Journal of AppliedMath 2025, 3(4), 2901.
https://doi.org/10.59400/jam2901

Article

The new Generalized Schwarzschild-spacetimes trivial Ricci solitons and the new smooth metric space

Orchidea Maria Lecian

Sapienza University of Rome, 00185 Rome, Italy; orchideamaria.lecian@uniroma1.it

CITATION

Lecian OM. The new Generalized Schwarzschild-spacetimes trivial Ricci solitons and the new smooth metric space. Journal of AppliedMath. 2025; 3(4): 2901. https://doi.org/10.59400/jam2901

ARTICLE INFO

Received: 6 March 2025
Revised: 10 June 2025
Accepted: 19 June 2025
Available online: 23 July 2025

Abstract: The Ricci flow of the Generalized-Schwarzschild spacetimes is newly studied. The soliton configurations are newly stated as trivial Ricci soliton of (Generalized)-Schwarzschild spacetimes. The new smooth metric space is written; the majorization theorem for the distance is given. The application of harmonic maps is presented. The definition of topological soliton as a Schwarzschild soliton of complete Riemannian manifold is newly provided with. New theorems about Generalized-Schwarzschild solitons which are extended from those about the Kaehler solitons are proven; the new theorems are given, which allow one to establish the differences with respect to Kaehler solitons. The new properties of the Generalized Schwarzschild metric are studied. As results, smooth metric spaces are newly exposed as ones endowed with bounded Bakry-Emery curvature; the initial conditions are newly studied: the weight function is majorized as consisting of a polynomial function of the distance(s) (from the initial condition) at most.The Generalized-Schwarzschild metric is now newly proven to be descending from a smooth function. The initial conditions are newly studied to depend only on the spherical neighborhood of the point. The trivial expanding Ricci Kaehler soliton is newly proven to be a Generalized-Schwarzschild soliton; accordingly, this soliton is newly proven to have only one end.

Keywords: Schwarzschild soliton; Generalized-Schwarzschild soliton; smooth metric space; isoperimetric inequalities

1. Introduction

In the work of Perelman [1], a monotonic expression of the Ricci flow is provided with, which is valid in any number of dimensions, and without any assumptions on the curvature.

The corresponding expression of the Ricci flow coincides with its interpretation of of the entropy formula of a prescribed canonical ensemble.

The Ricci flow equation is written after the work of Hamilton [2]; it is the time evolution of the equality

$$\frac{d}{dx}g_{\mu\nu} = -2R_{\mu\nu} \quad (1)$$

of a given Riemannian metric $R_{\mu\nu}$.

The solution of Equation (1) is ibidem proven to be unique on a closed manifold endowed with an arbitrary smooth metric. Accordingly, the time evolution of the metric tensor provides one with the time evolution of the Riemann tensor; ore precisely, prescriptions about the time evolution of the Ricci scalar are implied.

It is one of the aims of the present paper to expand the pioneering result form the work of Hamilton [2] and from the work of Hamilton [3], in which the proof is given of the fact that the Ricci flow conserves the positivity of the Ricci tensor in dimension 3.

Furthermore, the positivity of the curvature operator is proven after the work of Perelman [1] to be conserved in any number of dimensions.

It is the aim of the preset paper to start expanding the analysis of the work of Hamilton [3] about the conservation of the positivity of the Ricci tensor in a different number of dimensions.

The method of [1] is here followed. The Ricci flow of the Schwarzschild spacetime is newly established.

The results are here extended to a generic four-manifold, which is worked from the (Generalized)-Schwarzschild blackhole spacetime manifold of blackhole object surrounded of media [4]. The results overlap in the case of adiabatic perturbations of the pressure of the media [5].

In the presented methodology, the smooth metric space with bounded Bakry-Emery curvature is written. The initial conditions are studied; as a new result, for any initial conditions, the (proper) weight function is majorized as a polynomial function of the distances (calculated as a function the metric tensor of the smooth metric space).

The existence of trivial Ricci solitons in the newly-found smooth metric space is investigated as follows.

1) in the case of the Schwarzschild metric, the results from [4] are applied: the solitons are found to exist only in the inviscid media surrounding the blackhole, the radial velocity is found as constant in time, the thermodynamical entropy is found as constant in time, the thermodynamical differential enthalpy is determined as vanishing for the thermodynamically-isoentropic process, and the limit to the soliton as a 'small sphere' is not allowed;

2) in the case of generalized Schwarzschild spacetimes, whose line element is endowed with a Schwarzschild solid-angle element, where the $g_0 0$ component of the metric tensor is

$$g_{00} \equiv f(r) \equiv 1 - \frac{r_S}{r} + \Psi(r) \tag{2}$$

is found to admit solitons only for

$$\Psi(r) \equiv \Psi_0 \frac{r_0}{r} \tag{3}$$

where r_0 and Ψ_0 are integration constants, the radial velocity is constant in time.

For a different $\Psi(r)$, for the solitons to exist the media surrounding the blackhole object must be viscous, with non-vanishing vorticity and non-vanishing shear.

