

A note on Schur convexity for compositions of complete symmetric functions and new simple proofs

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Abstract: In this work, by applying the properties of Schur-convex functions, Schur-geometrically convex functions and Schur-harmonically convex functions directly, we will provide more direct and simple proofs for compositions of complete symmetric functions.

Keywords: complete symmetric function; Schur-convexity; Schur-geometric convexity; Schur-harmonic convexity; composite function

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1. Introduction and motivations

Convex analysis has been recognized as an important and powerful tool for many fields of science, such as applied mathematics, optimization, communications and networks, physics, and so on. A number of new concepts of generalized convexity and concavity have been introduced by many authors; for more details, see [1–13] and references therein. Over the past two decades. Great attention has been paid to the study of the Schur convexity of functions (see, e.g., [1–4, 9–12]).

Let $\mathbb{R} = \mathbb{R}^1 := (-\infty, +\infty)$, $\mathbb{R}_+ = \mathbb{R}_+^1 := (0, +\infty)$ and $\mathbb{R}_- = \mathbb{R}_-^1 := (-\infty, 0)$.

Definition 1. (see [5, 6])

(i) $\emptyset \neq D \subset \mathbb{R}^n$ is said to be convex if $\mathbf{a} = (a_1, a_2, \dots, a_n)$, $\mathbf{b} = (b_1, b_2, \dots, b_n) \in D$ and $0 \leq \beta \leq 1$ implies

$$\beta \mathbf{a} + (1 - \beta) \mathbf{b} = (\beta a_1 + (1 - \beta) b_1, \beta a_2 + (1 - \beta) b_2, \dots, \beta a_n + (1 - \beta) b_n) \in D.$$

(ii) Let $\emptyset \neq D \subset \mathbb{R}^n$ be a convex set. A function $\omega: D \rightarrow \mathbb{R}$ is said to be convex on D if

$$\omega(\beta \mathbf{a} + (1 - \beta) \mathbf{b}) \leq \beta \omega(\mathbf{a}) + (1 - \beta) \omega(\mathbf{b})$$

for all $\mathbf{a}, \mathbf{b} \in D$ and all $\beta \in [0, 1]$. The function ω is said to be concave on D if and only if $-\omega$ is convex on D .

Definition 2. (see [5, 6, 14]) Let $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$.

(i) $\mathbf{a} \geq \mathbf{b}$ means $a_j \geq b_j$ for all $j = 1, 2, \dots, n$.

(ii) \mathbf{a} is said to be majorized by \mathbf{b} (in symbols $\mathbf{a} \prec \mathbf{b}$) if $\sum_{r=1}^n a_r = \sum_{r=1}^n b_r$ and $\sum_{r=1}^k a_{[r]} \leq \sum_{r=1}^k b_{[r]}$ for $k = 1, 2, \dots, n - 1$, where $a_{[1]} \geq a_{[2]} \geq \dots \geq a_{[n]}$ and $b_{[1]} \geq b_{[2]} \geq \dots \geq b_{[n]}$ are rearrangements of \mathbf{a} and \mathbf{b} in a descending order.

Definition 3. (see [5,6]) Let $\emptyset \neq D \subset \mathbb{R}^n$ and $v : D \rightarrow \mathbb{R}$ be a function.

- (i) v is said to be increasing if $\mathbf{a}, \mathbf{b} \in D$ with $\mathbf{a} \geq \mathbf{b}$ implies $v(\mathbf{a}) \geq v(\mathbf{b})$.
- (ii) v is said to be decreasing if $-v$ is increasing.
- (iii) v is said to be a Schur-convex (abbreviated as **S-Conv**) function on D if $\mathbf{a} \prec \mathbf{b}$ on $D \Rightarrow v(\mathbf{a}) \leq v(\mathbf{b})$.
- (iv) v is said to be Schur-concave (abbreviated as **S-Conc**) on D if $-v$ is **S-Conv** on D .

The Schur-convexity described the ordering of majorization, the order-preserving functions were first comprehensively studied by Issai Schur in 1923. In Schur’s honor, such functions are said to be “Schur-convex”. It has many important applications in stochastic majorizations, matrix theory, combinatorial analysis, geometric inequalities, numerical analysis and other related fields (see [5]).

