

# On galaxies of sequences of matrix Pythagorean triples and completely Pythagorean maps

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**Abstract:** We develop an algorithm that allows us to construct the sequences of matrix Pythagorean triples of any size. We prove that there exists an infinite number of galaxies of sequences of matrix Pythagorean triples. We construct the semiring of sequences of matrix Pythagorean triples called the astral body of the set  $M_m(\mathbb{N})$  associated to a galaxy. We show that every galaxy of sequences of matrix Pythagorean triples is associated with a semiring, and every semiring is associated with a homomorphism of semirings. We construct the astral body of the set of complex polynomials over the unit disk  $\mathbb{D}$ . We construct the semiring of astral bodies of the set  $M_m(\mathbb{N})$  associated with several galaxies. We also introduce the sequences of completely Pythagorean maps over  $\mathbb{N}^3$ .

**Keywords:** matrices; polynomials; model theory; Mouanda's choice function; semiring

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## 1. Introduction and main result

The Pythagorean theorem states as follows: For any right-angled triangle, the square of the hypotenuse  $z$  equals the sum of squares of the two (shorter) leg lengths  $x$  and  $y$ , which is written as  $x^2 + y^2 = z^2$  [1–6]. It is well known that the equation  $x^2 + y^2 = z^2$  has many integer solutions, for instance (3, 4, 5) and (5, 12, 13). Around 1500 B.C., the Babylonians were aware of the solution (4961, 6480, 8161) and the Egyptians knew the solutions (148, 2736, 2740) and (514, 66048, 66050). Also Greek mathematicians were attracted to the solutions of this equation. We notice that this equation has sequences of complex number solutions

$$(1 + 2i \times a^k, 2i \times a^k - 2 \times a^{2k}, 1 + 2i \times a^k - 2 \times a^{2k}), a \in \mathbb{C}, k \in \mathbb{N}$$

and matrix solutions

$$\left( \left( \begin{pmatrix} 1 & 2 & 4 & 6 \\ 2 & 5 & 0 & 2 \\ 0 & 4 & 7 & 2 \\ 2 & 2 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 8 & 20 & 28 & 18 \\ 8 & 16 & 8 & 14 \\ 6 & 26 & 28 & 14 \\ 6 & 18 & 24 & 16 \end{pmatrix}, \begin{pmatrix} 9 & 20 & 28 & 18 \\ 8 & 17 & 8 & 14 \\ 6 & 26 & 29 & 14 \\ 6 & 18 & 24 & 17 \end{pmatrix} \right).$$

Pythagorean triples have been known and developed since ancient times with the oldest record dating back to 1900 BC [7]. Finding methods for generating Pythagorean triples has been of great interest to mathematicians since the Babylonians (from 1900 to 1600 BC). In the literature, there are three classical methods of generating Pythagorean

triples. Namely, Pythagoras’ method (c. 540 BC), Plato’s formula (c. 380 BC) [8, 9] and Euclid’s formula (c. 300 BC) [10]. There are several post-classical methods which are Stifel’s method (1544) [11] and Ozanam’s method (1694) [12]. There are two modern methods which are Portia’s method and Dickson’s method [13]. Several methods of generating Pythagorean triples have been introduced these recent years by many authors [14,15]. In 2019, Overmars, Ntogramatzidis and Venkatraman introduced a method that allowed them to generate all the Pythagorean triples by using a new different perspective [16]. In 2021, Mouanda introduced a new method of generating the galaxies of sequences of Pythagorean triples of positive integers. This led to the introduction of the new theory called ”Galaxy Number Theory”. This new theory provides us a better understanding of laws and structures of different universes [17]. In 1934, Berggren (1934) showed that there are  $3 \times 3$  matrices that may be used to generate one Pythagorean triple from another [18]. In 2019, Arnold and Eydelzon introduced a parametrization for the Pythagorean matrices, i.e., integer-valued matrices A, B, and C such that  $A^2 + B^2 = C^2$  [19].

Pythagorean triples have many applications in cryptography.

In this paper, we introduce a new method of generating matrix Pythagorean triples of any size. Mouanda’s choice function for matrices allows us to construct matrix Pythagorean triples.

**Theorem 1.** *There exists an infinite number of sequences of galaxies*

$$(X_k, Y_k, Z_k)_{k \in \mathbb{N}}$$

*of matrix triples with positive integers as entries such that*

$$X_k^2 + Y_k^2 = Z_k^2, k, m \in \mathbb{N}, X_k, Y_k, Z_k \in M_m(\mathbb{N}), X_k Y_k Z_k \neq 0.$$

We construct the semiring of sequences of matrix Pythagorean triples called the astral body of the set  $M_m(\mathbb{N})$  associated to a galaxy. We show that every galaxy of sequences of matrix Pythagorean triples is associated with a semiring and every semiring of sequences of matrix Pythagorean triples is associated with a homomorphism of semirings. We construct the astral body of the set of complex polynomials over the unit disk  $\mathbb{D}$ . We investigate the motion of galaxies in terms of jumps with no timing. We introduce, for the first time, the idea of completely Pythagorean maps. We construct sequences of completely Pythagorean maps over  $\mathbb{N}^3$ .

## 2. Preliminaries

**Definition 1.** *Let  $x, y, z \in \mathbb{C}$  be complex numbers. Denote by*

$$(x, y, z)^n = (x^n, y^n, z^n), n = \frac{p}{q}, p, q \in \mathbb{N}, q \neq 0.$$

*The triple  $(x^n, y^n, z^n)$  is called the triple  $(x, y, z)$  to the power  $n$ .*

**Definition 2.** *Let  $x, y, z \in \mathbb{C}$  be complex numbers. Denote by*

$$\alpha(x, y, z) = (\alpha x, \alpha y, \alpha z), (x, y, z) + (a, b, c) = (x + a, y + b, z + c).$$

Let  $f : \mathbb{N}^3 \rightarrow \mathbb{N}$  be a function of three variables. Define by

$$\mathcal{F}(\mathbb{N}^3) = \{(x, y, z) : f(x, y, z) = 0\}.$$

The set  $\mathcal{F}(\mathbb{N}^3)$  is called the universe of triples of positive integers. Every element of the set  $\mathcal{F}(\mathbb{N}^3)$  is called a planet. The equation  $f(x, y, z) = 0$  is called the stability law of the universe  $\mathcal{F}(\mathbb{N}^3)$ .

**Example 1.** Let  $f_{n,m,k} : \mathbb{N}^3 \rightarrow \mathbb{N}$  be a function of three variables such that

$$(x, y, z) \mapsto f_{n,m,k}(x, y, z) = x^n + y^m - z^k.$$

In this case,

$$\mathcal{F}_{n,m,k}(\mathbb{N}^3) = \{(x, y, z) \in \mathbb{N}^3 : f_{n,m,k}(x, y, z) = 0\} = \{(x, y, z) \in \mathbb{N}^3 : x^n + y^m = z^k\}.$$

Fermat's Last Theorem allows us to say that  $\mathcal{F}_{n,n,n}(\mathbb{N}^3) = \{\}$ , with  $n \geq 3$ .

**Definition 3.** A universe of degree  $\frac{p}{q}$  of the algebra  $\mathcal{B}$  is the set  $\mathcal{F}_{\frac{p}{q}, \frac{p}{q}, \frac{p}{q}}(\mathcal{B})$  of triples  $(x, y, z)$  of elements of  $\mathcal{B}$  that satisfy the law of stability

$$x^{\frac{p}{q}} + y^{\frac{p}{q}} = z^{\frac{p}{q}}, xyz \neq 0, p, q \in \mathbb{N}, q \neq 0.$$

The element  $(x, y, z)$  is called a star (or a planet) of the universe  $\mathcal{F}_{\frac{p}{q}, \frac{p}{q}, \frac{p}{q}}(\mathcal{B})$ . Every sequence  $(x_k, y_k, z_k)_{k \geq 0}$  of elements of the universe  $\mathcal{F}_{\frac{p}{q}, \frac{p}{q}, \frac{p}{q}}(\mathcal{B})$  is called a planet system of elements of  $\mathcal{B}$ .

Let

$$M_n(\mathbb{C}) = \left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \ddots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \ddots & a_{2,n} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_{n-1,1} & \ddots & \ddots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \ddots & \ddots & a_{n,n-1} & a_{n,n} \end{pmatrix} : a_{i,j} \in \mathbb{C} \right\},$$

be the set of  $n$ -by  $n$  complex matrices. It follows that

$$M_n(\mathbb{N}) = \left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \ddots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \ddots & a_{2,n} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_{n-1,1} & \ddots & \ddots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \ddots & \ddots & a_{n,n-1} & a_{n,n} \end{pmatrix} : a_{i,j} \in \mathbb{N} \right\},$$

and

$$M_n(\mathbb{Z}) = \left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1,1} & \cdots & \cdots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & \cdots & a_{n,n-1} & a_{n,n} \end{pmatrix} : a_{i,j} \in \mathbb{Z} \right\}.$$

Therefore,

$$M_n(\mathbb{N}) \subset M_n(\mathbb{Z}) \subset M_n(\mathbb{Q}) \subset M_n(\mathbb{R}) \subset M_n(\mathbb{C}).$$

The set

$$\mathcal{F}_{\frac{p}{q}, \frac{p}{q}, \frac{p}{q}}(M_m(\mathbb{C})) = \left\{ (X, Y, Z) \in M_m(\mathbb{C})^3 : X^{\frac{p}{q}} + Y^{\frac{p}{q}} = Z^{\frac{p}{q}}, XYZ \neq 0 \right\}, p, q \in \mathbb{N}, q \neq 0,$$

is called the complex matrix universe of degree  $\frac{p}{q}$ . In particular, the set

$$\mathcal{F}_{n,n,n}(M_m(\mathbb{N})) = \left\{ (X, Y, Z) \in M_m(\mathbb{N})^3 : X^n + Y^n = Z^n, XYZ \neq 0 \right\}, n \in \mathbb{N}, n \geq 2,$$

is called the natural matrix universe of degree  $n$ . We are going to construct the elements of the universe  $\mathcal{F}_{2,2,2}(M_m(\mathbb{N}))$ .

