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Article

Perturbation theory of the adiabatic pressure-less rarefied-fluid axisymmetric accretion objects

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Abstract: Pressure-less rarefied-fluid axisymmetric accretion objects are studied. The inviscid case and the rarefied viscous case with constant azimuthal component of the four-velocity in the pressure-less regime are addressed. The new General-Relativistic radial velocity slip flows expressions are analytically written in both the instances.

Keywords: axisymmetric accretion objects; inviscid fluid; rarefied viscous fluid

1. Introduction

In the work of Kandrup [1], the "local adiabatic stability of stationary axisymmetric perfect fluids" in General Relativity is questioned.

Ibidem, the problem is posed, about hat happen when the Eulerian perturbations of the pressure cannot be neglected.

Ibidem, the perturbations of the metric are neglected after the work of Seguin [2] and after the work of Tassoul [3].

The dynamics of accretion of unmagnetised fluids is recapitulated in the work of Vasyliunas [4]. In the analysis of Vasyliunas [4], the X-ray emission in the accretion process is studied as determined after the release of gravitational energy.

The present paper is aimed at studying adiabatic axisymmetric pressure-less fluid configurations of the axisymmetric accretion object. In the case of rarefied fluid, a new radial velocity slip flow is found. In particular, two cases are studied, i.e., the inviscid case, ad the viscous rarefied case $u^{\phi} = const$. In both cases, the dissipative mechanisms are found to act on the radial component of the velocity only.

The protocol here followed after the work of Kandrup [1], after the work of Friedman and Schutz [5] and after the work of Lecian [6] is one in which perturbation calculation is performed, where the perturbations do not necessarily consist of infinitesimal parameter, i.e., but also large perturbations are present, as the adiabatic perturbations of the pressure, which are next order with respect to the ordinary quantities without being necessarily infinitesimal.

The protocol here used can be compared with the homotopy-perturbation method. The homotopy-perturbation methods was introduced in the work of He [7], of Zheng [8] and of Chakraverty et al. [9] in order to modellize problems related to one here addresses without the introduction of infinitesimal parameters, and with the further properties of being apt also for semi-analytical solutions; nevertheless, in the present paper, the paradigms implemented follow those stated from Lecian [6] and are fully analytical. The von-Neumann approach is also introduced in the work of Lecian [6]. The Lagrangian perturbation technique for viscosity problem is studied in the work of

Shen et al. [10]; even though the methods of artificial viscosity [11–13] can be used to isolate the behaviour of shock (waves) (which are of interest in General-Relativistic Hydrodynamics in the most general setting), the methods here followed do not make use of the artificial-viscosity technique. The self-similar solutions of fluid dynamics which are reconducted to shock waves are not addressed in the present analysis.

As results, the new analytical expressions of the General-Relativistic slip flow radial velocities are written in the inviscid case and in the rarefied viscous case $u^{\phi} = const$. The new phenomena are dissipative mechanisms which are due to the fact that in the radial direction the temperature gradients is decreasing in the direction perpendicular to the boundary.

The variations of the gravitational potential are afterwords taken into account as well.

From the study of Vasyliunas [4], the study of unmagnetised fluids is motivated as for the understanding about which processes of the magnetic fields from the accreting objects can enter the accretion cylinder.

As a new result, the limit to a Keplerian azimuthal motion is newly calculated (in the absence of a magnetic field).

The novelty of the paper relies also in the new exact analytical calculation methods in 3 space dimensions (plus one time dimension): in the previous investigation presented in the literature, as more thoroughly recapitulated in the Outlook Section, the previous material is mostly about numerical simulations in 1-space dimension (plus one time dimension) and 2 space dimensions (plus one time dimension), while, in the present paper, the complete Schwarzschild spacetime in 3 space dimensions (plus one time dimension) is considered; as a new result which is here newly found after the items of bibliography in which the analytical apporach is followed, a constant u_{ϕ} velocity is proven not to modify the canonical energy of the perturbation(s).

Furthermore, as a prospective investigation, the cases of the Generalized-Schwarzschild spacetimes are envisaged, as a result, new spacetime surfaces are individuated, whose role and properties will be further delineated in a forthcoming paper of the Author elsewhere.

The paper is organised as follows.

In Section 2, the perturbation theory of fluids is recapitulated from the work of Kandrup [1].

