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Opinion

A direct trial function approach for solving nonlinear evolution equations

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Abstract: The Klein equation, Infeld equation, and Sivashinsky equation not only start from realistic physical phenomena but can also be widely used in many physically significant fields such as plasma physics, fluid dynamics, crystal lattice theory, nonlinear circuit theory, and astrophysics. As a consequence, it is a very significant and challenging topic to research the explicit and accurate travelling wave solutions to these three equations. In this paper, in order to solve these three nonlinear partial differential equations (NPDEs), we have made some modifications to the trial function technique proposed by Xie and Tang by bringing in an ansatz solution containing two E-exponential functions. On this basis, we have developed a direct trial function technique to seek the explicit and accurate travelling wave solutions of nonlinear evolution equations (NEEs). We have illustrated its feasibility by applying it to the Klein equation, Infeld equation, and Sivashinsky equation. As a result, a lot of more general explicit and accurate travelling wave solutions of these three equations, including the solitary wave solutions and the singular travelling wave solutions, are successfully constructed in a straightforward and simple manner. The obtained solutions are quite equivalent to those given in the existing references. In addition, compared with the proposed approaches in the existing references, the approach described herein appears to be less calculative. Our technique may provide a novel way of thinking for solving NEEs. It is our firm conviction that the procedure used herein may also be utilized to explore the explicit and accurate travelling wave solutions of other NEEs. We try to generalize this approach to search for the explicit and accurate traveling wave solutions of other NEEs.

Keywords: direct trial function approach; Klein equation; Infeld equation; Sivashinsky equation; travelling wave solution; solitary wave solution

1. Introduction

Many phenomena in physics and other domains are often characterized by nonlinear evolution equations (NEEs for short). When people try to explore the physical mechanism of natural phenomena characterized by NEEs, explicit and accurate solutions for NEEs have to be searched. For this reason, the construction of the explicit and accurate solutions of the governing NEEs has become one of the most important and essential topics in nonlinear science, especially in nonlinear physics science. In recent years, a vast variety of powerful and practical methods have been proposed to construct the explicit and accurate solutions of NEEs. Among them are the homogeneous balance method [1], the hyperbolic tangent function expansion method [2], the Jacobi elliptic function expansion method [3], the F-function expansion method [4], the trial function method [5], the Lax pairs and dual Lax pairs of matrix eigenproblems method [6], the inverse scattering transform method [7], the superposition equation expansion method [8], the new extended auxiliary equation method and the generalized Kudryashov method [9], the modified Sine-Gordon equation expansion method [10], the generalized Riccati equation mapping and the modified F-function expansion method [11], the new Riccati equation expansion method [12], the generalized exponential rational function (GERF) expansion method [13], the modified Sardar sub-equation method [14], the novel analytical function expansion method [15], the method of use of simpler solutions for constructing exact solutions [16], and so forth. However, not all the above techniques are generally applicable to seeking the explicit and accurate solutions of all types of NEEs. As a result, it is still a very significant and challenging job to proceed to casting around for all kinds of more preferable and efficient approaches to construct the explicit and accurate solutions of NEEs.

In the present paper, by bringing in an ansatz solution containing two E-exponential functions e^y and e^z, we have made some modifications to the trial function technique of solving three NPDEs presented by Xie and Tang. On this basis, we have developed a direct trial function approach to search for the explicit and accurate travelling wave solutions for NEEs. In order to demonstrate the effectiveness of this approach, we take advantage of it to solve three physically significant NEEs, namely the Klein equation, the Infeld equation, and the Sivashinsky equation. As a result, a series of explicit and accurate travelling wave solutions of these three equations are successfully derived in a straightforward and simple manner.

The main structure of this paper is as follows.

In part 1, we present a brief review of methods of searching for the explicit and accurate solutions of NEEs.

In part 2, we briefly set forth our direct trial function approach.

In part 3, we employ this technique to construct a host of explicit and accurate travelling wave solutions of three physically significant NEEs, namely the Klein equation, Infeld equation, and Sivashinsky equation.

In part 4, we make a simple summary of this paper and give some necessary comments.

2. Basic technique

The main thought of our technique is as follows. Let us take into account the following known NEE with regard to two independent variables, x and t:

$$N(u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial t^2},\frac{\partial^2 u}{\partial x^2},\cdots)=0. \tag{1}$$

Generally speaking, the left-hand side of Equation (1) is a polynomial with respect to u and its various partial derivatives.