**Definition 4.** A matrix triple  $(X, Y, Z) \in M_m(\mathbb{N})^3$  is said to be Pythagorean if  $X^2 + Y^2 = Z^2$ .

The universe  $\mathcal{F}_{2,2,2}(M_m(\mathbb{N}))$  is the set of  $m \times m$ -matrix Pythagorean triples,  $m \in \mathbb{N}$ . Denote by

$$C_*(M_m(\mathbb{C})) = \{h/h : M_m(\mathbb{C}) \longrightarrow \mathbb{C}\},$$

the set of functions over  $M_m(\mathbb{C})$ .

Let  $\Omega(\mathcal{F}_{2,2,2}(M_m(\mathbb{C}))) = \{P : P \subseteq \mathcal{F}_{2,2,2}(M_m(\mathbb{C}))\}$  be the set of all subsets of  $\mathcal{F}_{2,2,2}(M_m(\mathbb{C}))$ .

**Definition 5.** Let  $\mathcal{B}$  be an algebra. A choice  $(m_0(k), n_0(k))$  is said to be an appropriate choice in  $\mathcal{B}$  if  $2m_0(k)(m_0(k) - n_0(k)) = b^2, b \in \mathcal{B}$ .

Let us observe that the appropriate choice of the values of  $m_0(k)$  and  $n_0(k)$  could lead to the construction of the sequences of matrix triples with positive (or negative) integers as entries that satisfy the equation

$$X^2 + Y^2 = Z^2.$$

In 2022, Mouanda proved that in the case of positive integers the terms of the sequence  $(x_k(m_0(k), n_0(k)), y_k(m_0(k), n_0(k)), z_k(m_0(k), n_0(k)))_{k \geq 0}$  defined by

$$\begin{cases} x_k(m_0(k), n_0(k)) = \frac{2(m_0(k)-n_0(k))+\sqrt{8m_0(k)(m_0(k)-n_0(k))}}{2} \\ y_k(m_0(k), n_0(k)) = \frac{2(m_0(k)-n_0(k))+\sqrt{8m_0(k)(m_0(k)-n_0(k))}}{2} + n_0(k) \\ z_k(m_0(k), n_0(k)) = \frac{2(m_0(k)-n_0(k))+\sqrt{8m_0(k)(m_0(k)-n_0(k))}}{2} + m_0(k) \end{cases}$$

satisfy  $x_k(m_0(k), n_0(k))^2 + y_k(m_0(k), n_0(k))^2 = z_k(m_0(k), n_0(k))^2$  with  $(m_0(k), n_0(k))$  as an appropriate choice [17]. In a similar way, we could generate matrix Pythagorean triples by using matrix-appropriate choices.

Let  $\mathcal{O}$  be an observer and let  $Ch(\mathcal{O})$  the set of the appropriate choices  $(m_0(k), n_0(k))$  of the observer  $\mathcal{O}$ . The elements of the set  $Ch(\mathcal{O})$  depend on the different appropriate choices of the observer  $\mathcal{O}$ . For example, if the observer  $\mathcal{O}$  chooses  $(m_0(k), n_0(k)) = (2A^{2k}, 2A^{2k} - I_m) \in Ch(\mathcal{O}), A \in M_m(\mathbb{C})$ , he can compute the fraction

$$\frac{2(m_0(k) - n_0(k)) \pm \sqrt{8m_0(k)(m_0(k) - n_0(k))}}{2} = \frac{2I_m \pm \sqrt{16A^{2k}}}{2} = I_m \pm 2A^k.$$

We can notice that the above choice allows us to construct the sequences  $(X_k(A), Y_k(A), Z_k(A))_{k \in \mathbb{N}}$  defined by

$$\begin{cases} X_k(A) = I_m + 2A^k \\ Y_k(A) = X_k(A) + n_0(k) \\ Z_k(A) = X_k(A) + m_0(A) \\ A \in M_m(\mathbb{C}), k \in \mathbb{N}. \end{cases}$$

In other words,

$$\begin{cases} X_k(A) = I_m + 2A^k \\ Y_k(A) = 2A^k + 2A^{2k} \\ Z_k(A) = I_m + 2A^k + 2A^{2k} \\ A \in M_m(\mathbb{C}), k \in \mathbb{N}. \end{cases}$$

We can check that  $X_k^2(A) + Y_k^2(A) = Z_k^2(A), A \in M_m(\mathbb{C}), k \in \mathbb{N}$ . It is now possible to construct a choice function that depends on the appropriate matrix choices of the observer  $\mathcal{O}$ . Let  $f_{\mathbb{M}} : Ch(\mathcal{O}) \rightarrow \Omega(\mathcal{F}_{2,2,2}(M_m(\mathbb{C})))$  be the model galaxy-valued function defined by

$$f_{\mathbb{M}}(m_0(k), n_0(k)) = \left[ \begin{array}{l} m_0(k) = A^{\beta(k)}, \beta(k), k \in \mathbb{N}, \beta \in C_*(\mathbb{N}) \\ A, m_0(k) - n_0(k) \in M_m(\mathbb{C}) \\ X_k(m_0(k), n_0(k)) = \frac{2(m_0(k) - n_0(k)) + \sqrt{8m_0(k)(m_0(k) - n_0(k))}}{2} \in M_m(\mathbb{C}) \\ Y_k(m_0(k), n_0(k)) = \frac{2(m_0(k) - n_0(k)) + \sqrt{8m_0(k)(m_0(k) - n_0(k))}}{2} + n_0(k) \\ Z_k(m_0(k), n_0(k)) = \frac{2(m_0(k) - n_0(k)) + \sqrt{8m_0(k)(m_0(k) - n_0(k))}}{2} + m_0(k) \end{array} \right],$$

$(m_0(k), n_0(k)) \in Ch(\mathcal{O})$ . This type of function is called Mouanda's choice function for matrices [17].

**Definition 6.** Let  $\mathbb{A}$  be a unital Banach algebra. We say that  $a \in \mathbb{A}$  is invertible if there is an element  $b \in \mathbb{A}$  such that  $ab = ba = 1$ . In this case,  $b$  is unique and written as  $a^{-1}$ . The set

$$Inv(\mathbb{A}) = \{a \in \mathbb{A} : \exists b \in \mathbb{A}, ab = ba = 1\},$$

is a group under multiplication. If  $a$  is an element of  $\mathbb{A}$ , the spectrum of  $a$  is defined as

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda.1 \notin \text{Inv}(\mathbb{A})\},$$

and its spectral radius is defined to be

$$r(a) = \sup \{|\lambda| : \lambda \in \sigma(a)\}.$$

### 3. Proof of the main result

In this section, we develop an algorithm that allows us to construct sequences of matrix Pythagorean triples of any size. Mouanda’s choice function is the main tool. The structures of Pythagorean triples are not unique and some of them allow the construction of the structures of matrix Pythagorean triples. For example, it is straightforward to check that the triples of positive integers  $(x(\alpha, a), y(\alpha, a), z(\alpha, a))$  defined by

$$\begin{cases} x(\alpha, a) = \alpha^4 + 2\alpha^2 a \\ y(\alpha, a) = 2\alpha^2 a + 2a^2 \\ z(\alpha, a) = \alpha^4 + 2\alpha^2 a + 2a^2 \\ \alpha, a \in \mathbb{N}, \end{cases}$$

satisfy

$$x^2(\alpha, a) + y^2(\alpha, a) = z^2(\alpha, a).$$

It is now possible to construct matrix Pythagorean triples by using the construction structure of Pythagorean triples. It is straightforward to notice that the above construction structure of the triples  $(x(\alpha, a), y(\alpha, a), z(\alpha, a))$  allows the construction of a matrix Pythagorean triples structure of any size. For example, suppose that

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 0 & 1 \\ 0 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix}.$$

This matrix generates the construction of an infinite number of matrix Pythagorean triples depending on the values of  $\alpha$ . A small calculation shows that

$$2\alpha^2 A = \begin{pmatrix} 0 & 2\alpha^2 & 4\alpha^2 & 6\alpha^2 \\ 2\alpha^2 & 4\alpha^2 & 0 & 2\alpha^2 \\ 0 & 4\alpha^2 & 6\alpha^2 & 2\alpha^2 \\ 2\alpha^2 & 2\alpha^2 & 4\alpha^2 & 2\alpha^2 \end{pmatrix}, 2A^2 = \begin{pmatrix} 8 & 18 & 24 & 12 \\ 6 & 12 & 8 & 12 \\ 6 & 22 & 22 & 12 \\ 4 & 16 & 20 & 14 \end{pmatrix}.$$

Now, for every  $\alpha \in \mathbb{N}$ , we can compute

$$\begin{cases} X(\alpha) = \alpha^4 I_4 + 2\alpha^2 A \\ Y(\alpha) = 2\alpha^2 A + 2A^2 \\ Z(\alpha) = \alpha^4 I_4 + 2\alpha^2 A + 2A^2. \end{cases}$$