In Section 3, the inviscid pressure-less-fluid axisymmetric accretion object theory is newly introduced.

In Section 4, the inviscid pressure-less fluid methods are reported about.

In Section 5, the theory of the viscous pressure-less fluid i recalled.

More in detail, the axisymmetric rarefied pressure-less perfect inviscid fluid accretion objects are newly studied; the analytical expression of the General-Relativistic slip flow radial velocity is newly written.

Furthermore, the results are extended to the study of rarefied viscous fluids, i.e., ones with $u^{\phi} = const$.

In Section 6, the limit of the Keplerian azimuthal motion is calculated.

The results are discussed in Section 7.

The application to Generalized Schwarzschild spacetimes is prospected in Appendix A, where the geometrical objects for the calculation of the Einstein Field Equations are calculated.

The geometrical objects used in the case of the Schwarzschild spacetime are reported in Appendix B.

2. Introductory material

The introductory material to study the perturbations of the static inviscid dust cylinder is here gathered from the work of Kandrup [1]; to these purposes, the relevant items of steps are reported of the perfect viscous fluid, from which the wanted theory will be worked out analytically in the next Sections.

The perfect viscous fluid is taken as an example in order to implement the variation calculation.

The stress-energy tensor of the perfect fluid $T^{\mu\nu}$ is written as

$$T^{\mu\nu} = (\epsilon + p)U^{\mu}U^{\nu} + pg^{\mu\nu} \tag{1}$$

In the case of the perfect viscous fluid, the component of the 4-velocity u_{ϕ} is taken as vanishing in the work of Kandrup [1].

From pag. 691 ibidem, the canonical energy of the perturbation id defined to be negative unless the condition

$$\gamma p v^2/(\epsilon + p) \leq 1 \tag{2}$$

The existence of a timelike Killing field is hypothesized, after which the Lagrangean perturbations can be associated.

The timelike vector field ensures the existence of a Hilbert space where the (generic) operators A, B, and C live.

The canonical energy E_c is written as

$$E_c = \frac{1}{2} < \partial_t \xi, A\partial_t \xi > + \frac{1}{2} < \xi, C\xi > \tag{3}$$

where A and C are symmetric operators, and B is an anti-symmetric operator (the operator B will be used elsewhere).

Stationary configurations are studied as stable iff they are ascribed to purely dissipative mechanisms $dE_c/dt \leq 0$. The canonical energy E_c must be non-negative for all initial data.

Given ξ the quantity after which the perturbation is written, the following condition on the corresponding 3-vectors holds

$$\vec{u} \cdot \vec{\xi} = 0 \tag{4}$$

The variation are written as

$$\delta s = -\xi^\nu \nabla_\nu s, \tag{5a}$$

$$\delta p = -\gamma p \nabla_n u \xi^\nu - \left[1 + \frac{\gamma p}{\epsilon + p}\right] \xi^\nu \nabla_\nu p, \tag{5b}$$

$$\delta \epsilon = -(\epsilon + p)\nabla_\nu \xi^\nu - \xi^\nu \nabla_\nu (\epsilon + p) \tag{5c}$$

Equation (18) from the work of Kandrup [1] is here reported as

$$\nabla_\mu p = -(\epsilon + p) u^\nu \nabla_\nu u_\mu \tag{6}$$

From Equation (22) ibidem, the tensor $U^{\mu\nu\sigma\chi}$ is defined after the tensor $Q^{\mu\nu}$ as

$$Q^{\mu\nu} = g^{\mu\nu} + U^\mu U^\nu \tag{7}$$

from which the definition follows

$$U^{\mu\nu\sigma\chi} = (\epsilon + p) U^\mu U^\sigma Q^{\nu\chi} \tag{8}$$

From Equation (38) ibidem, the quantity L is defined for a viscous fluid

$$L \equiv -\frac{u_\phi}{u_t} \tag{9}$$

The tensor Λ_A is defined from Equation (65) ibidem as

$$\Lambda_A \equiv (u^t u_t)^2 \left[(1 - v^2) e^{-2\psi} (u_t)^2 \partial_A L \partial_A \Omega\right] \tag{10}$$

from which the condition must hold

$$\frac{dL}{d\Omega} \geq e^{2\psi} (u_t)^{-2} (1 - v^2)^{-1} \tag{11}$$

From the work of Abramowicz [14], the surfaces of constant L and the surfaces of constant Ω “necessarily” coincide in the “barotropic configurations”.