Our fundamental purpose is to research the explicit and accurate travelling wave solutions for Equation (1) in a systematic and concise manner. We have noticed that Xie and Tang [5] have developed a unified trial function method to construct the explicit and accurate solutions of three NEEs by introducing two trial functions recently. This approach seems to be slightly complicated because there are two trial functions in this method that are hard to select in a usual way. In order to improve this approach and to take advantage of it to solve Equation (1) more easily, here we directly suppose that Equation (1) is of the following ansatz solution containing two E-exponential functions e^y and e^z by our repeated and careful considerations:

$$u = \frac{be^z}{(a + e^y)^n}, \tag{2}$$

in which $y = k\xi$, $z = k'\xi$, $\xi = x - ct$ and a, b, c, k, k', n are undetermined constants with the constraint $0 \leq k' \leq kn$.

In the following, we briefly describe the main steps of applying Equation (2) to solve Equation (1).

Firstly, we determine the constant n by partial balance between the highest order derivative terms and the highest degree nonlinear terms in Equation (1). On account of $0 \leq k' \leq kn$, we may take the order of u as:

$$O(u) = n. \tag{3}$$

Then it is not hard to derive that:

$$O(\frac{\partial^j u}{\partial x^j}) = j + n. \tag{4}$$

Partial balance between the highest order derivative terms and the highest degree nonlinear terms in Equation (1) results in:

$$O(\frac{\partial^j u}{\partial x^j}) = O(u\frac{\partial u}{\partial x}), \tag{5}$$

i.e.,

$$n + j = n + n + 1, \tag{6}$$

from which it follows that:

$$n = j - 1. \tag{7}$$

So the ansatz solution (2) can be rewritten as:

$$u = \frac{be^z}{(a + e^y)^{(j-1)}}, \tag{8}$$

with the constraint $0 \leq k' \leq k(j-1)$.

Secondly, substituting the specific ansatz solution (8) into Equation (1) yields a set of algebraic equations because the coefficients of all e^y and e^z have to vanish.

Finally, the basic explicit and accurate solutions of Equation (1) can be readily found by solving this set of algebraic equations.

Because the Klein equation, Infeld equation, and Sivashinsky equation are real physical mode equations involving many different physical contexts, in the following, in order to demonstrate the effectiveness of this approach, we would like to utilize the procedure described above to search for the explicit and accurate travelling wave solutions of these three famous NEEs.

3. Applications

3.1. The Klein equation

The famous Klein equation in question reads:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^3 u}{\partial x^3}=0. \tag{9}$$

Here $j = 3$; in view of Equation (7), it is easily obtained that:

$$n=2. \tag{10}$$

So Equation (8) becomes:

$$u=\frac{be^{z}}{(a+e^{y})^2}, \tag{11}$$

with the constraint $0 \le k' \le 2k$.

From Equation (11), it is readily derived that:

$$\frac{\partial u}{\partial t}=\frac{bce^{k'\xi}[(2k-k')e^{k\xi}-bk']}{(a+e^{k\xi})^3}, \tag{12}$$

$$\frac{\partial u}{\partial x}=-\frac{be^{k'\xi}[(2k-k')e^{k\xi}-bk')}{(a+e^{k\xi})^3}, \tag{13}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{be^{k'\xi}[2b(k'^2-k^2-2kk')e^{k\xi}+(4k^2+k'^2-4kk')e^{2k\xi}+b^2k_1^2]}{(a+e^{k\xi})^4}, \tag{14}$$

$$\begin{aligned}\frac{\partial^3 u}{\partial x^3}&=\frac{be^{k'\xi}[b^3k'^3+(3b^2k'^3-2b^2k^3-6b^2k^2k'-6b^2kk'^2)e^{k\xi}]}{(a+e^{k\xi})^5}\\&+\frac{be^{k'\xi}[(14bk^3+6bk^2k'-12bkk'^2+3bk'^3)e^{2k\xi}+(12k^2k'-8k^3-6kk'^2+k'^3)e^{3k\xi}]}{(a+e^{k\xi})^5}.\end{aligned} \tag{15}$$

