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Opinion

Explicit and accurate solutions for the Benney equation

Yuan-Xi Xie

School of Physics and Electronic Science, Hunan Institute of Science and Technology, Yueyang 414000, China; xieyuanxi66@126.com

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Abstract: The Benney equation arises from many different physical contexts as an appropriately real physical model equation involving a lot of effects of dispersion, dissipation, nonlinearity, and instability. As a result, it is a very important and challenging theme to search for the explicit and accurate traveling wave solutions of the Benney equation. In this paper, by introducing an ansatz solution with two E-exponential functions, we have made some improvements to the trial function approach for solving three NPDEs proposed by Xie and Tang. On this basis, we have put forward a direct trial function approach to search for the explicit and accurate traveling wave solutions of NEEs. We have demonstrated its effectiveness by applying it to the Benney equation. Therefore, a series of more general explicit and accurate traveling wave solutions to the Benney equation, comprising the solitary wave solutions and the singular traveling wave solutions, are successfully derived in a forthright and concise way. The obtained results are completely consistent with those given in the existing references. In addition, compared with the proposed approaches in the existing references, the technique described herein seems to be less calculative. Our approach may provide a novel way of thinking for solving NEEs. We firmly believe that the method used herein may also be applied to search for the explicit and accurate traveling wave solutions to other NEEs. We plan to extend this technique to search for the explicit and accurate traveling wave solutions of other NEEs.

Keywords: Benney equation; direct trial function approach; traveling wave solution; solitary wave solution

1. Introduction

The explicit and accurate solutions for nonlinear evolution equations (NEEs for short) play a very important part in nonlinear science, especially in nonlinear physical science, since these solutions not only may well characterize various real natural phenomena, such as propagation with a finite speed, solitons, and vibrations, but also may give us insight into the physical essences of the problems. On account of this reason, the construction of the explicit and accurate solutions of NEEs has become one of the most important and essential tasks in nonlinear physics science. On account of the complexity of nonlinear systems, it is often very difficult to search for the explicit and accurate solutions of a real nonlinear physical model equation. Fortunately, a large number of powerful and efficient techniques of searching for the explicit and accurate solutions to NEEs have been developed. Among them are the hyperbolic tangent function expansion method [1,2], the Jacobi elliptic function expansion method [3,4], the trial function method [5], the combination equation expansion method [6], the function transformation expansion method [7], the trigonometric function expansion method [8], the new extended auxiliary equation method and the generalized Kudryashov method [9], the modified Sine-Gordon equation expansion method [10], the generalized Riccati equation mapping and the modified F-function expansion method [11], the new Riccati equation expansion method [12], the generalized exponential rational function (GERF) expansion method [13], the modified Sardar sub-equation method [14], the novel analytical function expansion method [15], the Hirota bilinear operator method [16], the Khater III and improved Kudryashov method [17], and so on. However, not all the above approaches are generally applicable to searching for the explicit and accurate solutions of all types of NEEs. As a consequence, it is still a very important and challenging task to proceed to casting around for all kinds of more preferable and efficient techniques to investigate the explicit and accurate solutions of NEEs.

In the present paper, by introducing an ansatz solution with two E-exponential functions $e^{k_1\xi}$ and $e^{k\xi}$, we have made some improvements to the trial function technique of solving three NPDEs presented by Xie and Tang. On this basis, we have put forward a direct trial function approach to search for the explicit and accurate traveling wave solutions of NEEs. As a consequence, a series of explicit and accurate traveling wave solutions to the Benney equation are easily obtained by virtue of this strategy.

This paper is arranged as follows:

In Section 1, we overview a lot of methods of searching for the explicit and accurate solutions of NEEs in a simple way.

In Section 2, we briefly describe our direct trial function technique.

In Section 3, we make use of this technique to construct a great deal of explicit and accurate traveling wave solutions for the Benney equation.

In Section 4, we make a summary of this paper and make some necessary remarks.

2. Outline of the presented method

The fundamental idea of our approach is as follows. Let us take into account the following known NEE with respect to two independent variables, x and t.

$$P(u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial t^2},\frac{\partial^2 u}{\partial x^2},\cdots)=0 \tag{1}$$

Generally speaking, the left-hand side of Equation (1) is a polynomial with regard to u and its various partial derivatives.