The canonical energy is written as

$$E_c \equiv K + W \equiv \frac{1}{2} \int dV \frac{1}{(\epsilon + p)} e^{2\mu[(\delta u^r)^2 + (\delta u^\theta)^2]} + \frac{1}{2} \int dV \epsilon e^{2\mu} \frac{\partial \epsilon}{\partial s} \frac{dP}{ds} \mid \xi^A \partial_A s \mid^2 + (u_t u^t)^2 \left[(1 - v^2) e^{-2\psi} (u_t)^2 - \frac{\partial \Omega}{\partial L} \mid \xi^A \partial_A L \mid^2\right] \tag{12}$$

with $dV = \sqrt{-g} d^3 x$ and $\vec{v}$ the 3-velocity of the fluid measured after an observer with vanishing angular momentum.

3. The inviscid pressure-less-fluid axisymmetric accretion object

The canonical energy if the perturbation of the pressure-less fluid is studied from Equation (2) as

$$lim_{p \to 0} \gamma p v^2 / (\epsilon + p) = 0 \leq 1 \tag{13}$$

The canonical energy of the perturbation is therefore here looked for as semi-positive definite.

The variations are specified for a pressure-less fluid whose nature is specified after the entropy s as

$$\delta s = -\xi^{\nu}\nabla_{\nu}s, \tag{14a}$$

$$= 0\delta p = -\xi^{\nu}\nabla_{\nu}p, \tag{14b}$$

$$\delta\epsilon = -(\epsilon)\nabla_{\nu}\xi^{\nu} - \xi^{\nu}\nabla_{\nu}(\epsilon) \tag{14c}$$

From Equation (14b), it (redundantly) follows that the saturated expression

$$\xi^{\nu}\nabla_{\nu}p = 0 \tag{15}$$

must hold.

The conditions on the pressure are now calculated.

From the metric tensor Equation (1), the following conditions are obtained, which must then be compared with among themselves and with the further conditions:

$$pg^{tt} = P\frac{r}{r - r_S + r\psi} << 1, \tag{16a}$$

$$p \mid g^{rr} \mid = p\frac{r - r_S + r\psi}{r} << 1, \tag{16b}$$

$$p \mid g^{\theta\theta} \mid = p\frac{1}{r^2} << 1, \tag{16c}$$

$$p \mid g^{\phi\phi} \mid = p\frac{1}{r^2(sin\theta)^2} << 1 \tag{16d}$$

and

$$p(u^0)^2 << 1, \tag{17a}$$

$$p(u^0u^r) << 1, \tag{17b}$$

$$p(u^r)^2 << 1 \tag{17c}$$

For vanishing p, Equation (18) from the work of Kandrup [1] is reconducted to

$$u^{\nu}\nabla_{\nu}u_{\mu} = 0 \tag{18}$$

From the mass flux conservation and fr0m the energy flux conservation the following expression of ρ is found

$$\rho = \frac{1}{2}\frac{1}{r^4(u^r)^2}\left(pr^4(u^r)^2 + c_1(u^r)^2 - c_2^2 + \left[p^2r^8(u^r)^4 + 2c_1^2pr^4(u^r)^4 + 2c_2^2pr^4(u^r)^2 + c_1^4(u^r)^4 - 2c_1^2c_2^2(u^r)^2 + c_2^4\right]^{1/2}\right) \tag{19}$$

where c_1 and c_2 are the integration constant, respectively. From Equation (19), the two conditions must be compared for the vanishing of the largest term containing p after the determination of u^0 and of u^r from $u_{\mu}u^{\mu}$.

4. The inviscid pressure-less fluid

In the case of inviscid pressure-less fluid, the component u_{ϕ} is vanishing.

As a result, the canonical energy of the perturbation is not negative.

The condition $\vec{U} \cdot \vec{\xi} = 0$ implies a vanishing ξ^t as

$$0 \equiv \xi^t = -\frac{u_\phi}{u_t}\xi^\phi \tag{20}$$

Because u^ϕ is vanishing, therefore L from Equation (9) is a vanishing quantity as well.

It is our aim to reconduct the expression of the canonical energy Equation (12) to that of the chosen instance.