3) In the case of generalized g_{tt} function $f(r)$, the Bunting theorem is requested to be obeyed, i.e. the Schwarzschild solution with a slight perturbation of the Schwarzschild radius is the only admitted one. The further implications will be addressed elsewhere.

The content of [6–9] is considered, and application is provided with.

The results of [10], those of [11] and those from [12] are questioned.

Throughout the present paper, the Einstein notation on summation is adopted, i.e. the saturated-over indices are summed over, and the summation symbol $\sum$ is omitted, as from [13]. The prescription of [13] is followed, according to which the matter is never put in the metric tensor, not even for the backreaction in the ultra-Relativistic limit.

Within this framework, it is possible to address the Schwarzschild spacetimes directly, in a manner complementary to [14]. More in detail, the metric tensor $g_{\mu\nu}$ from the Riemannian manifold $(\mathcal{M}, g)$ coincides with the metric tensor of the (to-be-metrized) soliton $(\mathcal{M}, g, \tilde{f})$ in the 4-dim case. More specifically, the Schwarzschild spacetime is considered, the Generalized-Schwarzschild (i.e., such as the Kottler) spacetimes can be treated as well, i.e. the spacetimes for which the $_{rr}$ component of the metric tensor $g_{rr} \equiv g_{tt}^{-1}$, where g_{rr} defines the radial distance, contains modification(s) addend(s) for the Schwarzschild radius, and the non-rotating spherically-symmetric blackhole spacetimes can be included, i.e. those spacetimes in which $g_{rr} \equiv g_{tt}^{-1}$ has a general dependence on the r component of the position 4-vector (for which the functional dependence on r of g_{rr} in more general than containing an addend $\frac{r_S}{r}$). It is worth recalling that for the Bunting Theorem [15], the Schoen-Yau Theorem [16], the Masood-ul-Alam Theorem [17], and the Ruback Theorem [18] are obeyed at the considered spacetimes.

The paper is organized as follows.

In Section 2, the introductory material is recalled.

In Section 3, the Schwarzschild-spacetime soliton is defined, and the definition of topological soliton as a Schwarzschild soliton of complete Riemannian manifold is newly given.

In Section 4, Schwarzschild trivial solitons are newly studied on the equatorial plane.

In Section 5, the Generalized-Schwarzschild spacetimes soliton is defined, and the generalized Schwarzschild spacetimes and the corresponding manifolds are here studied.

In Section 6, spherically-symmetric-non-rotating-spacetimes solitons are studied. The initial conditions are interrogated about.

In Section 7 the new theorems about Generalized-Schwarzschild solitons which are extended from those about the Kaehler solitons are proven; the new theorems are provided with, which allow one to introduce the following sections after studying the comparison with Kaehler solitons.

In Section 8, the new applications to harmonic maps are proven.

In Section 9, properties from GS metric are studied.

In Section 10, one of the new findings are presented. Smooth metric spaces are newly exposed as ones of bounded Bakry-Emery curvature; the initial conditions are studied: the weight function is majorized as a polynomial function of the distance(s) (from the initial condition) at most.

In Section 11, the trivial Kaehler soliton is proven to be connected at infinity.

In Section 12, the discussion is proposed.

In Section 13, the Outlook is presented. Here, the choice of complete manifold as a pseudo-Riemannian manifold is compared with the Schwarzschild spacetimes containing a blackhole object surrounded of media as from [4]: as results, the Generalized Schwarzschild soliton is demonstrated to be in a trivial gradient Ricci soliton.

In Appendix A, the geometrical objects are written; in particular, the Ricci Scalar and the Ricci tensor are written for the calculation of the Einstein Field Equations of the selected spacetimes.

2. Introductory material

From the work of Perelman [1], the Perelman Hamilton flow of the Ricci tensor is written from the work of Hamilton [2] as

Definition 1. *The Perelman Equation of the Hamilton flow of the Ricci tensor $R_{\mu\nu}$ is defined as*

$$R_{\mu\nu} = -\frac{1}{2}\frac{d}{dt}g_{\mu\nu} \tag{4}$$

where the indices μ, ν are as $0, 1, 2, 3$ of signature $(+, -, -, -)$.

From the work of Eminenti et al. [6], the notation is recalled for comparison with the present achievements of a Ricci soliton given after the existence of a one-form ω such that

$$R_{\mu\nu} + \nabla_\mu\omega_\nu + \nabla - \nu\omega_\mu = \frac{2\mu}{n}g_{\mu\nu} \tag{5}$$

where μ is a constant, $\mu \in \mathbb{R}$, in dimensions n.

The result is ibidem specified to a gradient Ricci soliton as $(\mathcal{M}, g)$ of a complete Riemannian manifold of metric $g_{\mu\nu}$ for which there exists a function $\hat{f}$ denominated a potential function such that

$$\mathcal{M} \to \mathbb{R} \tag{6}$$

according to the passage

$$R_{\mu\nu} + \nabla^2\hat{f} = \frac{\mu}{n}g_{\mu\nu} \tag{7}$$

The solitons are classified as contracting solitons, steady solitons and expanding solitons according to whether $\mu > 0$, $\mu = 0$ and $\mu < 0$, respectively.