Substituting Equations (12)–(15) into Equation (9) gives:

$$\begin{aligned}&ab^3k'^3(k'+k'^3-c)e^{k'\xi}+a^2bk'^3e^{2k'\xi}+ab(2bck-3bck'-2bk^2-4bkk'+\\&+3bk'^3-2bk^3-6bk^2k'-6bkk'^2+3bk'^3)e^{(k+k')\xi}+ab(4ck-3ck'+2k^2\\&-8kk'^3+3k'^2+14k^3+6k^2k'-12kk'^2+3k'^3)e^{(2k+k')\xi}+a^2(k'-2k)e^{(k+2k')\xi}\\&+a(2ck-ck_1+4k^2-4kk_1+k_1^2-8k^3+12k^2k'-6kk'^2+k'^3)e^{(3k+k')\xi}=0.\end{aligned} \tag{16}$$

Theoretically speaking, for Equation (16), there are three cases that need to be considered: namely, $k'=0$, $k'=k$, and $k'=2k$. However, by careful and repeated analysis and calculations, we find that only when $k'=0$, Equation (9) has an independently nontrivial solution.

Under this condition, Equation (16) can be rewritten as:

$$2bk(a^2c-b-ak-a^2k^2)e^{k\xi}+2abk(2c+k+7k^2)e^{2k\xi}+2k(c+2k-4k^2)e^{3k\xi}=0. \tag{17}$$

Because of $e^{jk\xi} \neq 0$ $(j = 1,2,3)$, Equation (17) yields the following system of algebraic equations:

$$b^2c - a - bk - a^2k^2 = 0, \tag{18}$$

$$2c + k + 7k^2 = 0, \tag{19}$$

$$c + 2k - 4k^2 = 0. \tag{20}$$

Solving the above system of equations with the help of Maple, we find that:

$$b = -\frac{12a^2}{25},\ a, c, k = \text{arbitrary constants.} \tag{21}$$

Plugging Equation (21) into Equation (10), we acquire the basic travelling wave solution to the Klein Equation (9) as follows:

$$u_1 = -\frac{12a^2}{25(a + e^{k\xi})^2}. \tag{22}$$

Making use of the following hyperbolic identical equation:

$$\frac{1}{e^{2x} + 1} = \frac{1}{2}(1 - \tanh x), \tag{23}$$

and taking $a = 1$ in Equation (22), we have the solitary wave solution to the Klein Equation (9) as follows:

$$u_2 = -\frac{3}{25}(1 - \tanh\frac{k}{2}\xi)^2. \tag{24}$$

Similarly, taking advantage of the following hyperbolic identical equation:

$$\frac{1}{e^{2x} - 1} = \frac{1}{2}(\coth x - 1), \tag{25}$$

and taking $a = -1$ in Equation (22), we obtain the singular travelling wave solution of the Klein Equation (9) as follows:

$$u_3 = -\frac{3}{25}(1 - \coth\frac{k}{2}\xi)^2. \tag{26}$$

Apparently, the solutions u_2 and u_3 are quite equivalent to those obtained in reference [16].

3.2. The Infeld equation

The celebrated Infeld equation in concern reads:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^3 u}{\partial x^3} + \frac{\partial^4 u}{\partial x^4} = 0. \tag{27}$$

Here $j = 4$; in view of Equation (7), it is easily derived that:

$$n = 3. \tag{28}$$

So Equation (8) becomes:

$$u = \frac{be^{z}}{(a + e^{y})^{3}}, \tag{29}$$

with the constraint $0 \leq k^{'} \leq 3k$.

In view of Equation (29), it is readily deduced that:

$$\frac{\partial u}{\partial t} = \frac{bce^{k^{'}\xi}[(3k - k^{'})e^{k\xi} - ak^{'}]}{(a + e^{k\xi})^{4}}, \tag{30}$$

$$\frac{\partial u}{\partial x} = -\frac{be^{k_1\xi}[(3k - k^{'})e^{k\xi} - ak^{'})}{(a + e^{k\xi})^{4}}, \tag{31}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{be^{k^{'}\xi}[(2bk^{'2} - 3bk^{2} - 6akk^{'})e^{k\xi} + (9k^{2} + k^{'2} - 6kk^{'})e^{2k\xi} + b^{2}k^{'2}]}{(a + e^{k\xi})^{5}}, \tag{32}$$

