}^{s}\mathscr{H}_{n-1} \\ &= \overline{Q}_{Dn}(|\psi_n\rangle + {}^{s}\mathscr{H}_n), \\ &= \overline{Q}_{Dn}J_n|\psi_n\rangle \end{aligned} \tag{A49}$$

proving Equation (80b).

We show next that the chain operator homotopy Equation (81) holds too. The proof is simple. The maps I_n, J_n turn out to be reciprocally inverse to one another. So, Equations (81a) and (81b) hold trivially with $\overline{W}_k = 0$, ${}^{c}W_k = 0$. The isomorphism Equation (79) follows.