Furthermore, the trivial soliton is one for which ω is chosen as vanishing, i.e. the function $\hat{f}$ is taken as constant.

The progress is recalled from $(\mathcal{M}, g)$ to be an Einsteinian manifold from th work of Eminenti et al. [6] when it has a constant curvature.

From Remark 1.2 ibidem, a Ricci soliton is a gradient soliton when the 1-form ω is exact.

The diffeomorphisms and the homeomorphisms of the initial metric can therefore be studied, such that the solitons are stationary points of the Ricci flow.

From the definition of smooth metric measure space, i.e. as those discussed in the work of Rimoldi et al. [19], the notion of 'smooth metric space' is worked out in the present paper.

3. Schwarzschild-spacetime solitons

In Schwarzschild spacetimes whose line element is spelled as

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2d\phi^2 - r^2(sin\theta)^2d\phi^2 \equiv \left(1 - \frac{r_S}{r}\right)dt^2 - \frac{1}{\left(1 - \frac{r_S}{r}\right)}dr^2 - r^2d\phi^2 - r^2(sin\theta)^2d\phi^2 \quad (8)$$

The Perelman equation of the Hamilton flow of the Ricci tensor Equation (4), the solutions are here studied. As from Appendix A, the Ricci tensor is vanishing. Equation (4) is specified as

$$\frac{d}{dt}g_{\mu\nu} \equiv 0 \quad (9)$$

Equation (9) is therefore rewritten as

$$\frac{dg_{\mu\nu}}{dr}\dot{r} + \frac{\partial g_{\mu\nu}}{\partial\theta}\frac{d\theta}{dt} \equiv 0 \quad (10)$$

where the $\dot{r}$ indicates the time derivative of the r component of the 4-position vector, i.e. it is the radial velocity u_r component of the 4-velocity vector.

The following definition is given

Definition 2. *A Schwarzschild soliton is a Riemannian manifold $\mathcal{M}$ endowed with the metric tensor $g_{\mu\nu}$ which obey the Perelman Equation of the Ricci flow Equation (9).*

The following extended definition is considered

Definition 3. *A Schwarzschild topological soliton is a Schwarzschild soliton.*

Accordingly,

Definition 4. *A Schwarzschild topological soliton is a Schwarzschild soliton of complete manifold.*

4. Schwarzschild trivial solitons

On the equatorial plane, Equation (10) requires an u_r constant in time. From the work of Lecian [4], the radial velocity is found as

$$u_r \equiv \frac{c_2}{r^{2+\sigma}} \quad (11)$$

with $\sigma \in \mathbb{R}$, thus u_r from Equation (11) form a family of solutions.

The Perelman equation of the Hamilton flow of the Riccitensor (9) implies that the Schwarzschild blackhole object of the blackhole spacetime is surrounded of (Astrophysical) media.

This way, after Definitions 2–4, the possibility is prepared, to define the weight of the weighted manifold of the soliton.

4.1. The case of constant r

In the case of constant r, in the case of constant θ, the motion of the test particle in the media is described as a process where the value of the thermodynamical entropy is fixed. It is here commented that therefore the differential thermodynamical enthalpy is vanishing, i.e. as from the work of Razdoburdin et al. [20].

In the case of non-constant θ, the process is not thermodynamically isoentropic; the value of the thermodynamical entropy depends on σ from Equation (11).

5. Generalized-Schwarzschild-spacetimes solitons

The case of the Generalized-Schwarzschild spacetimes and the corresponding manifolds are here studied.

The line elements is here spelled as

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2d\theta^2 - r^2(sin\theta)^2d\phi^2 \equiv \left(1 - \frac{r_S}{r} + \Psi(r)\right)dt^2 - \frac{1}{\left(1 - \frac{r_S}{r} + \Psi(r)\right)}dr^2 - r^2d\theta^2 - r^2(sin\theta)^2d\phi^2 \quad (12)$$

where the generalized potential $\Psi \equiv \Psi(r)$ is considered as well. The generalized potential $\tilde{\psi} \equiv -\frac{r_S}{r} + \Psi(r)$ is here requested to be non-vanishing as from the proof of the Birkhoff Theorem for these spacetimes provided in the work of Lecian [21].

The Perelman equations of the Hamilton flow of the Ricci tensor are spelled from Appendix A.

As a particular case, the initial condition (of the Einstein Field Equations) $\dot{r} = const$ is here studied as

$$R_{\mu\nu} = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial\theta}\frac{d\theta}{dt} \quad (13)$$

The Perelman equations is here solved for constant $\dot{\theta}$.

As a result, the solution

$$\Psi = \Psi_0\frac{r_0}{r} \quad (14)$$

is found, where r_0 and ψ_0 are integration constants.