$$\begin{aligned} \frac{\partial^3 u}{\partial x^3} &= \frac{be^{k^{'}\xi}[a^{3}k^{'3} + (3a^{2}k^{'3} - 3a^{2}k^{3} - 9a^{2}k^{2}k_{1} - 9b^{2}kk^{'2})e^{k\xi}]}{(a + e^{k\xi})^{6}} \\ &+ \frac{be^{k_1\xi}[(30bk^{3} + 18ak^{2}k^{'} - 18akk^{'2} + 3ak_{1}^{3})e^{2k\xi} + (27k^{2}k^{'} - 27k^{3} - 9kk^{'2} + k^{'3})e^{3k\xi}]}{(a + e^{k\xi})^{6}}, \end{aligned} \tag{33}$$

$$\begin{aligned} \frac{\partial^4 u}{\partial x^4} &= \frac{be^{k^{'}\xi}[a^{4}k^{'4} - (3a^{3}k^{4} + 12b^{3}k^{3}k^{'} + 18b^{3}k^{2}k^{'2} + 12b^{3}kk^{'3} - 4a^{3}k^{'4})e^{k\xi}]}{(a + e^{k\xi})^{7}} \\ &+ \frac{be^{k^{'}\xi}(75b^{2}k^{4} + 108b^{2}k^{3}k^{'} - 36b^{2}kk^{'3} + 18b^{2}k^{2}k^{'2} + 6b^{2}k^{'4})e^{2k\xi}}{(a + e^{k\xi})^{7}} \\ &+ \frac{be^{k^{'}\xi}(12bk^{3}k^{'} - 201bk^{4} + 90bk^{2}k^{'2} - 36bkk^{'3} + 4bk^{'4})e^{3k\xi}}{(a + e^{k\xi})^{7}} \\ &+ \frac{be^{k^{'}\xi}(81k^{4} - 108k^{3}k^{'} + 54k^{2}k^{'2} - 12kk^{'3} + k^{'4})e^{4k\xi}}{(a + e^{k\xi})^{7}}. \end{aligned} \tag{34}$$

Substituting Equations (30)–(34) into Equation (27) results in:

$$ab^4k^{'}(k^{'}+k^{'3}+k^{'4}-c)e^{k^{'}\xi}+a^2bk^{'}e^{2k^{'}\xi}+ab^3(3ck-4ck^{'}-3k^2-6kk^{'} \\ +4k^{'3}-3k^3-9k^2k^{'}-9kk^{'2}+4k^{'3}-3k^4-12k^3k^{'}-18k^2k^{'2}-12kk^{'3}+4k^{'4})e^{(k+k^{'})\xi} \\ +ab^2(9ck-6ck^{'}+3k^2-18kk^{'}+k^{'2}+27k^3+9k^2k^{'}-27kk^{'2}+6k^{'3}+75k^4+ \\ 108k^3k^{'}+18k^2k^{'2}-36kk^{'3}+6k^{'4})e^{(2k+k^{'})\xi}+ab(9ck-4ck^{'}+15k^2-18kk^{'}+4k^{'2}+3k^3 \\ +45k^2k^{'3}-27kk^{'2}+4k^{'3}-201k^4+12k^3k^{'}+90k^2k^{'2}-36kk^{'3}+4k^{'4})e^{(3k+k^{'})\xi} \\ +b(3ck-ck^{'}+9k^2-6kk+k^{'2}-27k^3+27k^2k^{'}-9kk^{'3}+k^{'4}+81k^4-108k^3+ \\ 54k^2k^{'2}-12kk^{'3}+k^{'4})e^{(4k+k^{'})\xi}+b^2(k^{'}-3k)e^{(k+2k^{'})\xi}=0. \tag{35}$$

Theoretically speaking, for Equation (35), there are four cases that need to be considered: namely, $k'=0$, $k'=k$, $k'=2k$, and $k'=3k$. However, by careful and repeated analysis and calculations, we find that only when $k'=0$ and $k'=k$, Equation (27) has an independently nontrivial solution. We shall give detailed discussions below.