Our fundamental aim is to search for the explicit and accurate traveling wave solutions for Equation (1) in a systematic and concise way. We have noticed that Xie and Tang [5] have put forward a unified trial function method to search for the explicit and accurate solutions of three NEEs by introducing two trial functions recently. This approach seems to be slightly complicated in that there are two trial functions in this approach that are hard to select in a usual manner. In order to modify this approach and to make use of it to solve Equation (1) more easily, here we straightforwardly presume that Equation (1) is of the following ansatz solution with two E-exponential functions $e^{k_1\xi}$ and $e^{k\xi}$ by our careful and repeated considerations.

$$u=u_0+\frac{ae^{k_1\xi}}{(b+e^{k\xi})^d},\qquad 0\le k_1\le kd \tag{2}$$

in which $\xi = x - ct$, and u_0, a, b, c, d, k, and k_1 are undetermined constants.

In what follows, let us briefly introduce the main steps of making use of Equation (2) to solve Equation (1).

Firstly, the constant d can be determined by means of substituting Equation (2) into Equation (1) and with the aid of partial balance between the highest order derivative term and the highest degree nonlinear term in Equation (1).

Secondly, the specific form of the ansatz solution Equation (2) can be easily get by means of substituting d into Equation (2).

Thirdly, substituting the specific form of the ansatz solution Equation (2) into Equation (1) gives rise to a set of algebraic equations because the coefficients of all $e^{k_1\xi}$ and $e^{k\xi}$ have to vanish.

Finally, the basic explicit and accurate solutions of Equation (1) can be readily found with the aid of solving this set of algebraic equations.

In the following, we would like to make use of the procedure described above to search for the explicit and accurate traveling wave solutions of the Benney equation.

3. Application to the Benney equation

The celebrated Benney equation in concern reads.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta u\frac{\partial^3 u}{\partial x^3} + \gamma\frac{\partial^4 u}{\partial x^4} = 0 \tag{3}$$

which arises from many different physical contexts as an appropriately real model equation involving a lot of effects of dispersion, dissipation, nonlinearity, instability, and so on, and where α, β, and γ are arbitrary constants with $\alpha\beta\gamma \neq 0$.

To start with, let us determine the constant d of the ansatz solution Equation (1) by means of partial balance between the highest order derivative term and the highest degree nonlinear term in Equation (3). Thanks to $0 \leq k_1 \leq kd$, we can take the order of u as:

$$O(u) = d \tag{4}$$

Then it is not hard to derive that:

$$O(\frac{\partial^n u}{\partial x^n}) = d + n \tag{5}$$

Partial balance between the highest order derivative term and the highest degree nonlinear term in Equation (3) gives rise to the following result:

$$O(\frac{\partial^4 u}{\partial x^4}) = O(u\frac{\partial u}{\partial x}) \tag{6}$$

From which it follows that:

$$d = 3 \tag{7}$$

So, the ansatz solution Equation (2) can be rewritten as:

$$u = \frac{ae^{k_1\xi}}{(b + e^{k\xi})^3}, \qquad 0 \leq k_1 \leq 3k \tag{8}$$

In view of Equation (8), it is not hard to deduce that:

$$\frac{\partial u}{\partial t}=\frac{ace^{k_1\xi}[(3k-k_1)e^{k\xi}-bk_1]}{(b+e^{k\xi})^4} \tag{9}$$

$$\frac{\partial u}{\partial x}=-\frac{ae^{k_1\xi}[(3k-k_1)e^{k\xi}-bk_1)}{(b+e^{k\xi})^4} \tag{10}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{ae^{k_1\xi}[(2bk_1^2-3bk^2-6bkk_1)e^{k\xi}+(9k^2+k_1^2-6kk_1)e^{2k\xi}+b^2k_1^2]}{(b+e^{k\xi})^5} \tag{11}$$

$$\begin{aligned}\frac{\partial^3 u}{\partial x^3}&=\frac{ae^{k_1\xi}[b^3k_1^3+(3b^2k_1^3-3b^2k^3-9b^2k^2k_1-9b^2kk_1^2)e^{k\xi}]}{(b+e^{k\xi})^6}\\&+\frac{ae^{k_1\xi}[(30bk^3+18bk^2k_1-18bkk_1^2+3bk_1^3)e^{2k\xi}+(27k^2k_1-27k^3-9k_1}{(b+e^{k\xi})^6}\end{aligned} \tag{12}$$