Definition 4. (see [7,8]) Let $\emptyset \neq D \subset \mathbb{R}_+^n$.

- (i) D is called a geometrically convex set (see [7] (p. 64)) if $(a_1^\beta b_1^{1-\beta}, a_2^\beta b_2^{1-\beta}, \dots, a_n^\beta b_n^{1-\beta}) \in D$ for all $\mathbf{a} = (a_1, a_2, \dots, a_n), \mathbf{b} = (b_1, b_2, \dots, b_n) \in D$ and $\beta \in [0, 1]$.
- (ii) The function $\omega : D \rightarrow \mathbb{R}_+$ is said to be Schur-geometrically convex (abbreviated as **S-G-Conv**) (see [7] (p. 107)) on D , for any $\mathbf{a}, \mathbf{b} \in D$, if

$$(\log a_1, \log a_2, \dots, \log a_n) \prec (\log b_1, \log b_2, \dots, \log b_n)$$

implies $\omega(\mathbf{a}) \leq \omega(\mathbf{b})$. The function ω is said to be Schur-geometrically concave (abbreviated as **S-G-Conc**) on D if and only if $-\omega$ is **S-G-Conv** on D .

The Schur-geometric convexity was proposed by Zhang [7] in 2004, and was investigated by Chu et al. [8]. Some authors use the term “Schur multiplicative convexity”.

In 2009, Chu [2, 9, 10] introduced the notion of Schur-harmonically convex function.

Definition 5. Let $\emptyset \neq D \subset \mathbb{R}_+^n$.

- (i) D is said to be harmonically convex if $(\frac{2a_1 b_1}{a_1 + b_1}, \frac{2a_2 b_2}{a_2 + b_2}, \dots, \frac{2a_n b_n}{a_n + b_n}) \in D$ for every $\mathbf{a} = (a_1, a_2, \dots, a_n), \mathbf{b} = (b_1, b_2, \dots, b_n) \in D$.
- (ii) A function $\omega : D \rightarrow \mathbb{R}_+$ is said to be Schur-harmonically convex (abbreviated as **S-H-Conv**) on D , for any $\mathbf{a}, \mathbf{b} \in D$, if $(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}) \prec (\frac{1}{b_1}, \frac{1}{b_2}, \dots, \frac{1}{b_n})$ implies $\omega(\mathbf{a}) \leq \omega(\mathbf{b})$. A function ω is said to be Schur-harmonically concave (abbreviated as **S-H-Conc**) on D if and only if $-\omega$ is **S-H-Conv** on D .

Let $n \in \mathbb{N}$ (the set of positive integers). The complete symmetric function $c_n(\mathbf{b}, t)$ is defined by

$$c_n(\mathbf{b}, t) = \sum_{k_1+k_2+\dots+k_n=t} b_1^{k_1} b_2^{k_2} \dots b_n^{k_n}, \tag{1}$$

where $\mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n := \{\mathbf{b} = (b_1, b_2, \dots, b_n) : b_j \in \mathbb{R}, j = 1, \dots, n\}$,

$t \in \{1, 2, \dots, n\}$ and k_1, k_2, \dots, k_n are non-negative integers. In particular, we define

$$c_n(\mathbf{b}, 0) = 1 \quad \text{for any } \mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n.$$

The completely symmetric functions are an important class of symmetric functions, and many scholars pay attention to this function and obtain some interesting results. For example, Guan [1] studied the Schur-convexity of this function and proved the following important result in 2006:

Proposition 1. $c_n(\mathbf{b}, t)$ is increasing in b_j for every $j = 1, 2, \dots, n$, and is **S-Conv** on $\mathbb{R}_+^n := \{\mathbf{b} = (b_1, b_2, \dots, b_n) : b_j > 0, j = 1, \dots, n\}$.

In 2011, Chu et al. [2] obtained the following result:

Proposition 2. $c_n(\mathbf{b}, t)$ is Schur-geometrically convex and Schur-harmonically convex on \mathbb{R}_+^n .