Solution Equation (14) obeys the Einstein filed equations from Appendix A.

For the soliton to exist, the motion of the test particle in the media surrounding the blackhole object is of constant thermodynamical entropy the work of Lecian [4] in an inviscid media.

For different soliton to exist, the process must be in 'viscous media' from the work of Lecian [5].

At the condition $\tilde{\psi} - \frac{r_S}{r} = 0$, the thermodynamical entropy is fixed, and the differential thermodynamical enthalpy is vanishing.

6. Spherically-symmetric non-rotating spacetimes solitons

The most general case of spherically-symmetric non-rotating spacetimes of line element

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2d\theta^2 - r^2(sin\theta)^2d\phi^2 \quad (15)$$

is here studied.

The Perelman equations in the case of constant θ are solved as

$$f(r) = 1 + (f_0 - 1)\frac{r_0}{r} \quad (16)$$

where f_0 and r_0 are integration constants. The Einstein Field Equations are obeyed from Appendix A. The Bunting Theorem from the work of Bunting et al. [22] is requested to be obeyed.

7. New theorems

The following new theorems are here given; the results from the work of Munteanu et al. [23] are newly considered, as the new definition of trivial Kaehler soliton is therefore allowed; it is explained as

Definition 5. *A Schwarzschild soliton is a particular case of Kaehler soliton of vanishing Bakry-Emery curvature.*

Definition 6. *A Generalized-Schwarzschild soliton is a particular case of Kaehler soliton of bounded Bakry-Emery curvature.*

Accordingly, the following new theorems are worked out from ibidem

Theorem 1. *A trivial gradient Kaehler shrinking soliton is found as a Generalized-Schwarzschild soliton of bounded curvature and of bounded Bakry-Emery curvature.*

Proof. *The Ricci curvature of the Schwarzschild-spacetime manifolds are written in Appendix A. The Proof is completed after Theorem 4.* □

Theorem 2. *Let $(\mathcal{M}, g)$ be a complete Kaehler manifold. The distance l is chosen as $l = \sqrt{ds^2}$, and the metric tensors from Sections 3, 5 and 6 are now considered, i.e.the case of Generalized-Schwarzschild (GS) soliton is considered, where the distance is written as*

$$l_{GS} = \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \tag{17}$$

Let Ψ be a smooth real function on the manifold. The condition that the complex structure $J(\nabla_\mu \Psi)$ be a Killing vector field X^μ is requested. Because the manifold is not Minkowskian, the Killing vectors are not trivial.

Proof. *The requested Killing vector X^μ is worked from the four-velocity u^μ, which is the derivative of the 4-position vector x^μ; particular, it is taken as the Lie derivative of the 4-velocity u^μ as*

$$X^\mu \equiv \nabla_\mu u_\nu - \nabla_\nu u_\mu. \tag{18}$$

□

8. New harmonic maps

It is now our purpose to study the harmonic maps. Harmonic maps are used to newly develop the requests from the work of Rimoldi et al. [19].

For this purpose, the initial condition which defines the 4-velocity vector is here newly studied.

Theorem 3. *The distance is calculated from the initial point $\bar{x}^\mu$. The proper function Ψ between topological spaces is considered The u_μ found is constant in time; it is to be requested that u_μ must be constant as ∇U must be vanishing, being $U \equiv \sqrt{u_\mu u^\mu}$.*

Proof. *The request is accomplished because the 4-velocities here calculated as from Equation (11) are in L^2.*

Therefore, the request from Theorem 0.2 from the work of Munteanu et al. [23] is *accomplished to prove Theorem 0.1 from ibidem as*

$$\int_{\mathcal{M}} \mid \nabla U \mid e^{-\psi} \leq \infty \tag{19}$$

being ψ the angular component of the 4-position.
Therefore, the constant C exists, for which the distance from the initial condition is calculated as $l_{GS}(\bar{X}^{\mu}) \equiv g_{\mu\nu}dx^{\mu}dx^{\nu}$ as

$$0 \leq Cl_{GS}(\bar{X}^{\mu}) \tag{20}$$

being $C \equiv 0$. □

9. Properties from the GS metric

Theorem 0.3 from the work of Munteanu et al. [23] is now newly generalized; the proof is therefore newly written after the work of Munteanu et al. [23].
Theorem 4. *The trivial expanding Ricci Kaehler soliton is a Generalized-Schwarzschild soliton; for this reason, this soliton has only one end.*

Proof. The GS metric is now from a smooth metric space. Also Theorem 0.4 from the work of Munteanu et al. [23] can now be newly proven. □

10. Smooth metric spaces

Now the weight function Ψ is proper; it is now indicated as $\mathcal{F}$.
Theorem 5. *$(\mathcal{M}, G_{\mu}, e^{-\mathcal{F}})$ is a smooth metric space with bounded Bakry-Emery curvature. The metric tensor $G_{\mu\nu}$ is a 'Riemannian manifold with a measure conformal to the Riemannian measure' (as classified from the work of WEY et al. [24] as written in the introductory paragraph of Section 5 ibidem).*

Further Corollaries will be given elsewhere, as to complete the study of the new implications from Theorem 5.