Case (A): $k'=0$

Under this condition, Equation (35) can be converted to:

$$3bk(b^3c-a-a^3k-a^3k^2-a^3k^3)e^{k\xi}+3ba^2k(3c+k+9k^2+25k^3)e^{2k\xi} \\ +3abk(3c+5k+k^2-67k^3)e^{3k\xi}+3bk(c+3k-9k^2+27k^3)e^{4k\xi}=0. \tag{36}$$

Thanks to $e^{jk\xi}\neq 0(j=1,2,3,4)$, we acquire the following system of nonlinear algebraic equations from Equation (36):

$$a^3c-b-a^3k-a^3k^2-a^3k^3=0, \tag{37}$$

$$3c+k+9k^2+25k^3=0, \tag{38}$$

$$3c+5k+k^2-67k^3=0,(\tag{39}$$

$$c+3k-9k^2+27k^3=0.(\tag{40}$$

Solving the above nonlinear algebraic equation system with the aid of Maple, we obtain the following results:

$$b=-\frac{56a^3}{72}, a, c, k = \text{arbitrary constants}. \tag{41}$$

Plugging Equation (41) into Equation (29), we acquire the basic travelling wave solution of the Infeld Equation (27) as follows:

$$u_1=-\frac{5a^3}{72\gamma^2(a+e^{k\xi})^3}. \tag{42}$$

Taking advantage of the previous hyperbolic identical Equation (23) and taking $a=1$ in Equation (42), we obtain the solitary wave solution of the Infeld Equation (27) as follows:

$$u_2 = -\frac{5}{576}(1-\tanh\frac{k}{2}\xi)^3. \quad (43)$$

In the same way, taking advantage of the previous hyperbolic identical Equation (25) and taking $a = -1$ in Equation (42), we possess the so-called singular travelling wave solution for the Infeld Equation (27) as follows:

$$u_3 = -\frac{5}{576}(1-\coth\frac{k}{2}\xi)^3. \quad (44)$$

Case (B): $k' = k$.

Utilizing the similar program as before, we find the basic travelling wave solution to the Infeld Equation (27) as follows:

$$u_4 = \frac{15a^2e^{k\xi}}{8(a+e^{k\xi})^3}. \quad (45)$$

Taking advantage of the previous hyperbolic identical Equation (23) and the following hyperbolic identical equation:

$$\frac{e^{2x}}{(e^{2x}+1)^2} = \frac{1}{4}\sec h^2 x, \quad (46)$$

and taking a = 1 in Equation (45), we acquire the solitary wave solution of the Infeld Equation (27) as follows:

$$u_5 = \frac{15}{64}\sec h^2\frac{k}{2}\xi(1-\tanh\frac{k}{2}\xi). \quad (47)$$

In the same way, taking advantage of the foregoing hyperbolic identical Equation (25) and the following hyperbolic identical equation:

$$\frac{e^{2x}}{(e^{2x}-1)^2} = \frac{1}{4}\csc h^2 x, \quad (48)$$

and taking $a = -1$ in Equation (45), we get the so-called singular travelling wave solution for the Infeld Equation (27) as follows:

$$u_6 = -\frac{15}{64}\csc h^2\frac{k}{2}\xi(1-\coth\frac{k}{2}\xi). \quad (49)$$

Apparently, the solutions u_2 and u_5 are quite equivalent to those obtained in reference [16].

3.3. The Sivashinsky equation

The celebrated Sivashinsky equation under consideration reads:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} - \frac{\partial^5 u}{\partial x^5} = 0. \quad (50)$$

Here j = 5; in view of Equation (7), it is easily derived that:

$$n = 4. \tag{51}$$

So Equation (8) becomes:

$$u = \frac{be^{z}}{(a+e^{y})^{4}}, \tag{52}$$

with the constraint $0 \leq k' \leq 4k$.

In view of Equation (52), it is readily deduced that:

$$\frac{\partial u}{\partial t} = \frac{bce^{k'\xi}[(4k-k')e^{k\xi} - bk']}{(a+e^{k\xi})^{5}}, \tag{53}$$

$$\frac{\partial u}{\partial x} = -\frac{be^{k'\xi}[(4k-k')e^{k\xi} - bk')}{(a+e^{k\xi})^{5}}, \tag{54}$$