The expression of the canonical energy is here enwly spelled out is reconducted to the expression

$$E_c \equiv K + W \equiv \frac{1}{2}\int dV \frac{1}{(\epsilon)} e^{2\mu[(\delta u^r)^2 + (\delta u^\theta)^2}] + \frac{1}{2}\int dV (u_t u^t)^2 \left[(1-v^2)e^{-2\psi}(u_t)^2\right] : \tag{21}$$

it is our aim to stress that the canonical energy is not changed after the presence of a constant u_ϕ.

4.1. The new General Relativistic radial velocity slip flow of the inviscid fluid

The condition $\vec{u} \cdot \vec{\xi} = 0$ is now implemented as

$$\delta u^t = -\frac{u_\phi}{u_t}\delta u^\phi \tag{22}$$

in the case of inviscid fluid,

$$\delta u^\phi \equiv 0 \tag{23}$$

i.e., also from Equation (60) from the work of Kandrup [1].

One therefore has

$$\epsilon u^\mu u_\nu u^\sigma \nabla_\sigma \delta u^\mu = 0 \tag{24}$$

As

$$\delta u^t = 0 \tag{25}$$

then from Equation (61) ibidem

$$\delta \epsilon = 0 \tag{26}$$

and the canonical energy of the perturbation becomes

$$E_c(u^\phi = 0) = \frac{1}{2}\int dV \epsilon e^{2\mu}\left[(\delta u^r)^2 + (\delta u^\theta)^2\right] \tag{27}$$

From $\delta u^t = 0$ and from

$$\left(\frac{\partial \epsilon}{\partial s}\right)_p = 0 \tag{28}$$

one has that

$$\xi^B \partial_B L = 0 \tag{29}$$

The perturbation equations therefore become

$$\epsilon u^\mu u_i u^\sigma \nabla_\sigma \delta u^i + \epsilon u^\sigma \nabla_\sigma \delta u^j = 0 \tag{30}$$

with $i, j = r, \theta, \phi$. Equation (30) is now split as

$$\epsilon u^0 u_i u^\sigma \nabla_\sigma \delta u^i = 0, \tag{31a}$$

$$\epsilon u^j u_i u^\sigma \nabla_\sigma \delta u^i + \epsilon u^\sigma \nabla_\sigma \delta u^j = 0 \tag{31b}$$

From Equation (31a), the new condition is in this case found

$$u^0 = 0 \tag{32}$$

from which

$$\epsilon u^j u_i u^\sigma \nabla_\sigma \delta u^i + \epsilon u^\sigma \nabla_\sigma \delta u^j = 0 \tag{33}$$

From the condition $u^\mu u_\mu = -1$ one finds that, for vanishing u^ϕ and for a fixed quote $u^\theta = const$ at the fixed θ^* such that

$$u_\mu u^\mu = u^0 u_0 + u^r u_r + u^\theta u_\theta = -1; \tag{34}$$

For the axisymmetric configuration the component of the metric tensor are reported in appendix A. The component u^θ is posed as $u^\theta = \sqrt{C_5}$. The radial component of the velocity u^r is found as

$$u^r = \left[e^{2\lambda} \left(1 + C_5 r^2 e^{2\lambda} \right) \right]^{1/2} \tag{35}$$

Equation (17c) must be controlled.

4.2. The variations of the gravitational potential

From Equation (21), after the methodology suggested in the work of Friedmann and Schutz [5], the equations of motion of unperturbed inviscid fluid are given as

$$\rho \partial_t^2 \xi_\mu + 2\rho u^\nu \nabla_\nu \partial_t \xi_\mu + \rho (u^\nu \nabla_\nu)^2 \xi_\mu - \nabla_\mu (\gamma \rho \nabla_\nu \xi^\nu) + \nabla_\mu p \nabla_\nu \xi^\nu - \nabla_\nu p \nabla_\mu \xi^\nu + \rho \xi^\nu \nabla_\nu \nabla_\mu \Phi + \rho \nabla_\mu \delta \Phi = 0 \tag{36}$$

For a pressure-less inviscid fluid, the equations of motion are modified after posing $p = 0$ and $\partial_t xi^\mu = 0$ in Equation (36) as

$$\rho (u^\nu \nabla_\nu)^2 \xi - \nabla_\mu (\gamma \rho \nabla_\nu \xi^\nu) + \rho \xi^\nu \nabla_\nu \nabla_\mu \Phi + \rho \nabla_\mu \delta \Phi = 0 \tag{37}$$

After the study of Vasyliunas [4], Equation (eqy1a) is split as

$$\rho (u^\nu \nabla_\nu)^2 \xi - \nabla_\mu (\gamma \rho \nabla_\nu \xi^\nu) + \rho \xi^\nu \nabla_\nu \nabla_\mu \Phi = 0, \tag{38a}$$

$$\rho \nabla_\mu \delta \Phi = 0 \tag{38b}$$

5. The viscous pressure-less fluid

In the simplest instance, the case $u^\phi \equiv const$ can be examined.