At the moment, the initial conditions are studied, which depend only on the spherical neighborhood of the point.
Corollary 1. *For any initial condition, the weight function $\mathcal{F}$ is majorized as a polynomial function of the distances $l(G_{\mu\nu})$.*

11. Further results

The further Theorem is here stated
Theorem 6. *A trivial Kaehler soliton $(\mathcal{M}, g)$ is connected at infinity.*

12. Discussion

In the work of Cao et al. [7], the growth of the potential function of the complete non-compact shrinking solitons is proven to be at most Euclidean.

The achievement is framed within the implication of the Bishop Theorem about non-compact Riemannian manifolds endowed with non-negative Ricci curvature which admits the same described growth.

The methods followed in the work of Cao et al. [7] consist in finding the suitable majorization and the suitable minorization of the potential function with functions of the radial coordinate (issued form the position vector).

In the work of Perelman [8], a canonical Ricci flow is constructed. The range of change of the radial variable is studied.

In the work of Cao et al. [9], the canonical neighborhood assumptions are scrutinized.

A long-time solution of the Ricci flow with surgery is constructed by induction for the scenario depicted in the work of Perelmann [8].

In the work of Munteanu et al. [23], the topology of gradient Kaehler Ricci solitons is studied. Shrinking Ricci solitons are shown to be connected at infinity, while Kaehler Ricci solitons are demonstrated to obey the same property under the assumption that the potential function be proper.

Furthermore, a 'sharp pointwise lower bound' is found for the weight function as a function of the associated Backry-Emery curvature. The Backry-Emery curvature is appreciated to admit generalizations to more general metric spaces.

The topology of manifolds whose Backry-Emery curvature is bounded from below is studied.

In the work of Wu et al. [25], the classification of the 4-dimensional simply-connected non-compact non-flat shrinking solitons is scrutinized. The analysis is based on the evolution of the eigenvalues of the Ricci curvature.

Some of the methodologies to investigate the here-found objects are recalled.

In the work of Cao et al. [26], curvature estimates of 4-dimensional gradient steady Ricci solitons are calculated.

The case of non-compact 4-dimensional gradient steady Ricci soliton is afforded with $R_{\mu\nu} > 0$ such that the Ricci scalar R reaches its maximum at a given positive $x \in \mathcal{M}^4$. The metric space $(\mathcal{M}^4, g_{\mu\nu}, \hat{f})$ is initially considered. Then, the metrizable space $(\mathcal{M}^4, g_{\mu\nu})$ is considered: in this case, in Theorem 1.1 ibidem, the upper bound of the absolute value of the Riemann tensor is considered as

$$sup_{x\in\mathcal{M}^4} \mid R_{\mu\nu\rho\sigma} \mid\leq C \tag{21}$$

being $C > 0$ a constant

The further hypothesis is requested, that the Ricci scalar R should admit at most a linear decay; in this case, one also has that

$$sup_{x\in\mathcal{M}} \frac{\mid R_{\mu\nu\rho\sigma} \mid}{R} \leq C. \tag{22}$$

The non-flat complete non-compact 4-dimensional gradient steady Ricci solitons are investigated in Theorem 1.2 ibidem.

The existence of a constant a, $0 < a < 1$ is requested such that

$$\mid R_{\mu\nu} \mid\leq CR^a \tag{23}$$

and

$$sup_{x\in\mathcal{M}^4} \mid R_{\mu\nu\rho\sigma} \mid\leq C. \tag{24}$$

The Ricci scalar is then further supposed to admit at most a polynomial decay $\forall 0 < a < 1$ the property is ibidem proven that

$$\mid R_{\mu\nu\rho\sigma} \mid^2\leq CR^a. \tag{25}$$

It is now my purpose to generalize Theorem 1.2 from the work of Cao et al. [26]:
Theorem 7. *Already form the properties of the non-metrized space, one proves that for a non-metrized space the constant* a *qualifying the non-flat complete non-compact steady Ricci soliton equals zero, i.e.,*

$$\mid R_{\mu\nu} \mid\leq CR, \tag{26}$$

and

$$\mid R_{\mu\nu\rho\sigma} \mid^2\leq CR. \tag{27}$$

Theorem 8. *This properties applies also to compact steady Ricci solitons.*

In Lemma 2.1 form the work of Cao et al. [26], the complete gradient steady soliton $(\mathcal{M}^4, g_{\mu\nu}, \hat{f})$ is studied. The following properties are found

$$R = -\Delta\hat{f}, \tag{28a}$$
$$\nabla_\mu R = 2R_\mu{}^\nu\nabla_\nu\hat{f}, \tag{28b}$$
$$R+ \mid \nabla\hat{f} \mid^2= C_0 \tag{28c}$$

for some constant C_0.