$$\begin{aligned}&\frac{\partial^4 u}{\partial x^4}\\&=\frac{ae^{k_1\xi}[b^4k_1^4-(3b^3k^4+12b^3k^3k_1+18b^3k^2k_1^2+12b^3kk_1^3-4b^3k_1^4)e^{k\xi}]}{(b+e^{k\xi})^7}\\&+\frac{ae^{k_1\xi}(75b^2k^4+108b^2k^3k_1-36b^2kk_1^3+18b^2k^2k_1^2+6b^2k_1^4)e^{2k\xi}}{(b+e^{k\xi})^7}\\&+\frac{ae^{k_1\xi}(12bk^3k_1-201bk^4+90bk^2k_1^2-36bkk_1^3+4bk_1^4)e^{3k\xi}}{(b+e^{k\xi})^7}\\&+\frac{ae^{k_1\xi}(81k^4-108k^3k_1+54k^2k_1^2-12kk_1^3+k_1^4)e^{4k\xi}}{(b+e^{k\xi})^7}\end{aligned} \tag{13}$$

Substituting Equations (8)–(13) into Equation (3) results in:

$$\begin{aligned}&ab^4k_1(k_1\alpha+k_1^2\beta+k_1^3\gamma-c)e^{k_1\xi}+a^2bk_1e^{2k_1\xi}+ab^3(3ck-4ck_1-3k^2\alpha-6kk_1\alpha\\&+4k_1^2\alpha-3k^3\beta-9k^2k_1\beta-9kk_1^2\beta+4k_1^3\beta-3k^4\gamma-12k^3k_1\gamma-18k^2k_1^2\gamma-12kk_1^3\gamma+4k_1^4\gamma)e^{(k+k_1)\xi}\\&+ab^2(9ck-6ck_1+3k^2\alpha-18kk_1\alpha+k_1^2\alpha+27k^3\beta+9k^2k_1\beta-27kk_1^2\beta+6k_1^3\beta+75k^4\gamma+\\&108k^3k_1\gamma+18k^2k_1^2\gamma-36kk_1^3\gamma+6k_1^4\gamma)e^{(2k+k_1)\xi}+ab(9ck-4ck_1+15k^2\alpha-18kk_1\alpha+4k_1^2\alpha+3k^3\beta\\&+45k^2k_1\beta-27kk_1^2\beta+4k_1^3\beta-201k^4\gamma+12k^3k_1\gamma+90k^2k_1^2\gamma-36kk_1^3\gamma+4k_1^4\gamma)e^{(3k+k_1)\xi}\\&+a(3ck-ck_1+9k^2\alpha-6kk_1\alpha+k_1^2\alpha-27k^3\beta+27k^2k_1\beta-9kk_1^2\beta+k_1^3\beta+81k^4\gamma-108k^3k_1\gamma+\\&54k^2k_1^2\gamma-12kk_1^3\gamma+k_1^4\gamma)e^{(4k+k_1)\xi}+a^2(k_1-3k)e^{(k+2k_1)\xi}+u_0=0\end{aligned} \tag{14}$$

For Equation (14), there are four cases that need to be considered: Case (A): $k_1=0$, Case (B): $k_1=k$, Case (C): $k_1=2k$, Case (D): $k_1=3k$. We shall give detailed discussions below.

Case (A): $k_1=0$.

In this case, Equation (13) can be converted to:

$$\begin{aligned}&3ak(b^3c-a-b^3k\alpha-b^3k^2\beta-b^3k^3\gamma)e^{k\xi}+3ab^2k(3c+k\alpha+9k^2\beta+25k^3\gamma)e^{2k\xi}\\&+3abk(3c+5k\alpha+k^2\beta-67k^3\gamma)e^{3k\xi}+3ak(c+3k\alpha-9k^2\beta+27k^3\gamma)e^{4k\xi}+u_0=0\end{aligned} \tag{15}$$

Because of $e^{jk\xi}\neq 0$ ($j=1,2,3,4$), we acquire the following system of nonlinear algebraic equations from Equation (15).

$$b^3c - a - b^3k\alpha - b^3k^2\beta - b^3k^3\gamma = 0 \tag{16}$$

$$3c + k\alpha + 9k^2\beta + 25k^3\gamma = 0 \tag{17}$$

$$3c + 5k\alpha + k^2\beta - 67k^3\gamma = 0 \tag{18}$$

$$c + 3k\alpha - 9k^2\beta + 27k^3\gamma + u_0 = 0 \tag{19}$$

Solving the above nonlinear algebraic equation system with the aid of Maple, we obtain the following results:

$$a = -\frac{5b^3\beta^3}{72\gamma^2},\ c = -\frac{5\beta^3}{144\gamma^2},\ k = \frac{\beta}{12\gamma},\ u_0 = 0,\ b = \text{arbitraryconstant} \tag{20}$$

with the constraint $\alpha = \frac{47\beta^2}{144\gamma}$.