The canonical energy E_C is rewritten as

$$E_c \equiv K + W \equiv \frac{1}{2}\int dV \frac{1}{(\epsilon)} e^{2\mu}[(\delta u^r)^2 + (\delta u^\theta)^2] \tag{39}$$

It is understood that in this case the role of L in the case of constant non-vanishing U^ϕ is not comprehended in the definition of the canonical energy.

5.1. The new General Relativistic radial velocity slip flow of the viscous fluid with constant U^ϕ

The considerations developped in the case of the inviscid fluid can be here reconducted.

From pag 701 of the work of Kandrup [1], Δu_ϕ is a constant of motion; the instance from the work of Lebovitz [15] can be here taken, that a restriction on the possible initial data is taken in order to have $\Delta u_\phi = 0$. The constant ϕ ensures no perturbations on u^ϕ i.e., such that Equation (38) from the work of Kandrup [1] is still well-posed.

For constant $U^\phi \equiv \sqrt{C_6}$, one has that the new General-relativistic radial velocity slip flow of the $U^\phi = const$ pressure-less fluid as

$$u^r = \left[\frac{1}{e^{2\lambda}}\left(1 + r^2 e2\lambda C_5 + r^2 e^{2\lambda}(sin\theta^*)^2 C_6\right)\right]^{1/2} \tag{40}$$

where the integration constant C_5 and C_6 are present.

6. The limit to Keplerian azimuthal motion

The present analysis is also motivated at introducing the prospective studies envisaged in the work of Vasyliunas [4]; more in detail, the X-ray emission of accretion objects is ascribed in the work of Vasyliunas [4] from the change of matter content of the accretion object. More specifically, the accretion objects considered in the work of Vasyliunas [4] are discs; in the present analysis, the model of axisymmetric accretion objects is developped: as a prospective investigation, the magnetosphere of a blackhole object with axisymmetric accretion object with varying matter content will be extended with the sake of finding the proper description of the variation of the gravitational field of the accretion object. As a most direct application of the model, the magnetosphere of these systems can be studied, i.e. , with respect to the General-Relativistic Magneto-Hydro-Dynamical description. The azimuthal component of the momentum equation can therefore be considered for the complete problem which, in the axisymmetric approach, is of less direct implementation with respect to that of the spherical accretion. More in detail, the tangential stress $\mathcal{T}_{\mathcal{R}\phi}$ of the cylindrical accretion object of radius $\mathcal{R}$ is calculated, in absence of magnetic field, as the viscous stress with η_{vis} coefficient of viscosity as

$$\mathcal{T}_{\mathcal{R}\phi} = \eta_{vis}\mathcal{R}\frac{d}{d}\left(\frac{V_\Phi}{\mathcal{R}}\right) \tag{41}$$

From the tangential stress $\mathcal{T}_{\mathcal{R}\phi}$ Equation (41), from the azimuthal component of the momentum equation, the follwing equality is found

$$\dot{M}\frac{d}{d\mathcal{R}}(d\mathcal{R}V_\Phi) = \frac{d}{d\mathcal{R}}2\pi\mathcal{R}\int d\mathcal{R}\int dz\mathcal{T}_{\mathcal{R}\phi} \tag{42}$$

In the case the forces obtained after viscosity are neglected, the accretion object becomes a Newtonian one, i.e., with Keplerian azimuthal motion with strong shear-related forces as

$$V_\Phi = \sqrt{\frac{M_x G}{\mathcal{R}}} \tag{43}$$

The energy radiated from the disc at each distances is calculated after the equation of energy conservation; more in detail, the radiation equation indicates the temperature changes and the pressure, i.e., the vertical structure of the accretion object and the related thickness. The presence of a magnetic field will be addressed in a following paper of Lecian.