Lemma 2.1 from the work of Cao et al. is now here specified as follows
Lemma 1. *A complete gradient steady soliton* $(\mathcal{M}^4, g_{\mu\nu}, \hat{f})$ *admits the following properties*

$$R = -\Delta\hat{f}, \tag{29a}$$
$$\nabla_\mu R = 2R_\mu{}^\nu\nabla_\nu\hat{f}, \tag{29b}$$
$$R+ \mid \nabla\hat{f} \mid^2= 0 \tag{29c}$$

i.e., the constant C_0 *is zero.*

In the work of Chan [27], the n-dimensional complete non-Ricci flat gradient steady Ricci solitons are considered, whose potential function $\hat{f}$ is bounded from above after a constant, and whose Riemann tensor $R_{\mu\nu\rho\sigma}$ is such that

$$lim_{r\rightarrow\infty} \mid R_{\mu\nu\rho\sigma} \mid\leq \frac{1}{5} \tag{30}$$

is considered: in Theorem 2 ibidem, under these hypotheses, the following inequality is proven

$$\mid R_{\mu\nu\rho\sigma} \mid\leq \mathcal{C}e^{-r} \tag{31}$$

with $\mathcal{C}$ a constant, $\mathcal{C} > 0$. It is now my aim to further study the implications of the hypotheses of Theorem 2 from the work of Cao [27]. More in detail, the following new Theorem is given

Theorem 9. *Given an n-dimensional complete non-Ricci flat gradient steady Ricci soliton whose potential function $\hat{f}$ is bounded from above after a constant, and whose Riemann tensor $R_{\mu\nu\rho\sigma}$ is such that*

$$lim_{r\to\infty} \mid R_{\mu\nu\rho\sigma} \mid \leq \mathcal{C} \tag{32}$$

with $\mathcal{C} > 0$ a constant, then

$$lim_{r\to\infty} \mid R_{\mu\nu\rho\sigma} \mid < \mathcal{C} \tag{33}$$

Furthermore, from the work of Chan [27], the estimate is proven that, for a 4-dimensional complete non-Ricci flat gradient steady Ricci soliton whose Ricci scalar R is such that

$$lim_{r\to\infty} = 0 \tag{34}$$

then the Riemann tensor is majorized as

$$\mid R_{\mu\nu\rho\sigma} \mid \leq cR \tag{35}$$

for c a constant, $c > 0$.

It is now my aim to generalize this result with the following new:

Theorem 10. *Given a 4-dimensional complete non-Ricci flat gradient steady Ricci soliton whose Ricci scalar R is such that*

$$lim_{r\to\infty} = 0 \tag{36}$$

then the Riemann tensor is majorized as

$$\mid R_{\mu\nu\rho\sigma} \mid < cR \tag{37}$$

for c a constant, $c > 0$.

Lemma 2 From the work of Chan [27], is here studied; ibidem, the n-dimensional non-Ricci-flat complete steady gradient Ricci soliton with

$$lim_{r\to\infty}\hat{F} = -\infty \tag{38}$$

for which there exists a constant $\mathcal{C}$, $\mathcal{C} > 0$ then the Ricci scalar exhibits the behaviour

$$R \leq \mathcal{C}e^{\hat{F}} \tag{39}$$

on $\mathcal{M}$.

It is now my aim to newly generalize Lemma 2 from the work of Chan [27] as with the following new

Theorem 11. *Given an n-dimensional complete steady gradient Ricci soliton with*

$$lim_{r\to\infty}\hat{F} = -\infty \tag{40}$$

for which there exists a constant $\mathcal{C}$, $\mathcal{C} > 0$ then the Ricci scalar is selected

$$R \leq 0 \tag{41}$$

on $\mathcal{M}$.

and

Theorem 12. *Given an n-dimensional non-Ricci-flat complete steady gradient Ricci soliton with*

$$lim_{r\to\infty}\hat{F} = -\infty \tag{42}$$

for which there exists a constant $\mathcal{C}$, $\mathcal{C} > 0$ then the Ricci scalar is selected

$$R < 0 \tag{43}$$

on $\mathcal{M}$.

In the work of Deng et al. [28], the role of a linear decay in the curvature in the investigation of the rotational invariance is studied.

In the work of Chen et al. [29], the importance of the existence of a maximal Ricci flow trajectory for the work of Hamilton [3] on a compact manifold is underlined as far as the position of the initial conditions of the metric tensor is concerned.

ibidem, the role of the Ricci soliton as fixed point of the Ricci flow is reviewed.

It is one of the aims of the work of Chen et al. [29] to draw the attention on the role of the macroscopic matter fields in the definition of a soliton from a blackhole solution of the Einstein field equations of a blackhole object surrounded of media.

In the work of Qu et al. [30], the connectedness of gradient Ricci solitons is studied; the result is proven, that for any complete gradient Ricci soliton $(\mathcal{M}^4, g_{\mu\nu}, \hat{f})$, with $n > 3$, if

$$\mid R_{\mu\nu} \mid < \frac{n-2}{2\sqrt{n}} \tag{44}$$

then the soliton has one end.