Additionally, in 2016, Shi et al. [3] considered the Schur-convexity of $c_n(\mathbf{b}, t)$ on $\mathbb{R}_-^n := \{\mathbf{b} = (b_1, b_2, \dots, b_n) : b_j < 0, j = 1, \dots, n\}$ as follows:

Proposition 3. If t is even integer (or odd integer, resp.), then $c_n(\mathbf{b}, t)$ is **S-Conv** and decreasing (or **S-Conc** and increasing, resp.) on \mathbb{R}_-^n .

In 2017, Sun et al. [4] studied the Schur, Schur-geometric and Schur-harmonic convexities of the following composite function $S_n(\mathbf{b}, t)$:

$$S_n(\mathbf{b}, t) = \sum_{k_1+k_2+\dots+k_n=t} \left(\frac{1+b_1}{1-b_1}\right)^{k_1} \left(\frac{1+b_2}{1-b_2}\right)^{k_2} \dots \left(\frac{1+b_n}{1-b_n}\right)^{k_n}. \quad (2)$$

As an application, some inequalities have been established using majorization theory, particularly new geometric inequalities in high-dimensional spaces(refer to reference [4] for details).

Using Lemmas 1–3 in the next section respectively, Sun et al. [4] established the following Theorems 1–3, respectively.

Theorem 1. For $\mathbf{b} = (b_1, b_2, \dots, b_n) \in (1, +\infty)^n \cup [0, 1]^n$ and $t \in \mathbb{N}$, the following hold:

- (i) $S_n(\mathbf{b}, t)$ is increasing in b_j ($j = 1, 2, \dots, n$), and is **S-Conv** on $[0, 1]^n$;
- (ii) If t is even integer (or odd integer, resp.), then $S_n(\mathbf{b}, t)$ is decreasing (or increasing, resp.) in b_j ($j = 1, 2, \dots, n$), and is **S-Conv** (or **S-Conc**, resp.) on $(1, +\infty)^n$.

Theorem 2. For $\mathbf{b} = (b_1, b_2, \dots, b_n) \in (1, +\infty)^n \cup [0, 1]^n$ and $t \in \mathbb{N}$, the following hold:

- (i) $S_n(\mathbf{b}, t)$ is Schur-geometrically convex on $[0, 1]^n$;
- (ii) If t is even integer (or odd integer, resp.), then $S_n(\mathbf{b}, t)$ is Schur-geometrically convex (or Schur-geometrically concave, resp.) on $(1, +\infty)^n$.

Theorem 3. For $\mathbf{b} = (b_1, b_2, \dots, b_n) \in (1, +\infty)^n \cup [0, 1]^n$ and $t \in \mathbb{N}$, the following hold:

- (i) $S_n(\mathbf{b}, t)$ is Schur-harmonically convex on $[0, 1]^n$;
- (ii) If t is even integer (or odd integer, resp.), then $S_n(\mathbf{b}, t)$ is Schur-harmonically

convex (or Schur-harmonically concave, resp.) on $(1, +\infty)^n$.

In fact, the original proof of Theorems 1–3 is more complicated by using Lemmas 1–3. (for more details, see [4]). In this paper, applying the properties of Schur-convex functions, Schur-geometrically convex functions and Schur-harmonically convex functions directly, we will provide more direct and simple proofs of these results.

2. Preliminaries

In this section, we review some definitions and known results that will be used in this article.

Definition 6. (see [5,6]) Let $\emptyset \neq D \subset \mathbb{R}^n$.

- (i) D is called symmetric, if $\mathbf{a} \in D \Rightarrow \mathbf{a}Q \in D$ for every $n \times n$ permutation matrix Q .
- (ii) A function $\omega : D \rightarrow \mathbb{R}$ is called symmetric if for each permutation matrix Q , $\omega(\mathbf{a}Q) = \omega(\mathbf{a})$ for all $\mathbf{a} \in D$.