Therefore, for every  $\alpha \in \mathbb{N}$ , one has

$$\left\{ \begin{array}{l} X(\alpha) = \begin{pmatrix} \alpha^4 & 2\alpha^2 & 4\alpha^2 & 6\alpha^2 \\ 2\alpha^2 & \alpha^4 + 4\alpha^2 & 0 & 2\alpha^2 \\ 0 & 4\alpha^2 & \alpha^4 + 6\alpha^2 & 2\alpha^2 \\ 2\alpha^2 & 2\alpha^2 & 4\alpha^2 & \alpha^4 + 2\alpha^2 \end{pmatrix} \\ Y(\alpha) = \begin{pmatrix} 8 & 2\alpha^2 + 18 & 4\alpha^2 + 24 & 6\alpha^2 + 12 \\ 2\alpha^2 + 6 & 4\alpha^2 + 12 & 8 & 2\alpha^2 + 12 \\ 6 & 4\alpha^2 + 22 & 6\alpha^2 + 22 & 2\alpha^2 + 12 \\ 2\alpha^2 + 4 & 2\alpha^2 + 16 & 4\alpha^2 + 20 & 2\alpha^2 + 14 \end{pmatrix} \\ Z(\alpha) = \begin{pmatrix} \alpha^4 + 8 & 2\alpha^2 + 18 & 4\alpha^2 + 24 & 6\alpha^2 + 12 \\ 2\alpha^2 + 6 & \alpha^4 + 4\alpha^2 + 12 & 8 & 2\alpha^2 + 12 \\ 6 & 4\alpha^2 + 22 & \alpha^4 + 6\alpha^2 + 22 & 2\alpha^2 + 12 \\ 2\alpha^2 + 4 & 2\alpha^2 + 16 & 4\alpha^2 + 20 & \alpha^4 + 2\alpha^2 + 14 \end{pmatrix} \end{array} \right.$$

The triples  $(X(\alpha), Y(\alpha), Z(\alpha))$  also satisfy

$$X^2(\alpha) + Y^2(\alpha) = Z^2(\alpha), \alpha \in \mathbb{N}.$$

• Suppose  $\alpha = 1$ . Let us compute the triple  $(X(1), Y(1), Z(1))$ . In this case, we have

$$\left\{ \begin{array}{l} X(1) = \begin{pmatrix} 1 & 2 & 4 & 6 \\ 2 & 5 & 0 & 2 \\ 0 & 4 & 7 & 2 \\ 2 & 2 & 4 & 3 \end{pmatrix} \\ Y(1) = \begin{pmatrix} 8 & 20 & 28 & 18 \\ 8 & 16 & 8 & 14 \\ 6 & 26 & 28 & 14 \\ 6 & 18 & 24 & 16 \end{pmatrix} \\ Z(1) = \begin{pmatrix} 9 & 20 & 28 & 18 \\ 8 & 17 & 8 & 14 \\ 6 & 26 & 29 & 14 \\ 6 & 18 & 24 & 17 \end{pmatrix} \end{array} \right.$$

We can check that  $X(1)^2 + Y(1)^2 = Z(1)^2$ . We can now introduce some examples of sequences of matrix Pythagorean triples and construct galaxies of sequences of matrix Pythagorean triples by using the following galaxy of galaxies model of Pythagorean triples introduced in 2021 by Mouanda [17].

$$Gala(\mathbb{N}, \mathbb{N}) = \left[ \begin{array}{l} x_k(\alpha, a) = \alpha + 2\sqrt{\alpha} \times a^k \\ y_k(\alpha, a) = 2\sqrt{\alpha} \times a^k + 2 \times a^{2k} \\ z_k(\alpha, a) = \alpha + 2\sqrt{\alpha} \times a^k + 2 \times a^{2k} \\ \alpha = r^2, k, a, r \in \mathbb{N}, a \neq 0, r \geq 2 \end{array} \right].$$

For matrices of the set  $M_m(\mathbb{N})$ , one has a galaxy model of matrix Pythagorean triples

$$Gala(\mathbb{N}, M_m(\mathbb{N})) = \left[ \begin{array}{l} X_k(\alpha, A) = \alpha I_m + 2\sqrt{\alpha} \times A^k \\ Y_k(\alpha, A) = 2\sqrt{\alpha} \times A^k + 2 \times A^{2k} \\ Z_k(\alpha, A) = \alpha I_m + 2\sqrt{\alpha} \times A^k + 2 \times A^{2k} \\ \alpha = r^2, k, r \in \mathbb{N}, r \geq 2, A \in M_m(\mathbb{N}) \end{array} \right].$$

**Example 2.** Let us choose  $\alpha = 4, A \in M_m(\mathbb{N})$ . The triples of the sequence

$$(X_k(4, A), Y_k(4, A), Z_k(4, A))_{k \in \mathbb{N}} = (4I_m + 4A^k, 4A^k + 2A^{2k}, 4I_m + 4A^k + 2A^{2k})_{k \in \mathbb{N}},$$

satisfy

$$X_k^2(4, A) + Y_k^2(4, A) = Z_k^2(4, A), k \in \mathbb{N}.$$

We can construct the galaxy

$$Gala(4, M_m(\mathbb{N})) = \left[ \begin{array}{l} X_k(4, A) = 4I_m + 4A^k \\ Y_k(4, A) = 4A^k + 2A^{2k} \\ Z_k(4, A) = 4I_m + 4A^k + 2A^{2k} \\ k \in \mathbb{N}, A \in M_m(\mathbb{N}), \end{array} \right], m \in \mathbb{N}.$$

**Example 3.** Let us choose  $\alpha = 9, A \in M_m(\mathbb{N})$ . The triples of the sequence

$$(X_k(9, A), Y_k(9, A), Z_k(9, A))_{k \in \mathbb{N}} = (9I_m + 6A^k, 6A^k + 2A^{2k}, 9I_m + 6A^k + 2A^{2k})_{k \in \mathbb{N}},$$

satisfy

$$X_k^2(9, A) + Y_k^2(9, A) = Z_k^2(9, A), k \in \mathbb{N}.$$

We can construct the galaxy

$$Gala(9, M_m(\mathbb{N})) = \left[ \begin{array}{l} X_k(9, A) = 9I_m + 6A^k \\ Y_k(9, A) = 6A^k + 2A^{2k} \\ Z_k(9, A) = 9I_m + 6A^k + 2A^{2k} \\ k \in \mathbb{N}, A \in M_m(\mathbb{N}) \end{array} \right],$$

and its associated galaxy of eigenvalues

$$Gala(9, \sigma(M_m(\mathbb{N}))) = \left[ \begin{array}{l} X_k(9, \lambda) = 9 + 6\lambda^k \\ Y_k(9, \lambda) = 6\lambda^k + 2\lambda^{2k} \\ Z_k(9, \lambda) = 9 + 6\lambda^k + 2\lambda^{2k} \\ k \in \mathbb{N}, \lambda \in \sigma(A), A \in M_m(\mathbb{N}) \end{array} \right].$$

**Example 4.** Let us choose  $\alpha = 16, A \in M_m(\mathbb{N})$ . The triples of the sequence

$$(X_k(16, A), Y_k(16, A), Z_k(16, A))_{k \in \mathbb{N}} = (16I_m + 8A^k, 8A^k + 2A^{2k}, 16I_m + 8A^k + 2A^{2k})_{k \in \mathbb{N}},$$

satisfy

$$X_k^2(16, A) + Y_k^2(16, A) = Z_k^2(16, A), k \in \mathbb{N}.$$

We can construct the galaxy

$$Gala(16, M_m(\mathbb{N})) = \left[ \begin{array}{l} X_k(16, A) = 16I_m + 8A^k \\ Y_k(16, A) = 8A^k + 2A^{2k} \\ Z_k(16, A) = 16I_m + 8A^k + 2A^{2k} \\ k \in \mathbb{N}, A \in M_m(\mathbb{N}) \end{array} \right]$$

and its associated galaxy of eigenvalues

$$Gala(16, \sigma(M_m(\mathbb{N}))) = \left[ \begin{array}{l} X_k(16, \lambda) = 16 + 8\lambda^k \\ Y_k(16, \lambda) = 8\lambda^k + 2\lambda^{2k} \\ Z_k(16, \lambda) = 16 + 8\lambda^k + 2\lambda^{2k} \\ k \in \mathbb{N}, \lambda \in \sigma(A), A \in M_m(\mathbb{N}) \end{array} \right].$$

**Example 5.** Let us choose  $A \in M_m(\mathbb{N})$ . The triples of the sequence

$$(X_k(A), Y_k(A), Z_k(A))_{k \in \mathbb{N}} = (2I_m + 2A^k, 2A^k + A^{2k}, 2I_m + 2A^k + A^{2k})_{k \in \mathbb{N}},$$

satisfy

$$X_k^2(A) + Y_k^2(A) = Z_k^2(A), k \in \mathbb{N}.$$

We can construct the galaxy

$$Fa(M_m(\mathbb{N})) = \left[ \begin{array}{l} X_k(A) = 2I_m + 2A^k \\ Y_k(A) = 2A^k + A^{2k} \\ Z_k(A) = 2I_m + 2A^k + A^{2k} \\ k \in \mathbb{N}, A \in M_m(\mathbb{N}) \end{array} \right],$$

and its associated galaxy of eigenvalues

$$Fa(\sigma(M_m(\mathbb{N}))) = \left[ \begin{array}{l} X_k(\lambda) = 2 + 2\lambda^k \\ Y_k(\lambda) = 2\lambda^k + \lambda^{2k} \\ Z_k(\lambda) = 2 + 2\lambda^k + \lambda^{2k} \\ k \in \mathbb{N}, \lambda \in \sigma(A), A \in M_m(\mathbb{N}) \end{array} \right].$$

**Example 6.** Let us choose  $A \in M_m(\mathbb{N})$ . The triples of the sequence

$$(X_k(A), Y_k(A), Z_k(A))_{k \in \mathbb{N}} = (32I_m + 8A^{2k}, 8A^{2k} + A^{2^{k+1}}, 32I_m + 8A^{2k} + A^{2^{k+1}})_{k \in \mathbb{N}},$$

satisfy

$$X_k^2(A) + Y_k^2(A) = Z_k^2(A), k \in \mathbb{N}.$$

We can construct the galaxy

$$Ha(M_m(\mathbb{N})) = \left[ \begin{array}{l} X_k(A) = 32I_m + 8A^{2^k} \\ Y_k(A) = 8A^{2^k} + A^{2^{k+1}} \\ Z_k(A) = 32I_m + 8A^{2^k} + A^{2^{k+1}} \\ k \in \mathbb{N}, A \in M_m(\mathbb{N}) \end{array} \right],$$

and its associated galaxy of eigenvalues

$$Ha(\sigma(M_m(\mathbb{N}))) = \left[ \begin{array}{l} X_k(\lambda) = 32 + 8\lambda^{2^k} \\ Y_k(\lambda) = 8\lambda^{2^k} + \lambda^{2^{k+1}} \\ Z_k(\lambda) = 32 + 8\lambda^{2^k} + \lambda^{2^{k+1}} \\ k \in \mathbb{N}, \lambda \in \sigma(A), A \in M_m(\mathbb{N}) \end{array} \right].$$

We can now prove our main result.