$$\begin{aligned}\frac{\partial^3 u}{\partial x^3} &= \frac{be^{k'\xi}[b^3k'^3 + (3a^2k'^3 - 4b^2k^3 - 12b^2k^2k'^2 - 12a^2kk'^3)e^{k\xi}]}{(a+e^{k\xi})^7} \\ &+ \frac{be^{k'\xi}[(52ak^3 + 36ak^2k' - 24bkk'^2 + 3ak'^3)e^{2k\xi} + (48k^2k' - 64k^3 - 12kk'^2 + k'^3)e^{3k\xi}]}{(a+e^{k\xi})^7},\end{aligned} \tag{55}$$

$$\begin{aligned}\frac{\partial^5 u}{\partial x^5} &= \frac{be^{k'\xi}[b^5k'^5 - (4b^4k^5 + 20a^4k^4k^5 - 40a^4k^3k'^2 - 40a^4k^2k'^3 - 20b^4kk'^4 - 5b^4k'^5)e^{k\xi}]}{(a+e^{k\xi})^9} \\ &+ \frac{be^{k'\xi}(284b^3k^5 + 620b^3k^4k' + 40b^3k^2k'^2 + 440a^3k^3k'^3 - 80a^3kk'^4 + 10b^3k'^5)e^{2k\xi}}{(a+e^{k\xi})^9} \\ &+ \frac{be^{k'\xi}(-1620a^2k^4k' - 2124b^2k^5 + 360b^2k^3k'^2 + 360a^2k^2k'^3 - 120b^2kk'^4 + 10a^2k'^5)e^{3k\xi}}{(a+e^{k\xi})^9} \\ &+ \frac{be^{k'\xi}(3284bk^5 - 980bk^4k' - 760ak^3k'^2 + 440bk^2k'^3 - 80akk'^4 + 5bk'^5)e^{4k\xi}}{(a+e^{k\xi})^9} \\ &+ \frac{be^{k'\xi}(-1024k^5 + 1280k^4k' - 640k^3k'^2 + 160k^2k'^3 - 20kk'^4 + k'^5)e^{5k\xi}}{(a+e^{k\xi})^9}.\end{aligned} \tag{56}$$

Substituting Equations (53)–(56) into Equation (50) yields:

$$
\begin{aligned}
& ab^5k'(k'^2-k'^5-c)e^{k'\xi}+b^2ak'e^{2k'\xi}+ba^4(4ck-5ck'-4k^3-12k^2k' \\
& -12kk'^2+5k'^3+4k^5+20k^4k'^5+40k^3k'^2+40k^2k'^3+20kk'^4-5k'^5)e^{(k+k')\xi} \\
& +2ab^3(8ck-5ck_1+22k^3+6k^2k'-24kk'^2+5k'^3-142k^5-310k^4k' \\
& -220k^3k'^2-20k^2k'^3+40kk'^4-5k'^5)e^{(2k+k')\xi}+2ab^2(12ck-5ck_1+18k^3 \\
& +54k^2k'-36kk'^2+5k'^4+1062k'^5+810k^4k'^5-180k^3k'^3-180k^2k'^5 \\
& +60kk'^4-5k')e^{(3k+k')\xi}+ab(16ck-5ck'-76k^3+132k^2k'-48kk'^2+ \\
& 5k'^4-3284k^5+980k^4k'+760k^3k'^2-440k^2k'^3+80kk'^4-5k'^5)e^{(4k+k')\xi} \\
& a(4ck-ck'-64k^3+48k^2k'-12kk'^2+k'^3+1024k^5-1280k^4k' \\
& +640k^3k'^2-160k^2k'^3+20kk'^4-k'^5)e^{(5k+k')\xi}+b^2(k'-4k)e^{(k+2k')\xi}=0.
\end{aligned} \tag{57}
$$

Theoretically speaking, for Equation (57), there are five cases that need to be considered: namely $k'=0$, $k'=k$, $k'=2k$, $k'=3k$, and $k'=4k$. However, by careful and repeated analysis and calculations, we find that only when $k'=2k$, Equation (50) has an independently nontrivial solution.