It the work of Cao et al. [31], 4-dimensional gradient steady Ricci solitons are studied to be characterized as either

$$\mid R_{\mu\nu} \mid > 0 \tag{45}$$

or

$$lim_{x\to\infty}R(x) = 0 \tag{46}$$

In the work of Li et al. [32], the result is proven, that any Kaehler Ricci shrinking soliton surface has bounded sectional curvature.

13. Outlook

From Definition 4, the choice of complete manifold as a pseudo-Riemannian manifold is here compared with the Schwarzschild spacetimes containing a blackhole object surrounded of media; as from [4], the media can be the macroscopic solutions of the Einstein Field Equations from the work of Landau et al. [13] dust, perfect fluid, gaseous material: mixtures of them can be considered as well as they are solutions of the Einstein Field Equations.

For these purposes, the following Theorem is here added

Theorem 13. *A Generalized Schwarzschild soliton is in a trivial gradient Ricci soliton. The way is now open to study charged blackhole solutions: in these case, the structure of the Ricci tensor is modified with respect to the non-charged cases, in a manner such that all the definitions will be newly framed.*

Conflict of interest: The author declares no conflict of interest.

References

1. Perelman G. The entropy formula for the Ricci flow and its geometric applications. Available online: https://arxiv.org/abs/math/0211159 (accessed on 20 May 2025).
2. Hamilton RS. Three-manifolds with positive Ricci curvature. Journal of Differential Geometry. 1982; 17(2): 255–306. doi: 10.4310/jdg/1214436922
3. Hamilton RS. Four manifolds with positive curvature operator. Journal of Differential Geometry. 1986; 24(2): 153–179. doi: 10.4310/JDG/1214440433
4. Lecian OM. Non-rotating Spherically-symmetric Blackhole Spacetimes: The Relativistic-Astrophysical configurations. LAP Lambert Academic Publishing; 2024.
5. Lecian OM. Transient Phenomena of Inviscid Accretion Gas-radiation Slim Disc in A Gravitational Potential after Adiabatic Perturbations of the Velocities. International Journal of Mathematics and Computer Research. 2024; 12(11): 4562.
6. Eminenti M, La Nave G, Mantegazza C. Ricci Solitons-The Equation Point of View. Manuscripta Mathematica. 2008; 127: 345–367. doi: 10.1007/s00229-008-0210-y
7. Cao H-D, Zhou D. On complete gradient shrinking Ricci solitons. Journal of Differential Geometry. 2010; 85(2): 175–186. doi: 10.4310/jdg/1287580963
8. Perelmann G. Ricci flow with surgery on three manifolds. Available online: https://www.arxiv.org/abs/math/0303109 (accessed on 20 May 2025).
9. Cao H-D, Zhu XP. A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton-Perelman theory of the Ricci flow. Asian J. Math. 2006; 10: 165. doi: 10.4310/AJM.2006.v10.n2.a2
10. Ali M, Ahsan Z. Geometry of Schwarzschild Soliton. Journal of the Tensor Society. 2013; 7(01): 49–57. doi: 10.56424/jts.v7i01.10467
11. Akbar MM, Woolger E. Ricci soliton and Einstein-scalar field theory. Classical and Quantum Gravity. 2009; 26(5): 55015–55034. doi: 10.1088/0264-9381/26/5/055015
12. Borgiel W. The gravitational field of the Schwarzschild spacetime. Differential Geometry and its Applications. 2011; 29: 5207–5210.
13. Landau LD, Lifshitz EM. The Classical Theory of Fields, 3rd Revised Edition. Pwegamon Press; 1971.
14. Pundeer NA, Ghosh P, Shah HM, et al. Some Solitons on Homogeneous Almost α-Cosymplectic 3-Manifolds and Harmonic Manifolds. Math Notes. 2024; 116: 717–728. doi: 10.1134/S0001434624090293
15. Bunting GL, Masood-ul-Alam AKM. Nonexistence of multiple black holes in asymptotically Euclidean static vacuum space-time. Gen Relat Gravit. 1987; 19: 147–154.
16. Schoen R, Yau S-T. On the proof of the positive mass conjecture in general relativity. Commun. Math. Phys. 1979; 65: 45–76.