**Proof.** Let  $\mathcal{O}$  be an observer and let  $Ch(\mathcal{O})$  be the set of the appropriate matrix choices  $(m_0(k), n_0(k))$  of the observer  $\mathcal{O}$ . Let us consider Mouanda's choice function

$$f_{\mathbb{M}} : Ch(\mathcal{O}) \longrightarrow \Omega(\mathbb{F}_2(M_m(\mathbb{C}))),$$

defined by

$$f_{\mathbb{M}}(m_0(k), n_0(k)) = \left[ \begin{array}{l} m_0(k) = A^{\beta(k)}, k, \beta(k) \in \mathbb{N}, \beta \in C_*(\mathbb{N}) \\ A, m_0(k) - n_0(k) \in M_m(\mathbb{C}) \\ X_k(m_0(k), n_0(k)) = \frac{2(m_0(k) - n_0(k)) + \sqrt{8m_0(k)(m_0(k) - n_0(k))}}{2} \in M_m(\mathbb{C}) \\ Y_k(m_0(k), n_0(k)) = \frac{2(m_0(k) - n_0(k)) + \sqrt{8m_0(k)(m_0(k) - n_0(k))}}{2} + n_0(k) \\ Z_k(m_0(k), n_0(k)) = \frac{2(m_0(k) - n_0(k)) + \sqrt{8m_0(k)(m_0(k) - n_0(k))}}{2} + m_0(k) \end{array} \right].$$

Let us notice that for every  $(m_0(k), n_0(k)) \in Ch(\mathcal{O})$ , the matrix  $8m_0(k)(m_0(k) - n_0(k))$  can be written as a square of another matrix. The appropriate choice of the values of  $m_0(k)$  and  $n_0(k)$  allows the construction of the structures of sequences of matrix Pythagorean triples. Let us introduce several structures of sequences of matrix Pythagorean triples.

- Let us choose  $m_0(k) = 2A^{2^k}$ ,  $m_0(k) - n_0(k) = I_m$  with  $A \in M_m(\mathbb{N})$ . The triples of the sequence

$$(X_k(A), Y_k(A), Z_k(A))_{k \in \mathbb{N}} = (I_m + 2A^k, 2A^k + 2A^{2^k}, I_m + 2A^k + 2A^{2^k})_{k \in \mathbb{N}},$$

satisfy

$$X_k^2(A) + Y_k^2(A) = Z_k^2(A), k \in \mathbb{N}.$$

We can construct the galaxy

$$Ta(M_m(\mathbb{N})) = \left[ \begin{array}{l} X_k(A) = I_m + 2A^k \\ Y_k(A) = 2A^k + 2A^{2^k} \\ Z_k(A) = I_m + 2A^k + 2A^{2^k} \\ k \in \mathbb{N}, A \in M_m(\mathbb{N}) \end{array} \right].$$

- Let  $\alpha \in \mathbb{N}$  be a positive integer. Let us choose  $A \in M_m(\mathbb{N})$  and

$$m_0(k) = 18A^{2k}, m_0(k) - n_0(k) = \alpha^2 I_m.$$

The triples of the sequence

$$(X_k(\alpha, A), Y_k(\alpha, A), Z_k(\alpha, A))_{k \in \mathbb{N}},$$

defined by

$$\begin{cases} X_k(\alpha, A) = \alpha^2 I_m + 6\alpha A^k \\ Y_k(\alpha, A) = 6\alpha A^k + 18A^{2k} \\ Z_k(\alpha, A) = \alpha^2 I_m + 6\alpha A^k + 18A^{2k}, \end{cases}$$

satisfy

$$X_k^2(\alpha, A) + Y_k^2(\alpha, A) = Z_k^2(\alpha, A), k \in \mathbb{N}.$$

We have an infinite number of sequences of matrix Pythagorean triples. We can construct the galaxy

$$Za(\alpha, M_m(\mathbb{N})) = \left[ \begin{array}{l} X_k(\alpha, A) = \alpha^2 I_m + 6\alpha A^k \\ Y_k(\alpha, A) = 6\alpha A^k + 18A^{2k} \\ Z_k(\alpha, A) = \alpha^2 I_m + 6\alpha A^k + 18A^{2k} \\ k \in \mathbb{N}, A \in M_m(\mathbb{N}) \end{array} \right], \alpha \in \mathbb{N}.$$

The number  $\alpha$  is called the base of the galaxy  $Za(\alpha, M_m(\mathbb{N}))$  and the elements of  $M_m(\mathbb{N})$  are called leads of the galaxy.

- Let  $\alpha \in \mathbb{N}$  be a positive integer. Let us choose  $A \in M_m(\mathbb{N})$  and

$$m_0(k) = A^{2^{k+1}}, m_0(k) - n_0(k) = 2\alpha^2 I_m.$$

The triples of the sequence

$$(X_k(\alpha, A), Y_k(\alpha, A), Z_k(\alpha, A))_{k \in \mathbb{N}},$$

defined by

$$\begin{cases} X_k(\alpha, A) = 2\alpha^2 I_m + 2\alpha A^{2^k} \\ Y_k(\alpha, A) = 2\alpha A^{2^k} + A^{2^{k+1}} \\ Z_k(\alpha, A) = 2\alpha^2 I_m + 2\alpha A^{2^k} + A^{2^{k+1}}, \end{cases}$$

satisfy

$$X_k^2(\alpha, A) + Y_k^2(\alpha, A) = Z_k^2(\alpha, A), k \in \mathbb{N}.$$

We have an infinite number of sequences of matrix Pythagorean triples. We can construct the galaxy

$$Qa(\alpha, M_m(\mathbb{N})) = \left[ \begin{array}{l} X_k(\alpha, A) = 2\alpha^2 I_m + 2\alpha A^{2k} \\ Y_k(\alpha, A) = 2\alpha A^{2k} + A^{2k+1} \\ Z_k(\alpha, A) = 2\alpha^2 I_m + 2\alpha A^{2k} + A^{2k+1} \\ k \in \mathbb{N}, A \in M_m(\mathbb{N}) \end{array} \right], \alpha \in \mathbb{N}.$$

- Let  $\alpha \in \mathbb{N}$  be a positive integer. Let us choose  $A \in M_m(\mathbb{N})$  and

$$m_0(k) = A^{2k}, m_0(k) - n_0(k) = 2\alpha^2 I_m.$$

The triples of the sequence

$$(X_k(\alpha, A), Y_k(\alpha, A), Z_k(\alpha, A))_{k \in \mathbb{N}},$$

defined by

$$\begin{cases} X_k(\alpha, A) = 2\alpha^2 I_m + 2\alpha A^k \\ Y_k(\alpha, A) = 2\alpha A^k + A^{2k} \\ Z_k(\alpha, A) = 2\alpha^2 I_m + 2\alpha A^k + A^{2k}, \end{cases}$$

satisfy

$$X_k^2(\alpha, A) + Y_k^2(\alpha, A) = Z_k^2(\alpha, A), k \in \mathbb{N}.$$

We have an infinite number of sequences of matrix Pythagorean triples. We can construct the galaxy

$$Pa(\alpha, M_m(\mathbb{N})) = \left[ \begin{array}{l} X_k(\alpha, A) = 2\alpha^2 I_m + 2\alpha A^k \\ Y_k(\alpha, A) = 2\alpha A^k + A^{2k} \\ Z_k(\alpha, A) = 2\alpha^2 I_m + 2\alpha A^k + A^{2k} \\ k \in \mathbb{N}, A \in M_m(\mathbb{N}) \end{array} \right], \alpha \in \mathbb{N}.$$

- Let  $\alpha \in \mathbb{N}$  be a positive integer. Let us choose  $A \in M_m(\mathbb{N})$  and

$$m_0(k) = 2A^{2k}, m_0(k) - n_0(k) = \alpha^4 I_m.$$

The triples of the sequence

$$(X_k(\alpha, A), Y_k(\alpha, A), Z_k(\alpha, A))_{k \in \mathbb{N}},$$

defined by

$$\begin{cases} X_k(\alpha, A) = \alpha^4 I_m + 2\alpha^2 A^k \\ Y_k(\alpha, A) = 2\alpha^2 A^k + 2A^{2k} \\ Z_k(\alpha, A) = \alpha^4 I_m + 2\alpha^2 A^k + 2A^{2k}, \end{cases}$$

satisfy

$$X_k^2(\alpha, A) + Y_k^2(\alpha, A) = Z_k^2(\alpha, A), k \in \mathbb{N}.$$

We have an infinite number of sequences of matrix Pythagorean triples. We can construct the galaxy

$$Wa(\alpha, M_m(\mathbb{N})) = \left[ \begin{array}{l} X_k(\alpha, A) = \alpha^4 I_m + 2\alpha^2 A^k \\ Y_k(\alpha, A) = 2\alpha^2 A^k + 2A^{2k} \\ Z_k(\alpha, A) = \alpha^4 I_m + 2\alpha^2 A^k + 2A^{2k} \\ k \in \mathbb{N}, A \in M_m(\mathbb{N}) \end{array} \right], \alpha \in \mathbb{N}.$$