7. Discussion

The rarefied perfect fluid in the pressure-less case in the axisymmetric configuration of accretion objects are studied. The new analytical General-Relativistic expression of the radial component of the velocity is found, which has the features of the 'thermal-stress slip flow velocity' in the radial direction only. The analysis of the pressure-less case is obtained after restricting the most general initial data to the wanted characterisation such that $\Delta u_\phi \equiv 0$, which implies $\Delta L \equiv 0$. The use of the operators A, B and C is found in the present analysis redundant.

The new characterisations of the radial velocities indicate that the dissipative mechanisms according to which the temperature is descending are present only in the radial direction in both the instances.

The topic of shock waves in accretion objects has not been investigated much further. More in detail, numerical simulation were performed about shock waves in accretion objects, and, in particlur, about shock waves in plasma cylindrical accretion objects. As an example, in the work of Giri [16], numerical simulations are run to verify the existence of self-consistent generalized viscous transonic flow shock solutions. Furthemore, in the work of Kim [17], the sub-Keplerian transonic accretion flows are investigated numerically: as pointed out ibidem, no complete theoretical analysis is present in the literature. A pioneering investigation is provided with in the work of El Mellah [18].

It is the aim of further studies of forthcoming papers of the Author to include the formalisms frm the work of Spruit [19] and the work of Kato and Fukue [20].

Conflict of interest: The author declares no conflict of interest.

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Appendix A

Generalized-Schwarzschild spacetimes

It is here recalled that it isis possible to extend the study in the case of Generalized-Schwarzschild spacetimes.

The generalization considered is the presence of a new generalized potential $\Psi(r)$ in the g_{tt} component of the metric tensor (and in that of the g_{rr} component of the metric tensor as consequence.) The non-vanishing components of the Generalized Schwarzschild spacetimes of the metric tensor $g^{\mu\nu}$ are written as

$$g^{tt} = \frac{r}{r - r_S + r\Psi}, \tag{A1a}$$

$$g^{rr} = -\frac{r - r_S + r\Psi}{r}, \tag{A1b}$$

$$g^{\theta\theta} = -\frac{1}{r^2}, \tag{A1c}$$

$$g^{\phi\phi} = -\frac{1}{r^2(sin\theta)^2} \tag{A1d}$$

The Einstein Field Equations are requested to be obeyed, i.e., new analytical relations arise between the components o the metric tensor.

The Einstein Field Equations are written after the geometrical objects R the Ricci scalar and $R_{\mu\nu}$ the Ricci tensor of generalized Schwarzschild spacetimes of line element

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2 theta^2 - r^2(sin\theta)^2\phi^2 \tag{A2}$$

as

$$R = -\frac{1}{r^2}\left[r^2\frac{d^2f}{dr^2} + 4\frac{df}{dr} + 2f - 2\right] \tag{A3}$$

and

$$R_{tt} = -\frac{1}{2}\frac{1}{r^2}\left[r\frac{d^2f}{dr^2} + 2\frac{df}{dr}\right], \tag{A4a}$$

$$R_{rr} = \frac{1}{2}\frac{1}{rf}\left[r\frac{d^2f}{dr^2} + 2\frac{df}{dr}\right], \tag{A4b}$$

$$R_{\theta\theta} = r\frac{df}{dr} + f - 1, \tag{A4c}$$

$$R_{\phi\phi} = (sin\theta)^2 R_{\theta\theta}. \tag{A4d}$$

As a new result, new hyper-surfaces are defined in vacuum, which have to be studied (in a forthcoming paper). Furthermore, the presence of matter induces the definition of new manifolds, whose properties have to be investigated accordingly ibidem.

Appendix B

Geometrical objects

The geometrical objects employed in the present paper are here given.

The alternative form here used in isotropic coordinates of the metric tensor is given as for the component of the line element

$$ds^2 = e^{2\nu}dt^2 - e^{2\lambda}\left(dr^2 + r^2 d\theta^2 + r^2 (sin\theta)^2 d\phi^2\right) \tag{B1}$$

with

$$e^{2\nu} = \left(\frac{1 - \frac{r_S}{4r}}{1 + \frac{r_S}{4r}}\right)^2 \tag{B2}$$

and

$$e^{2\lambda} = \left(1 + \frac{r_S}{4r}\right)^4. \tag{B3}$$