Plugging Equation (20) into Equation (8), we acquire the basic traveling wave solution of the Benney Equation (3) in the following form:

$$u_1 = -\frac{5b^3\beta^3}{72\gamma^2(b + e^{\frac{\beta}{12\gamma}\xi})^3} + c + \frac{5\beta^3}{144\gamma^2} \tag{21}$$

Taking advantage of the following hyperbolic identical equation

$$\frac{1}{e^{2x} + 1} = \frac{1}{2}(1 - tanh\, x) \tag{22}$$

and letting $b = 1$ in Equation (21), we obtain the solitary wave solution of the Benney Equation (3) in the following form:

$$u_2 = -\frac{5\beta^3}{576\gamma^2}(1 - tanh\frac{\beta}{24\gamma}\xi)^3 + c + \frac{5\beta^3}{144\gamma^2} \tag{23}$$

The solitary wave solution Equation (23) of the Benney Equation (3) is of definite physical meaning in that it may aid in explaining nonlinear wave phenomena of diffusion in fluid mechanics. In addition, it is worth presenting the physical motivation by plotting a figure to the solitary wave solution Equation (23) of the Benney Equation (3). We shall leave this issue for further research in the future.

In the same way, taking advantage of the following hyperbolic identical equation:

$$\frac{1}{e^{2x} - 1} = \frac{1}{2}(coth\, x - 1) \tag{24}$$

and letting $b = -1$ in Equation (21), we possess the so-called singular traveling wave solution for the Benney Equation (3) in the following form:

$$u_3 = -\frac{5\beta^3}{576\gamma^2}(1 - coth\frac{\beta}{24\gamma}\xi)^3 + c + \frac{5\beta^3}{144\gamma^2} \tag{25}$$

Case (B): $k_1 = k$

Utilizing the similar program as before, we find the basic traveling wave solution to the Benney Equation (3) in the following form:

$$u_4 = \frac{15b^2\beta^3 e^{\frac{\beta}{4\gamma}\xi}}{8\gamma^2(b + e^{\frac{\beta}{4\gamma}\xi})^3} + c - \frac{3\beta^3}{32\gamma^2} \tag{26}$$

with the constraint $\alpha = \frac{\beta^2}{16\gamma}$.

Taking advantage of the previous identical Equation (22) and the following hyperbolic identical equation:

$$\frac{e^{2x}}{(e^{2x}+1)^2} = \frac{1}{4} sec\, h^2\, x \tag{27}$$

and letting $b = 1$ in Equation (26), we acquire the solitary wave solution of the Benney Equation (3) as follows:

$$u_5 = \frac{15\beta^3}{64\gamma^2} sec\, h^2 \frac{\beta}{8\gamma}\xi(1 - tanh\frac{\beta}{8\gamma}\xi) + c - \frac{3\beta^3}{32\gamma^2} \tag{28}$$

Making use of the following hyperbolic identical equation:

$$sec\, h^2\, x = \frac{2}{cosh 2x + 1} \tag{29}$$

Then Equation (20) can be changed to:

$$u_6 = \frac{15\beta^3}{32\gamma^2} \frac{1}{cosh\frac{\beta}{8\gamma}\xi + 1}(1 - tanh\frac{\beta}{8\gamma}\xi) + c - \frac{3\beta^3}{32\gamma^2} \tag{30}$$

In the same way, taking advantage of the foregoing identical Equation (24) and the following hyperbolic identical equation:

$$\frac{e^{2x}}{(e^{2x}-1)^2} = \frac{1}{4} csc\, h^2\, x \tag{31}$$

and letting $b = -1$ in Equation (26), we get the so-called singular traveling wave solution for the Benney Equation (3) in the following form:

$$u_7 = -\frac{15\beta^3}{64\gamma^2} csc\, h^2 \frac{\beta}{8\gamma}\xi(1 - coth\frac{\beta}{8\gamma}\xi) + c - \frac{3\beta^3}{32\gamma^2} \tag{32}$$

Making use of the following hyperbolic identical equation

$$csc\, h^2\, x = \frac{2}{cosh 2x - 1} \tag{33}$$

Then Equation (32) can be rewritten as:

$$u_8 = -\frac{15\beta^3}{32\gamma^2} \frac{1}{cosh\frac{\beta}{4\gamma}\xi - 1} csc\, h^2 \frac{\beta}{8\gamma}\xi(1 - coth\frac{\beta}{8\gamma}\xi) + c - \frac{3\beta^3}{32\gamma^2} \tag{34}$$

Case (C): $k_1 = 2k$.