• Let  $\alpha \in \mathbb{N}$  be a positive integer. Let us choose  $A \in M_m(\mathbb{N})$  and

$$m_0(k) = 2\alpha A^{2k}, m_0(k) - n_0(k) = \alpha^3 I_m.$$

The triples of the sequence

$$(X_k(\alpha, A), Y_k(\alpha, A), Z_k(\alpha, A))_{k \in \mathbb{N}},$$

defined by

$$\begin{cases} X_k(\alpha, A) = \alpha^3 I_m + 2\alpha^2 A^k \\ Y_k(\alpha, A) = 2\alpha^2 A^k + 2\alpha A^{2k} \\ Z_k(\alpha, A) = \alpha^3 I_m + 2\alpha^2 A^k + 2\alpha A^{2k}, \end{cases}$$

satisfy

$$X_k^2(\alpha, A) + Y_k^2(\alpha, A) = Z_k^2(\alpha, A), k \in \mathbb{N}.$$

We have an infinite number of sequences of matrix Pythagorean triples. We can construct the galaxy

$$Ma(\alpha, M_m(\mathbb{N})) = \left[ \begin{array}{l} X_k(\alpha, A) = \alpha^3 I_m + 2\alpha^2 A^k \\ Y_k(\alpha, A) = 2\alpha^2 A^k + 2\alpha A^{2k} \\ Z_k(\alpha, A) = \alpha^3 I_m + 2\alpha^2 A^k + 2\alpha A^{2k} \\ k \in \mathbb{N}, A \in M_m(\mathbb{N}) \end{array} \right], \alpha \in \mathbb{N}.$$

This yields the desired result. □

There are many ways of generating matrix Pythagorean triples. Matrix solutions of the Diophantine equations

$$X^{2n} + Y^{2m} = Z^{2k}, n, m, k \in \mathbb{N},$$

generate matrix Pythagorean triples.

#### 4. Semirings of sequences of matrix Pythagorean triples and homomorphisms

In this section, we show that every galaxy of sequences of matrix Pythagorean triples is associated with a semiring and every semiring of sequences of matrix Pythagorean triples is associated with a semiring homomorphism. Let us consider the sequence of matrix Pythagorean triples

$$\Omega(\alpha, A) = (X_k(\alpha, A), Y_k(\alpha, A), Z_k(\alpha, A))_{k \in \mathbb{N}},$$

of the galaxy

$$Wa(\alpha, M_m(\mathbb{N})) = \left[ \begin{array}{l} X_k(\alpha, A) = \alpha^4 I_m + 2\alpha^2 A^k \\ Y_k(\alpha, A) = 2\alpha^2 A^k + 2A^{2k} \\ Z_k(\alpha, A) = \alpha^4 I_m + 2\alpha^2 A^k + 2A^{2k} \\ k \in \mathbb{N}, A \in M_m(\mathbb{N}) \end{array} \right], \alpha \in \mathbb{N}.$$

Let

$$\mathcal{V}_{\alpha,m} = \{\Omega(\alpha, A) : A \in M_m(\mathbb{N})\},$$

be the set of all sequences of the galaxy  $Wa(\alpha, M_m(\mathbb{N}))$ . The set  $\mathcal{V}_{\alpha,m}$  is called the astral body of the set  $M_m(\mathbb{N})$  associated to the galaxy  $Wa(\alpha, M_m(\mathbb{N}))$ . Every element  $\Omega(\alpha, A)$  of the set  $\mathcal{V}_{\alpha,m}$  is called an astral cell of  $\mathcal{V}_{\alpha,m}$ . The universe is a complex environment. There are births and deaths of galaxies (stars, planets). The same phenomenon happens inside the body of every living creature. It is possible to create genetic astral laws that could help us to generate new astral cells of the astral bodies of  $M_m(\mathbb{N})$  associated with different galaxies. Let

$$\Omega(\alpha, A) = (X_k(\alpha, A), Y_k(\alpha, A), Z_k(\alpha, A))_{k \in \mathbb{N}},$$

and

$$\Omega(\alpha, B) = (X_k(\alpha, B), Y_k(\alpha, B), Z_k(\alpha, B))_{k \in \mathbb{N}},$$

be two astral cells of the astral body  $\mathcal{V}_{\alpha,m}$  of the set  $M_m(\mathbb{N})$  associated to the galaxy  $Wa(\alpha, M_m(\mathbb{N}))$ . Define two laws  $\perp$  and  $*$  in  $\mathcal{V}_{\alpha,m}$  which act only on the leads of the galaxy  $Wa(\alpha I_m, M_m(\mathbb{N}))$  by setting

$$\Omega(\alpha, A) \perp \Omega(\alpha, B) = \Omega(\alpha, A + B) = (X_k(\alpha, A + B), Y_k(\alpha, A + B), Z_k(\alpha, A + B))_{k \in \mathbb{N}},$$

and

$$\Omega(\alpha, A) * \Omega(\alpha, B) = \Omega(\alpha, A \times B) = (X_k(\alpha, A \times B), Y_k(\alpha, A \times B), Z_k(\alpha, A \times B))_{k \in \mathbb{N}}.$$

The sequence

$$\Omega(\alpha, I_m) = (X_k(\alpha, I_m), Y_k(\alpha, I_m), Z_k(\alpha, I_m))_{k \in \mathbb{N}},$$

is the multiplicative identity element of  $(\mathcal{V}_{\alpha,m}, *)$ . The sequence

$$\Omega(\alpha, 0) = (X_k(\alpha, 0), Y_k(\alpha, 0), Z_k(\alpha, 0))_{k \in \mathbb{N}},$$

is the additive identity element of  $(\mathcal{V}_{\alpha,m}, \perp)$ .

**Definition 7.** A semiring is a ringoid  $(S, *, \circ)$  in which:

- 1)  $(S, *)$  forms a semigroup.
- 2)  $(S, \circ)$  forms a semigroup.

That is, such that  $(S, *, \circ)$  has the following properties:

- 1)  $\forall a, b \in S : a * b \in S$ .
- 2)  $\forall a, b, c \in S : (a * b) * c = a * (b * c)$ .
- 3)  $\forall a, b \in S : a \circ b \in S$ .
- 4)  $\forall a, b, c \in S : (a \circ b) \circ c = a \circ (b \circ c)$ .
- 5)  $a, b, c \in S : (a * b) \circ c = (a \circ c) * (b \circ c)$ .

6)  $a, b, c \in S : a \circ (b * c) = (a \circ b) * (a \circ c).$

These are called the semiring axioms. The set  $(\mathcal{V}_{\alpha,m}, \perp, *)$  satisfies the semiring axioms. Indeed, let

$$\Omega(\alpha, A) = (X_k(\alpha, A), Y_k(\alpha, A), Z_k(\alpha, A))_{k \in \mathbb{N}}, A \in M_m(\mathbb{N}),$$

$$\Omega(\alpha, B) = (X_k(\alpha, B), Y_k(\alpha, B), Z_k(\alpha, B))_{k \in \mathbb{N}}, B \in M_m(\mathbb{N}),$$

and

$$\Omega(\alpha, C) = (X_k(\alpha, C), Y_k(\alpha, C), Z_k(\alpha, C))_{k \in \mathbb{N}}, C \in M_m(\mathbb{N}),$$

be three astral cells the astral body  $\mathcal{V}_{\alpha,m}$  of the set  $M_m(\mathbb{N})$  associated to the galaxy  $Wa(\alpha, M_m(\mathbb{N}))$ .

1)  $\Omega(\alpha, A) \perp \Omega(\alpha, B) = \Omega(\alpha, A + B) \in \mathcal{V}_{\alpha,m}.$

2)  $(\Omega(\alpha, A) \perp \Omega(\alpha, B)) \perp \Omega(\alpha, C) = \Omega(\alpha, A + B) \perp \Omega(\alpha, C) = \Omega(\alpha, A + B + C).$

We know that

$$\Omega(\alpha, A + B + C) = \Omega(\alpha, A) \perp \Omega(\alpha, B + C) = \Omega(\alpha, A) \perp (\Omega(\alpha, B) \perp \Omega(\alpha, C)).$$

Therefore,

$$(\Omega(\alpha, A) \perp \Omega(\alpha, B)) \perp \Omega(\alpha, C) = \Omega(\alpha, A) \perp (\Omega(\alpha, B) \perp \Omega(\alpha, C)).$$

3)  $\Omega(\alpha, A) * \Omega(\alpha, B) = \Omega(\alpha, A \times B) \in \mathcal{V}_{\alpha,m}.$

4)  $(\Omega(\alpha, A) * \Omega(\alpha, B)) * \Omega(\alpha, C) = \Omega(\alpha, A \times B) * \Omega(\alpha, C) = \Omega(\alpha, A \times B \times C).$

We know that

$$\Omega(\alpha, A \times B \times C) = \Omega(\alpha, A) * \Omega(\alpha, B \times C) = \Omega(\alpha, A) * (\Omega(\alpha, B) * \Omega(\alpha, C)).$$

Therefore,

$$(\Omega(\alpha, A) * \Omega(\alpha, B)) * \Omega(\alpha, C) = \Omega(\alpha, A) * (\Omega(\alpha, B) * \Omega(\alpha, C)).$$

5)  $(\Omega(\alpha, A) \perp \Omega(\alpha, B)) * \Omega(\alpha, C) = \Omega(\alpha, A + B) * \Omega(\alpha, C) = \Omega(\alpha, (A + B)C).$

It is straightforward to see that

$$\Omega(\alpha, (A + B)C) = \Omega(\alpha, AC + BC) = \Omega(\alpha, AC) \perp \Omega(\alpha, BC).$$

It is clear that

$$\Omega(\alpha, AC) = \Omega(\alpha, A) * (\Omega(\alpha, C), \Omega(\alpha, BC) = \Omega(\alpha, B) * \Omega(\alpha, C).$$

Therefore,

$$(\Omega(\alpha, A) \perp \Omega(\alpha, B)) * \Omega(\alpha, C) = (\Omega(\alpha, A) * \Omega(\alpha, C)) \perp (\Omega(\alpha, B) * \Omega(\alpha, C)).$$

$$\Omega(\alpha, A) * (\Omega(\alpha, B) \perp \Omega(\alpha, C)) = \Omega(\alpha, A) * \Omega(\alpha, B + C) = \Omega(\alpha, A(B + C)).$$

It is straightforward to see that

$$\Omega(\alpha, A(B + C)) = \Omega(\alpha, AB + AC) = \Omega(\alpha, AB) \perp \Omega(\alpha, AC).$$

It is clear that

$$\Omega(\alpha, AB) = \Omega(\alpha, A) * (\Omega(\alpha, B), \Omega(\alpha, AC) = \Omega(\alpha, A) * \Omega(\alpha, C).$$

Therefore,

$$\Omega(\alpha, A) * (\Omega(\alpha, B) \perp \Omega(\alpha, C)) = (\Omega(\alpha, A) * (\Omega(\alpha, B))) \perp (\Omega(\alpha, A) * \Omega(\alpha, C)).$$

Finally, the astral body  $\mathcal{V}_{\alpha,m}$ , with the two laws  $\perp$  and  $*$ , is a semiring. In other words, the set  $(\mathcal{V}_{\alpha,m}, \perp, *)$  is a semiring of sequences of matrix Pythagorean triples associated to the galaxy  $Wa(\alpha, M_m(\mathbb{N}))$ . We can conclude that. These laws allow us to study the astral genetics inside the astral body  $\mathcal{V}_{\alpha,m}$ . It is well known that the set  $(M_m(\mathbb{N}), +, \times)$  is a non-commutative semiring.