Lemma 1. (see [5] (p. 84), [6]) Let $D \subset \mathbb{R}^n$ be symmetric convex set with nonempty interior. D° is the interior of D . The function $\omega : D \rightarrow \mathbb{R}$ is continuous on D and continuously differentiable on D° . Then ω is **S-Conv** (or **S-Conc**, resp.) if and only if ω is symmetric on D and

$$(a_1 - a_2) \left(\frac{\partial \omega}{\partial a_1} - \frac{\partial \omega}{\partial a_2} \right) \geq 0 \text{ (or } \leq 0, \text{ resp.)} \tag{3}$$

holds for any $\mathbf{a} \in D^\circ$.

The following is obvious from Definition 4.

Proposition 4. Let $\emptyset \neq D \subset \mathbb{R}_+^n$, and let

$$\log D = \{(\log a_1, \log a_2, \dots, \log a_n) : (a_1, a_2, \dots, a_n) \in D\}.$$

Then $h : D \rightarrow \mathbb{R}_+$ is a **S-G-Conv** (or **S-G-Conc**, resp.) function on D if and only if $h(e^{a_1}, e^{a_2}, \dots, e^{a_n})$ is a **S-Conv** (or **S-Conc**, resp.) function on $\log D$.

Lemma 2. (see [7] (p. 108), [8]) Let $\emptyset \neq D \subset \mathbb{R}_+^n$ be a geometrically convex and symmetric set with a nonempty interior D° . Let $\omega : D \rightarrow \mathbb{R}_+$ be continuous on D and differentiable in D° . If ω is symmetric on D and

$$(\log a_1 - \log a_2) \left(a_1 \frac{\partial \omega}{\partial a_1} - a_2 \frac{\partial \omega}{\partial a_2} \right) \geq 0 \text{ (or } \leq 0, \text{ resp.)} \tag{4}$$

holds for any $\mathbf{a} = (a_1, a_2, \dots, a_n) \in D^\circ$, then ω is a **S-G-Conv** (or **S-G-Conc**, resp.) function.

By Definition 5, the following is obvious.

Proposition 5. Let $D \subset \mathbb{R}_+^n$ be a nonempty set, and let

$$\frac{1}{D} = \left\{ \left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n} \right) : (a_1, a_2, \dots, a_n) \in D \right\}.$$

Then $\omega : D \rightarrow \mathbb{R}_+$ is a **S-H-Conv** (or **S-H-Conc**, resp.) function on D if and only if $\omega \left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n} \right)$ is a **S-Conv** (or **S-Conc**, resp.) function on $\frac{1}{D}$.

Lemma 3. (see [2,9,10]) Let $\emptyset \neq D \subset \mathbb{R}_+^n$ be a symmetric and harmonically convex set with inner points and let $\omega : D \rightarrow \mathbb{R}_+$ be a continuous symmetric function which is differentiable on D° . Then ω is **S-H-Conv** (or **S-H-Conc**, resp.) on D if and only if

$$(a_1 - a_2) \left(a_1^2 \frac{\partial \omega}{\partial a_1} - a_2^2 \frac{\partial \omega}{\partial a_2} \right) \geq 0 \quad (\text{or } \leq 0, \text{ resp.}), \quad \mathbf{a} \in D^\circ \quad (5)$$

Lemma 4. (see [5] (p. 91), [6] (pp. 64–65)) Let sets $C, D \subset \mathbb{R}$, $\omega : D^n \rightarrow \mathbb{R}$, $h : C \rightarrow D$ and $\psi(a_1, a_2, \dots, a_n) = \omega(h(a_1), h(a_2), \dots, h(a_n)) : C^n \rightarrow \mathbb{R}$. Then the following hold :

- (i) If ω is **S-Conv** and increasing and h is increasing and convex, then ψ is **S-Conv** and increasing.
- (ii) If ω is **S-Conv** and decreasing and h is increasing and concave, then ψ is **S-Conv** and decreasing.
- (iii) If ω is **S-Conc** and increasing and h is increasing and concave, then ψ is **S-Conc** and increasing.
- (iv) If ω is **S-Conv** and decreasing and h is decreasing and concave, then ψ is **S-Conv** and increasing.
- (v) If ω is **S-Conc** and increasing and h is decreasing and concave, then ψ is **S-Conc** and decreasing.