Making use of the similar program as above, we find the basic traveling wave solution to the Benney Equation (3) in the following form:

$$u_9 = \frac{15b\beta^3 e^{-\frac{\beta}{2\gamma}\xi}}{8\gamma^2(b + e^{-\frac{\beta}{4\gamma}\xi})^3} + c - \frac{3\beta^3}{32\gamma^2} \tag{35}$$

with the constraint $\alpha = \frac{\beta^2}{16\gamma}$.

Taking advantage of the former identical Equation (27) and the following hyperbolic identical equation:

$$\frac{e^{2x}}{e^{2x} + 1} = \frac{1}{2}(1 + tanh\, x) \tag{36}$$

and letting $b = 1$ in Equation (35), we obtain the same solution as Equation (28).

In the same way, taking advantage of the former identical Equation (31) and the following hyperbolic identical equation:

$$\frac{e^{2x}}{e^{2x} - 1} = \frac{1}{2}(1 + coth\, x) \tag{37}$$

and letting $b = -1$ in Equation (35), we get the same solution as Equation (32).

Case (D): $k_1 = 3k$.

Taking advantage of the same procedure as above, we find the basic traveling wave solution for the Benney Equation (3) in the following form:

$$u_{10} = -\frac{5\beta^3 e^{-\frac{\beta}{4\gamma}\xi}}{72\gamma^2(b + e^{-\frac{\beta}{12\gamma}\xi})^3} + c + \frac{5\beta^3}{144\gamma^2} \tag{38}$$

with the constraint $\alpha = \frac{47\beta^2}{144\gamma}$.

Taking advantage of the previous identical Equation (22) and letting $b = 1$ in Equation (38), we acquire the same solution as Equation (23).

In the same way, taking advantage of the former identical Equation (24) and letting $b = -1$ in Equation (38), we get the same solution as Equation (25).

Apparently, the solutions u_1, u_2 and u_3 are in agreement with those obtained in reference [7]. It is worth noting that there is an error (maybe a misprint) in reference [7], namely, $\frac{\beta}{8\gamma}\xi$ in Equations (50) and (51) in reference [7] should be $\frac{\beta}{24\gamma}\xi$. The solution u_5 is equivalent to one given in reference [8] as well.

Finally, it is worth pointing out that if all "c" in the above solutions is replaced with "$-c$", in this way we can obtain other lots of explicit and accurate traveling wave solutions to the Benney equation. Here we do not list them one by one for the limit of length.

4. Conclusions and remarks

In a word, by presuming an ansatz solution with two E-exponential functions $e^{k_1\xi}$ and $e^{k\xi}$, we have made some modifications to the trial function approach of solving three NPDEs put forward by Xie and Tang. On this basis, we have come up with a straightforward trial function approach to looking for the explicit and accurate traveling wave solutions to NEEs. We have demonstrated its effectiveness by applying it to the Benney equation. Therefore, a series of more general explicit and accurate traveling wave solutions of the Benney equation, comprising the solitary wave solutions and the singular traveling wave solutions, are successfully obtained in a forthright and concise way. The obtained results are the same as those given in the existing reference. On account of the Benney equation being real physical models, solving it has received much attention from many authors, and quite a lot of papers have investigated its explicit and accurate traveling wave solutions. However, compared with some proposed approaches in the literature, the above technique described herein seems to be relatively concise and straightforward and less calculative. For example, there are two trial functions in the trial function method put forward by Xie and Tang [5], and it needs to perform more tedious calculations. However, there is one trial function in our approach, and it certainly needs to perform fewer calculations. In addition, it should be pointed out that some solutions found herein are of definite physical meaning. For example, the solitary wave solution Equation (23) of the Benney Equation (3) may aid in explaining nonlinear wave phenomena of diffusion in fluid mechanics. Furthermore, our technique may provide a new way of thinking for solving NEEs. We are sure that the approach used herein may also be used to search for the explicit and accurate traveling wave solutions of other NEEs. We intend to develop this procedure to look for the explicit and accurate traveling wave solutions for other NEEs. Unfortunately, the approach used herein cannot currently be applied to solve NEEs with variable coefficients. Therefore, how to make some improvements to this technique so that it can be used to solve NEEs with variable coefficients is worth further research in the future.

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