**Theorem 2.** Define the map  $\varphi^{\mathcal{V}_{\alpha,m}} : (M_m(\mathbb{N}), +, \times) \longrightarrow (\mathcal{V}_{\alpha,m}, \perp, *)$  via the formula

$$\varphi^{\mathcal{V}_{\alpha,m}}(A) = (X_k(\alpha, A), Y_k(\alpha, A), Z_k(\alpha, A))_{k \in \mathbb{N}}, \alpha, m \in \mathbb{N}.$$

The maps  $\varphi^{\mathcal{V}_{\alpha,m}}$  are homomorphisms of semirings.

**Proof.** Let  $A, B \in M_m(\mathbb{N})$ . It is straightforward to check that

$$\varphi^{\mathcal{V}_{\alpha,m}}(A + B) = (X_k(\alpha, A + B), Y_k(\alpha, A + B), Z_k(\alpha, A + B))_{k \in \mathbb{N}} = \varphi^{\mathcal{V}_{\alpha,m}}(A) \perp \varphi^{\mathcal{V}_{\alpha,m}}(B)$$

$$\varphi^{\mathcal{V}_{\alpha,m}}(A \times B) = (X_k(\alpha, AB), Y_k(\alpha, AB), Z_k(\alpha, AB))_{k \in \mathbb{N}} = \varphi^{\mathcal{V}_{\alpha,m}}(A) * \varphi^{\mathcal{V}_{\alpha,m}}(B), \alpha, m \in \mathbb{N}$$

$$\varphi^{\mathcal{V}_{\alpha,m}}(I_m) = (X_k(\alpha, I_m), Y_k(\alpha, I_m), Z_k(\alpha, I_m))_{k \in \mathbb{N}},$$

and

$$\varphi^{\mathcal{V}_{\alpha,m}}(0) = (X_k(\alpha, 0), Y_k(\alpha, 0), Z_k(\alpha, 0))_{k \in \mathbb{N}}.$$

□

In fact, the homomorphism  $\varphi^{\mathcal{V}_{\alpha,m}}$  is bijective for every  $\alpha \in \mathbb{N}$ . Indeed, every sequence of matrix Pythagorean triples of the set  $\mathcal{V}_{\alpha,m}$  is generated by a matrix of the set  $M_m(\mathbb{N})$ . Therefore,

$$\varphi^{\mathcal{V}_{\alpha,m}}(M_m(\mathbb{N})) = \mathcal{V}_{\alpha,m}.$$

Also, two elements  $\Omega(\alpha, A)$  and  $\Omega(\alpha, B)$  of the set  $\mathcal{V}_{\alpha,m}$  are two different elements if and only if the matrices A and B are different too.

### 5. Astral body of the set of complex polynomials over $\mathbb{D}$ associated to the galaxies of sequences of Pythagorean triples of complex polynomials over $\mathbb{D}$

We notice that, we can construct the semirings of sequences of Pythagorean triples of many mathematical objects with their astral bodies. For instance, from the construction structure of the galaxy  $Wa(\alpha, M_m(\mathbb{N}))$ , we can construct the galaxy of sequences of polynomial Pythagorean triples

$$Wa(\alpha, \mathcal{P}(\mathbb{N})) = \left[ \begin{array}{l} X_k(\alpha, f) = \alpha^4 + 2\alpha^2 f^k \\ Y_k(\alpha, f) = 2\alpha^2 f^k + 2f^{2k} \\ Z_k(\alpha, f) = \alpha^4 + 2\alpha^2 f^k + 2f^{2k} \\ k \in \mathbb{N}, f \in \mathcal{P}(\mathbb{N}) \end{array} \right], \alpha \in \mathbb{N},$$

with

$$\mathcal{P}(\mathbb{N}) = \left\{ f(z) = \sum_{k=0}^n a_k z^k : a_k, n \in \mathbb{N}, z \in \mathbb{D} \right\},$$

the algebra of polynomials over  $\mathbb{D}$ . It follows that

$$\mathcal{V}_{\alpha, \mathcal{P}(\mathbb{N})} = \left\{ \left( \alpha^4 + 2\alpha^2 f^k, 2\alpha^2 f^k + 2f^{2k}, \alpha^4 + 2\alpha^2 f^k + 2f^{2k} \right)_{k \in \mathbb{N}} : f \in \mathcal{P}(\mathbb{N}) \right\},$$

is the astral body of  $\mathcal{P}(\mathbb{N})$  associated with the galaxy  $Wa(\alpha, \mathcal{P}(\mathbb{N}))$ . In other words,

$$\mathcal{V}_{\alpha, \mathcal{P}(\mathbb{N})} = \{ \Omega(\alpha, f) : f \in \mathcal{P}(\mathbb{N}) \}.$$

A simple calculation shows that if  $\Omega(\alpha, f), \Omega(\alpha, g) \in \mathcal{V}_{\alpha, \mathcal{P}(\mathbb{N})}$  are two astral cells, in general,

$$\Omega(\alpha, f) \perp \Omega(\alpha, g) = \Omega(\alpha, f + g) \in \mathcal{V}_{\alpha, \mathcal{P}(\mathbb{N})},$$

and

$$\Omega(\alpha, f) * \Omega(\alpha, g) = \Omega(\alpha, f \times g) \in \mathcal{V}_{\alpha, \mathcal{P}(\mathbb{N})}.$$

In this case, we can deduce that  $(\mathcal{V}_{\alpha, \mathcal{P}(\mathbb{N})}, \perp, *)$  is a semiring of sequences of polynomial Pythagorean triples. The same thing can be done for the set  $C(X)$  of complex continuous functions on the compact set  $X$ . The map  $\varphi^{\mathcal{V}_{\alpha, \mathcal{P}(\mathbb{N})}} : (\mathcal{P}(\mathbb{N}), +, \times) \rightarrow (\mathcal{V}_{\alpha, \mathcal{P}(\mathbb{N})}, \perp, *)$  defined by

$$\varphi^{\mathcal{V}_{\alpha, \mathcal{P}(\mathbb{N})}}(f) = \Omega(\alpha, f) = (X_k(\alpha, f), Y_k(\alpha, f), Z_k(\alpha, f))_{k \in \mathbb{N}}.$$

is the associated homomorphism of semirings.

## 6. Astral body of complex polynomials over $\mathbb{D}$ associated to the galaxies of sequences of Pythagorean triples of complex polynomials over $\mathbb{D}$

Let

$$f(z) = \sum_{k=0}^n a_k z^k : a_k, n \in \mathbb{N},$$

be a complex polynomial over  $\mathbb{D}$ . From the structure of the galaxy  $Wa(\alpha, \mathcal{P}(\mathbb{N}))$ , we can construct the galaxy

$$Wa(\alpha, f) = \left[ \begin{array}{l} X_k(\alpha, f(z)) = \alpha^4 + 2\alpha^2 f(z)^k \\ Y_k(\alpha, f(z)) = 2\alpha^2 f(z)^k + 2f(z)^{2k} \\ Z_k(\alpha, f(z)) = \alpha^4 + 2\alpha^2 f(z)^k + 2f(z)^{2k} \\ k \in \mathbb{N}, z \in \mathbb{D} \end{array} \right], \alpha \in \mathbb{N}.$$

Assume that

$$\Omega(\alpha, f(z)) = \left( \alpha^4 + 2\alpha^2 f(z)^k, 2\alpha^2 f(z)^k + 2f(z)^{2k}, \alpha^4 + 2\alpha^2 f(z)^k + 2f(z)^{2k} \right)_{k \in \mathbb{N}},$$

is the sequence of complex numbers associated with the complex polynomial  $f(z)$ . It follows that

$$\mathcal{V}_{\alpha, f} = \{ \Omega(\alpha, f(z)) : z \in \mathbb{D} \},$$

is the astral body of  $f$  associated with the galaxy  $Wa(\alpha, f)$ . It is possible from the astral cells of the astral body  $\mathcal{V}_{\alpha, f}$  of the complex polynomial  $f$  to create new astral cells that are not astral cells of the astral body  $\mathcal{V}_{\alpha, f}$ . Indeed, let  $\Omega(\alpha, f(x)), \Omega(\alpha, f(y)) \in \mathcal{V}_{\alpha, f}$

be two different astral cells. In general,

$$\Omega(\alpha, f(x)) \perp \Omega(\alpha, f(y)) = \Omega(\alpha, f(x) + f(y)) \notin \mathcal{V}_{\alpha, f},$$

and

$$\Omega(\alpha, f(x)) * \Omega(\alpha, f(y)) = \Omega(\alpha, f(x) \times f(y)) \notin \mathcal{V}_{\alpha, f}.$$

In this case,  $(\mathcal{V}_{\alpha, f}, \perp, *)$  is not a semiring of sequences of polynomial Pythagorean triples. In conclusion, the astral body of a complex polynomial is not a semiring. This phenomenon could be used to create new astral cells that never existed before or that were never seen before. This is a great opportunity in chemistry. The properties of the astral bodies of complex polynomials can be used to diversify our environment and improve our lives. The property of adding or multiplying the leads of the galaxies of sequences of complex polynomial Pythagorean triples could create new environmental astral cells which can be used in agricultural farming, biology, chemistry, industry. For example, the tree that gives the soap as a fruit does not exist but mixing banana leaf powder and palm oil under the heat conditions could create the banana soap which has many skin-healing benefits.