17. Masood-ul-Alam AKM. Uniqueness proof of static charged black holes revisited. Classical and Quantum Gravity. 1992; 9(5): L53.
18. Ruback P. A new uniqueness theorem for charged black holes. Quantum Grav. 1988; 5(10): L155.
19. Rimoldi M, Veronelli G. Topology of steady and expanding gradient Ricci solitons via f-harmonic maps. Differential Geometry and its Applications. 2013; 31(5): 623–638. doi: 10.1016/j.difgeo.2013.06.001
20. Razdoburdin DN, Zhuravlev VV. Transient dynamics of perturbations in astrophysical disks. Physics Uspekhi. 2015; 58(11): 1031–1058.
21. Lecian OM. The Generalized Schwarzschild spacetimes with a linear term and a cosmological constant. Universe MPDI. 2024; 10(11): 408. doi: 10.3390/universe10110408
22. Bunting GL. Masood-ul-Alam AKM. Nonexistence of multiple black holes in asymptotically Euclidean static vacuum space-time. Gen. Relat. Grav. 1987; 19: 147–154. doi: 10.1007/BF00770326
23. Munteanu O, Wang J. Topology of Kahler Ricci solitons. Journal of Differential Geometry. 2015; 100(1): 109–128. doi: 10.4310/jdg/1427202765
24. Wey G, Wylie W. Comparison Geometry for the Smooth Metric Measure Spaces. ICCM II, 1-4 (2007). Available online at https://web.math.ucsb.edu/w̃ei/paper/07ICCM.pdf (accessed on 11 May 2025)
25. Wu G, Wu J-Y. Four-dimensional shrinkers with nonnegative Ricci curvature. arXiv preprint. 2025. doi: 10.48550/arXiv.2505.02315
26. Cao H-D, Cui X. Curvature Estimates for Four-Dimensional Gradient Steady Ricci Solitons. The Journal of Geometric Analysis. 2020; 30(1): 511–525.
27. Chan P-Y. Curvature estimates for steady Ricci solitons. Transactions of the American Mathematical Society. 2019; 372(12): 8985–9008.
28. Deng Y, Zhu X. Classification of gradient steady Ricci solitons with linear curvature decay. Sci. China Math. 2020; 63: 135–154. doi: 10.1007/s11425-019-1548-0
29. Chen B-Y, Choudhary MA, Mohammed N, Siddiqi MD. A Comprehensive Review of Solitonic Inequalities in Riemannian Geometry. International Electronic Journal of Geometry. 2024; 17(2): 727–752. doi: 10.36890/iejg.1526047
30. Qu Y, Wu G, When does gradient Ricci soliton have one end? Ann. Glob. Anal. Geom. 2022; 62: 679–691. doi: 10.1007/s10455-022-09868-8
31. Cao H-D, Cui X. Curvature Estimates for Four-Dimensional Gradient Steady Ricci Solitons. J. Geom. Anal. 2020; 30: 511–525. doi: 10.1007/s12220-019-00152-z
32. Li Y, Wang B. On Kaehler Ricci shrinker surfaces. e-print. 2025. doi: 10.48550/arXiv.2301.09784

Appendix A

Geometrical objects

The geometrical objects used in the present paper are here studied.

For the Schwarzschild spacetime of line element 8, the Ricci scalar and the Ricci tensor are vanishing.

For the Generalized Schwarzschild spacetime of line element 12, i.e. as in the proof of the Birkhoff Theorem for Generalized Schwarshild spacetimes as in the work of Lecian [21] the geometrical objects are written as the Ricci scalar

$$R = -\frac{1}{2}\frac{1}{r^2}(r\Psi + r - r_S)\left[r^2\frac{d^2\Psi}{dr^2} + 4r\frac{d\psi}{dr} + 2\Psi\right] \tag{A1}$$

and the non-vanishing components of the Ricci tensor are

$$R_{tt} = -\frac{1}{2}\frac{1}{r^2}\left[r\frac{d^2\Psi}{dr^2} + 2\frac{d\Psi}{dr}\right], \tag{A2a}$$

$$R_{rr} = \frac{1}{2}\frac{1}{r\Psi + r - r_S}\left[r\frac{d^2\Psi}{dr^2} + 2\frac{d\Psi}{dr}\right], \tag{A2b}$$

$$R_{\theta\theta} = r\frac{d\Psi}{dr} + \Psi, \tag{A2c}$$

$$R_{\phi\phi} = (sin\theta)^2 R_{\theta\theta} \tag{A2d}$$

The constraint on the parameter space of the model

$$r\Psi + r - r_S \neq 0 \tag{A3}$$

must be imposed (from the component R_{rr} Equation (A2b)).

For the spherically-symmetric non-rotating blackhole endowed with a Schwarzschild solid angle Equation (15), the Ricci scalar is

$$R = -\frac{1}{r^2}\left[r^2\frac{d^2 f}{dr^2} + 4\frac{df}{dr} + 2f - 2\right] \tag{A4}$$

and the non-vanishing components of the Riemann tensor are

$$R_{tt} = -\frac{1}{2}\frac{1}{r^2}\left[r\frac{d^2 f}{dr^2} + 2\frac{df}{dr}\right], \tag{A5a}$$

$$R_{rr} = \frac{1}{2}\frac{1}{rf}\left[r\frac{d^2 f}{dr^2} + 2\frac{df}{dr}\right], \tag{A5b}$$

$$R_{\theta\theta} = r\frac{df}{dr} + f - 1, \tag{A5c}$$

$$R_{\phi\phi} = (sin\theta)^2 R_{\theta\theta} \tag{A5d}$$