Under this condition, Equation (57) can be rewritten as:

$$
\begin{aligned}
& 2ba^5k(4k^2-c-16k^4)e^{2k\xi}-6ba^4k(c+6k^2-114k^4)e^{3k\xi} \\
& +2abk(b-2a^2c-22a^2k^2-1322a^2k^4)e^{4k\xi} \\
& -2ak(a-2b^2c-22a^2k^2-1322a^2k^4)e^{5k\xi} \\
& +6abk(c+6k^2-114k^4)e^{6k\xi}-2bk(4k^2-c-16k^4)e^{7k\xi}=0.
\end{aligned} \tag{58}
$$

Because of $e^{jk\xi}\neq 0\ (j=2,3,4,5,6,7)$, Equation (58) engenders the following system of algebraic equations:

$$4k^2-c-16k^4=0, \tag{59}$$

$$c+6k^2-114k^4=0, \tag{60}$$

$$b-2a^2c-22a^2k^2-1322a^2k^4=0. \tag{61}$$

Solving the above system of algebraic equations with the aid of Maple, we find that:

$$b=\frac{1680a^2}{169},\ \ a,\ \ c,\ \ k=\text{arbitrary constants}. \tag{62}$$

Plugging Equation (62) into Equation (52), we obtain the basic travelling wave solution to the Sivashinsky Equation (50) as follows:

$$u_1=\frac{16805a^2e^{2k\xi}}{169(a+e^{2k\xi})^4}. \tag{63}$$

Taking advantage of the foregoing hyperbolic identical Equation (46) and letting $a=1$ in Equation (63), we obtain the solitary wave solution to the Sivashinsky Equation (50) as follows:

$$u_2 = \frac{105}{169} \sec h^4 k\xi. \tag{64}$$

Similarly, making use of the previous hyperbolic identical Equation (48) and taking $a = -1$ in Equation (63), we get the singular travelling wave solution to the Sivashinsky Equation (50) as follows:

$$u_3 = \frac{105}{169} \csc h^4 k\xi. \tag{65}$$

Obviously, the solutions u_2 and u_3 are quite equivalent to those obtained in reference [16].

In addition, it should be pointed out that some solutions obtained herein are of definite physical meaning in that the solitary wave solution of these three equations may aid in explaining nonlinear wave phenomena of diffusion in fluid mechanics. Also, it is worth presenting the physical motivation by plotting some figures to their solitary wave solutions. We shall leave this problem for further study in the future. Furthermore, we think that it is very meaningful and necessary to give numerical simulations to illustrate the traveling wave solutions. We are planning to do this work in the light of the methods in references [17–19].

4. Conclusions and comments

To sum up, by supposing an ansatz solution containing two E-exponential functions e^y and e^z, we have made some improvements to the trial function approach for solving three NPDEs put forth by Xie and Tang. On this basis, we have developed a straightforward trial function technique for searching for the explicit and accurate travelling wave solutions of NEEs. We have illustrated its feasibility by applying it to the Klein equation, Infeld equation, and Sivashinsky equation. As a result, a series of more general explicit and accurate travelling wave solutions of these three equations, containing the solitary wave solutions and the singular travelling wave solutions, are successfully found in a forthright and simple manner. The obtained solutions are quite equivalent to those given in the existing references.

On account of these three equations being real physical models, solving them has received much attention from many authors, and quite a few papers have constructed their explicit and accurate travelling wave solutions. Nevertheless, compared with some proposed methods in the literature, the above approach described herein has certain advantages, which can be briefly explained as follows.

1) The hyperbolic tangent function expansion method proposed in reference [2] can only construct the tanh-type explicit and accurate travelling wave solutions of NEEs, while the approach described herein constructs not only the tanh-type explicit and accurate travelling wave solutions of NEEs but also the coth-type explicit and accurate travelling wave solutions of NEEs.

2) There are two trial functions in the unified trial function method proposed in reference [5], and it needs to perform more tedious calculations. However, there is one trial function in our technique, and it certainly needs to perform fewer calculations. Obviously, our technique appears to be relatively simple and direct.

3) Although the generalized Riccati equation mapping and the modified F-function expansion method proposed in reference [11] can construct the rich types of explicit and accurate travelling wave solutions to NEEs, this method is more complicated than our approach.

We shall not compare our approach with other methods one by one for the limit of length. More importantly, our approach may provide a new way of thinking for solving NEEs.

In addition, it is worth further research in the future whether our approach can be used to solve other NEEs, such as the KdV equation, the KdV-Burgers-Kuramoto equation, the sine-Gordon equation, and so on. Although this issue still needs to be solved, we fully believe that the procedure used herein may also be utilized to look for the explicit and accurate travelling wave solutions of other NEEs. We plan to popularize this strategy to search for the explicit and accurate travelling wave solutions of other NEEs.

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