### 7. Semiring of astral bodies

The triples

$$\begin{cases} x(\alpha, a) = \alpha^4 + 2\alpha^2 a \\ y(\alpha, a) = 2\alpha^2 a + 2a^2 \\ z(\alpha, a) = \alpha^4 + 2\alpha^2 a + 2a^2 \\ \alpha, a \in \mathbb{N}, \end{cases}$$

satisfy

$$x^2(\alpha, a) + y^2(\alpha, a) = z^2(\alpha, a).$$

Assume that  $\alpha_1, \alpha_2, a_1, a_2 \in \mathbb{N}$ . A simple calculation shows that

$$x^2(\alpha_1 + \alpha_2, a_1 + a_2) + y^2(\alpha_1 + \alpha_2, a_1 + a_2) = z^2(\alpha_1 + \alpha_2, a_1 + a_2),$$

and

$$x^2(\alpha_1 \alpha_2, a_1 a_2) + y^2(\alpha_1 \alpha_2, a_1 a_2) = z^2(\alpha_1 \alpha_2, a_1 a_2).$$

Also, we have

$$x^2(\alpha_1 + \alpha_2, A_1 + A_2) + y^2(\alpha_1 + \alpha_2, A_1 + A_2) = z^2(\alpha_1 + \alpha_2, A_1 + A_2),$$

and

$$x^2(\alpha_1 \alpha_2, A_1 A_2) + y^2(\alpha_1 \alpha_2, A_1 A_2) = z^2(\alpha_1 \alpha_2, A_1 A_2), A_1, A_2 \in M_m(\mathbb{N}).$$

Recall that

$$\mathcal{V}_{\alpha, m} = \{\Omega(\alpha, A) : A \in M_m(\mathbb{N})\} = \{(X_k(\alpha, A), Y_k(\alpha, A), Z_k(\alpha, A))_{k \in \mathbb{N}} : A \in M_m(\mathbb{N})\},$$

is the astral body of  $M_m(\mathbb{N})$  associated with the galaxy  $Wa(\alpha, M_m(\mathbb{N}))$ ,  $\alpha \in \mathbb{N}$ . Let

$$\mathcal{V}_m = \{\mathcal{V}_{\alpha, m} : \alpha \in \mathbb{N}\},$$

be the set of astral bodies of  $M_m(\mathbb{N})$  associated with the galaxies  $Wa(\alpha, M_m(\mathbb{N}))$ ,  $\alpha \in \mathbb{N}$ . Let

$$\mathcal{V}_{\alpha_1, m} = \{\Omega(\alpha_1, A) : A \in M_m(\mathbb{N})\} = \{(X_k(\alpha_1, A), Y_k(\alpha_1, A), Z_k(\alpha_1, A))_{k \in \mathbb{N}} : A \in M_m(\mathbb{N})\},$$

and

$$\mathcal{V}_{\alpha_2, m} = \{\Omega(\alpha_2, B) : B \in M_m(\mathbb{N})\} = \{(X_k(\alpha_2, B), Y_k(\alpha_2, B), Z_k(\alpha_2, B))_{k \in \mathbb{N}} : B \in M_m(\mathbb{N})\},$$

be two astral bodies of the set  $\mathcal{V}_m$ . Define two laws  $\odot$  and  $\oplus$  in  $\mathcal{V}_m$  that operate on the bases and leads of the associated galaxies by setting

$$\mathcal{V}_{\alpha_1, m} \odot \mathcal{V}_{\alpha_2, m} = \{\Omega(\alpha_1 + \alpha_2, A + B) : A, B \in M_m(\mathbb{N})\},$$

and

$$\mathcal{V}_{\alpha_1, m} \oplus \mathcal{V}_{\alpha_2, m} = \{\Omega(\alpha_1 \times \alpha_2, A \times B) : A, B \in M_m(\mathbb{N})\}.$$

It is very simple to check that

$$\mathcal{V}_{\alpha_1, m} \odot \mathcal{V}_{\alpha_2, m}, \mathcal{V}_{\alpha_1, m} \oplus \mathcal{V}_{\alpha_2, m} \in \mathcal{V}_m.$$

Therefore,  $(\mathcal{V}_m, \odot, \oplus)$  is also a semiring of astral bodies. These two laws  $\odot$  and  $\oplus$  allow us to investigate the astral genetics of astral bodies of the set  $M_m(\mathbb{N})$ .

### 8. Motions of galaxies of sequences of matrix Pythagorean triples

The semiring of astral bodies allows us to observe the motion of galaxies. In this section, we describe how to travel from one galaxy to another. Let us consider the galaxy of sequences of triples of positive integers

$$H(\alpha, \mathbb{N}) = \left[ \begin{array}{l} x_k(\alpha, a) = \alpha^2 + 2 \times \alpha \times a^{2^k} \\ y_k(\alpha, a) = 2 \times \alpha \times a^{2^k} + a^{2^{k+1}} \\ z_k(\alpha, a) = \alpha^2 + 2 \times \alpha \times a^{2^k} + a^{2^{k+1}} \\ k, a \in \mathbb{N}, a \neq 0 \end{array} \right], \alpha \in \mathbb{N}.$$

The triples of this galaxy satisfy

$$x_k^2(\alpha, a) + y_k^2(\alpha, a) = z_k^2(\alpha, a), k, \alpha \in \mathbb{N}. \tag{1}$$

The model

$$H(\mathbb{N}, \mathbb{N}) = \left[ \begin{array}{l} H(\alpha, \mathbb{N}) \\ \alpha \in \mathbb{N} \end{array} \right],$$

can be considered as the galaxy of galaxies. We can construct the galaxy

$$H(\alpha, M_m(\mathbb{N})) = \left[ \begin{array}{l} X_k(\alpha, A) = \alpha^2 + 2 \times \alpha \times A^{2^k} \\ Y_k(\alpha, A) = 2 \times \alpha \times A^{2^k} + A^{2^{k+1}} \\ Z_k(\alpha, A) = \alpha^2 + 2 \times \alpha \times A^{2^k} + A^{2^{k+1}} \\ k \in \mathbb{N}, A \in M_m(\mathbb{N}) \end{array} \right], \alpha \in \mathbb{N}.$$

A small perturbation of the constant  $\alpha$  allows the motion of one galaxy to another galaxy in terms of jumps. The map  $\pi : \mathbb{N} \rightarrow \Omega(\mathcal{F}_{2,2,2}(M_m(\mathbb{C})))$  defined by  $\alpha \rightarrow$

$\pi(\alpha) = H(\alpha, M_m(\mathbb{N}))$  with

$$\Omega(\mathcal{F}_{2,2,2}(M_m(\mathbb{C}))) = \{P : P \subseteq \mathcal{F}_{2,2,2}(M_m(\mathbb{C}))\},$$

determine the motions inside the galaxy

$$H(\mathbb{N}, M_m(\mathbb{N})) = \left[ \begin{array}{c} H(\alpha, M_m(\mathbb{N})) \\ \alpha \in \mathbb{N} \end{array} \right].$$

The motions inside these galaxies occurred in the form of jumps with no timing.

### 9. Sequences of completely Pythagorean maps

In this section, we introduce, for the first time, the idea of completely Pythagorean maps over  $\mathbb{N}^3$ . Let  $\mathcal{A}$  and  $\mathcal{B}$  be two  $C^*$ -algebras. Let  $M_n(\mathcal{A})$  be the  $C^*$ -algebra of  $n \times n$  matrices with entries from  $\mathcal{A}$ . Given a map  $\phi : \mathcal{A} \rightarrow \mathcal{B}$ . Define the map

$$\phi_n : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B}),$$

via the formula

$$\phi_n([a_{i,j}]_{i,j=1}^n) = [\phi(a_{i,j})]_{i,j=1}^n.$$

The map  $\phi$  is positive if it maps positive elements of  $\mathcal{A}$  to positive elements of  $\mathcal{B}$  and  $\phi$  is called completely positive if every  $\phi_n$  is a positive map. In a similar fashion, if  $\phi$  is a bounded map, then every  $\phi_n$  will be bounded and when  $\sup_n \|\phi_n\|$  is finite, we call  $\phi$  a completely bounded map [20].

**Definition 8.** Let  $\varphi : \mathbb{N}^3 \rightarrow \mathbb{N}^3$  be a map. The map  $\varphi$  is Pythagorean if

$$\varphi(a, b, c) \in \mathcal{F}_{2,2,2}(\mathbb{N}), \forall (a, b, c) \in \mathcal{F}_{2,2,2}(\mathbb{N}).$$

In the case of matrix Pythagorean triples, the definition of Pythagorean map is slightly different. The mapping set here is  $M_n(\mathbb{N})^3$  not  $M_n(\mathbb{N}^3)$ . Define the map  $\varphi_n : M_n(\mathbb{N})^3 \rightarrow M_n(\mathbb{N})^3$  via the formula

$$\varphi_n(A, B, C) = \varphi(A, B, C), \forall (A, B, C) \in M_n(\mathbb{N})^3.$$

**Definition 9.** Let  $\varphi_n : M_n(\mathbb{N})^3 \rightarrow M_n(\mathbb{N})^3$  be a map. The map  $\varphi$  is  $n$ -Pythagorean if

$$\varphi_n(A, B, C) \in \mathcal{F}_{2,2,2}(M_n(\mathbb{N})), \forall (A, B, C) \in \mathcal{F}_{2,2,2}(M_n(\mathbb{N})).$$

We call  $\varphi$  completely Pythagorean if every  $\varphi_n$  is a Pythagorean map.

Finding methods for generating matrix Pythagorean triples is not an easy task at all. On the other hand, completely Pythagorean maps are important mathematical tools capable of allowing the construction of matrix Pythagorean triples of any size. Let  $\mathcal{V}_{\alpha,m}$  be the semiring associated to the galaxy  $Gala(\alpha, M_m(\mathbb{N}))$  of matrix Pythagorean triples. Define the map

$$\pi^{\mathcal{V}_{\alpha,m}} : M_m(\mathbb{N})^3 \rightarrow \mathcal{V}_{\alpha,m},$$

via the formula

$$\pi^{\mathcal{V}_{\alpha,m}}(A, B, C) = (X_k(\alpha, A + B + C), Y_k(\alpha, A + B + C), Z_k(\alpha, A + B + C))_{k \in \mathbb{N}},$$

with  $(A, B, C) \in M_m(\mathbb{N})^3$  and  $(X_k(\alpha, A), Y_k(\alpha, A), Z_k(\alpha, A))_{k \in \mathbb{N}} \in \mathcal{V}_{\alpha, m}$ . The map

$$f_k^{\pi^{\nu_{\alpha, m}}} : M_m(\mathbb{N})^3 \longrightarrow M_m(\mathbb{N})^3,$$

defined by

$$f_k^{\pi^m}(A, B, C) = (X_k(\alpha, A + B + C), Y_k(\alpha, A + B + C), Z_k(\alpha, A + B + C)),$$

is Pythagorean. There exists an infinite number of completely Pythagorean maps.

**Theorem 3.** *There exists an infinite number of sequences  $(\varphi_k)_{k \in \mathbb{N}}$  of completely Pythagorean maps.*

**Proof.** Let  $\alpha \in \mathbb{N}$  be a positive integer. Mouanda's choice function, with  $m_0(k) = 2\alpha a^{2k}$  and  $m_0(k) - n_0(k) = \alpha^3$ , allows us to construct the sequences  $(x_k(\alpha, a), y_k(\alpha, a), z_k(\alpha, a))_{k \in \mathbb{N}}$  defined by

$$\begin{cases} x_k(\alpha, a) = \alpha^3 + 2\alpha^2 a^k \\ y_k(\alpha, a) = 2\alpha^2 a^k + 2\alpha a^{2k} \\ z_k(\alpha, a) = \alpha^3 + 2\alpha^2 a^k + 2\alpha a^{2k}. \end{cases}$$

It is straightforward to check that

$$x_k^2(\alpha, a) + y_k^2(\alpha, a) = z_k^2(\alpha, a), \alpha, k, a \in \mathbb{N}.$$

Define the map  $\varphi_k^\alpha : \mathbb{N}^3 \longrightarrow \mathbb{N}^3$  by

$$\varphi_k^\alpha(a, b, c) = (\alpha^3 + 2\alpha^2 d^k, 2\alpha^2 d^k + 2\alpha d^{2k}, \alpha^3 + 2\alpha^2 d^k + 2\alpha d^{2k}), k \in \mathbb{N},$$

with  $d = a + b + c, \forall (a, b, c) \in \mathbb{N}^3$ . The choice of d is not unique, we could choose  $d = abc$ . A simple calculation shows that

$$\varphi_k^\alpha(a, b, c) \in \mathcal{F}_{2,2,2}(\mathbb{N}), \forall (a, b, c) \in \mathbb{N}, k \in \mathbb{N}.$$

Finally, for every  $\alpha \in \mathbb{N}$ , the sequence  $(\varphi_k^\alpha)_{k \in \mathbb{N}}$  is a sequence of Pythagorean maps over  $\mathcal{F}_{2,2,2}(\mathbb{N})$ . For every  $\forall A \in M_m(\mathbb{N})$ , the triples of the sequence

$$(X_k(\alpha, A), Y_k(\alpha, A), Z_k(\alpha, A))_{k \in \mathbb{N}},$$

defined by

$$\begin{cases} X_k(\alpha, A) = \alpha^3 I_m + 2\alpha^2 A^k \\ Y_k(\alpha, A) = 2\alpha^2 A^k + 2\alpha A^{2k} \\ Z_k(\alpha, A) = \alpha^3 I_m + 2\alpha^2 A^k + 2\alpha A^{2k}, \end{cases}$$

also satisfy

$$X_k^2(\alpha, A) + Y_k^2(\alpha, A) = Z_k^2(\alpha, A), m, k \in \mathbb{N}.$$

Define the map  $\varphi_{k,m}^\alpha : M_m(\mathbb{N})^3 \longrightarrow M_m(\mathbb{N})^3$  by

$$\varphi_{k,m}^\alpha(A, B, C) = (\alpha^3 I_m + 2\alpha^2 X^k, 2\alpha^2 X^k + 2\alpha X^{2k}, \alpha^3 I_m + 2\alpha^2 X^k + 2\alpha X^{2k}), k \in \mathbb{N},$$

with  $X = A + B + C, \forall (A, B, C) \in M_m(\mathbb{N})^3$ . The choice of X is not unique. We could even choose  $X = 2A + 2B + C$ . A simple calculation shows that

$$\varphi_{k,m}^\alpha(A, B, C) \in \mathbb{F}_2(M_m(\mathbb{N})), \forall(A, B, C) \in M_m(\mathbb{N})^3, k \in \mathbb{N}.$$

Therefore, for every  $\alpha \in \mathbb{N}$ , the sequence  $(\varphi_{k,m}^\alpha)_{k \in \mathbb{N}}$  is the sequence of Pythagorean maps over  $M_m(\mathbb{N})^3, m \in \mathbb{N}$ . In other words, for every  $\alpha \in \mathbb{N}$ , the sequence  $(\varphi_k^\alpha)_{k \in \mathbb{N}}$  is a sequence of completely Pythagorean maps over  $\mathbb{N}^3$ .  $\square$

There exist Pythagorean maps that are not completely Pythagorean. For example, the maps  $\varphi_k^\alpha : \mathbb{N}^3 \rightarrow \mathbb{N}^3$  defined by

$$\varphi_k^\alpha(a, b, c) = (\alpha^3 + 2\alpha^2 d^k, 2\alpha^2 d^k + 2\alpha d^{2k}, \alpha^3 + 2\alpha^2 d^k + 2\alpha d^{2k}), k \in \mathbb{N},$$

with  $d = (c - a) + (c - b), \forall(a, b, c) \in \mathbb{N}^3, a \leq c, b \leq c$ , are Pythagorean maps but we cannot define the maps  $\varphi_{k,m}, m \in \mathbb{N}$  because if  $(A, B, C) \in M_m(\mathbb{N})^3$ , the matrix  $D = (C - A) + (C - B)$  is not necessarily an element of  $M_m(\mathbb{N})^3$ .

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## References

- Dickson LE. History of The Theory of Numbers: Diophantine Analysis, 2nd ed. Dover Publications; 2005.
- Barning FJM. On Pythagorean and near-Pythagorean triangles and a generation process using unimodular matrices (in Dutch). Stichting Mathematisch Centrum; 1963.
- Maor E. The Pythagorean Theorem: A 4,000-Year History. Princeton University Press; 2007.
- Trivedi RA, Bhanot SA. Pythagorean Triplets—Views, Analysis and Classification. IOSR Journal of Mathematics. 2015; 11: 54–63.
- Sierpinski W. Pythagorean triangles. Dover Publications; 1962.
- Cass D, Arpaia PJ. Matrix generation of Pythagorean n-tuples. Proceedings of The American Mathematical Society. 1990; 109(1): 1–7.
- Barnett JH. Generating Pythagorean triples: The methods of Pythagoras and of Plato via Gnomons. Number Theory. 2017.
- Proclus. A commentary on the first book of Euclid’s Elements. Princeton University Press; 1992.
- Joyce DE. Euclid’s Elements. Clark University; 1997.
- Bruno LC. Math and Mathematicians: The History of Math Discoveries Around the World. UXL; 2002.
- Swetz FJ, Katz VJ. Mathematical treasures—Michael Stifel’s Arithmetica Integra. Available online: <https://old.maa.org/press/periodicals/convergence/mathematical-treasures-michael-stifels-arithmetica-integra> (accessed on 18 January 2025).
- Ozanam J. Recreations in Mathematics and Natural Philosophy. Longman, Hurst, Rees, Orme, and Brown; 1814.
- Rukavicka J. Dickson’s method for generating Pythagorean triples revisited. European Journal of Pure and Applied Mathematics. 2013; 6(3).
- Dickson LE. History of the Theory of Numbers, Volume II: Diophantine Analysis. Dover Publication Inc.; 2013.

15. Hu J, Zhan Y. Pythagorean triples and quadratic residues modulo an odd prime. *AIMS Mathematics*. 2022; 7(1): 957–966.
16. Overmars A, Ntogramatzidis L, Venkatraman S. A new approach to generate all Pythagorean triples. *AIMS Mathematics*. 2019; 4(2): 242–253.
17. Moussounda Mouanda J. On Fermat’s Last Theorem and Galaxies of sequences of positive integers. *American Journal of Computational Mathematics*. 2022; 12(1): 162–189.
18. Berggren B. Pythagorean triangles (in Swedish). *Tidskrift för elementär matematik, fysik och kemi*. 1934; 17: 129–139.
19. Arnold M, Eydelzon A. On Matrix Pythagorean Triples. *The American Mathematical Monthly*. 2019; 126(2).
20. Paulsen VI. Completely bounded maps and dilations. Longman Scientific and Technical; 1986.