Lemma 5. (see [3]) Let $\emptyset \neq D \subset \mathbb{R}_+^n$ and the function $\omega : D \rightarrow \mathbb{R}_+$ is differentiable. Then the following hold:

- (i) If ω is **S-Conv** and increasing, then ω is **S-G-Conv**.
- (ii) If ω is **S-Conc** and decreasing, then ω is **S-G-Conc**.

Lemma 6. (see [3]) Let $\emptyset \neq D \subset \mathbb{R}_+^n$ and the function $\omega : D \rightarrow \mathbb{R}_+$ is differentiable. Then the following hold:

- (i) If ω is **S-Conv** and increasing, then ω is **S-H-Conv**.
- (ii) If ω is **S-Conc** and decreasing, then ω is **S-H-Conc**.

3. New simple proofs of Theorems

New simple proof of Theorem 1. Let $h(w) = \frac{1+w}{1-w}$. Then $h > 0$ on $(0, 1)$ and $h < 0$ on $(1, +\infty)$, from $h'(w) = 2(1-w)^{-2}$ and $h''(w) = 4(1-w)^{-3}$, we know that h is convex and increasing on $(0, 1)$ and h is concave and increasing on $(1, +\infty)$.

By Proposition 1, $c_n(\mathbf{b}, t)$ is **S-Conv** and increasing on \mathbb{R}_+^n . Using Lemma 4 (i), we have that $S_n(\mathbf{b}, t)$ is **S-Conv** and increasing on $(0, 1)^n$. By the continuity of $S_n(\mathbf{b}, t)$ on $[0, 1]^n$, we show that $S_n(\mathbf{b}, t)$ is **S-Conv** and increasing on $[0, 1]^n$.

If t is even integer, then, by Proposition 3, $c_n(\mathbf{b}, t)$ is **S-Conv** and decreasing. Moreover, we obtain that h is concave and increasing on $(1, +\infty)$. Applying Lemma 4 (ii), we get that $S_n(\mathbf{b}, t)$ is **S-Conv** and decreasing.

If t is odd integer, then, by Proposition 3, $c_n(\mathbf{b}, t)$ is increasing and **S-Conc**.

Moreover, we know that h is concave and increasing on $(1, +\infty)$. Using Lemma 4 (iii), we have that $S_n(\mathbf{b}, t)$ is **S-Conc** and increasing and hence all conclusions are proved. \square

New simple proof of Theorem 2. By applying Lemma 5 (i) and Theorem 1 (i), we can show that the conclusion (i) of Theorem 2 holds. Considering

$$S_n(e^{\mathbf{b}}, t) = \sum_{k_1+k_2+\dots+k_n=t} \left(\frac{1+e^{b_1}}{1-e^{b_1}}\right)^{k_1} \left(\frac{1+e^{b_2}}{1-e^{b_2}}\right)^{k_2} \dots \left(\frac{1+e^{b_n}}{1-e^{b_n}}\right)^{k_n}. \quad (6)$$

Define $g(w) = \frac{1+e^w}{1-e^w}$ for $w \in (0, +\infty)$. Then $g(w) < 0$ for all $w \in (0, +\infty)$. Since $g'(w) = \frac{2e^w}{(1-e^w)^2}$ and $g''(w) = \frac{2e^w(1+e^w)}{(1-e^w)^3}$, we see that g is increasing and concave on $(0, +\infty)$. If t is even integer (or odd integer, resp.), then, by applying Proposition 3 and Lemma 4 (ii) (or (iii), resp.), we show that $S_n(e^{\mathbf{b}}, t)$ is **S-Conc** (or **S-Conc**, resp.) on $(0, +\infty)^n$. Finally, by using Proposition 4, we can prove the conclusion (ii) of Theorem 2. Hence we complete the proof. \square

New simple proof of Theorem 3. By applying Lemma 6 (i) and Theorem 1 (i), we can show that the conclusion (i) of Theorem 3 holds. Considering

$$S_n\left(\frac{1}{\mathbf{b}}, t\right) = \sum_{k_1+k_2+\dots+k_n=t} \left(\frac{1+b_1}{b_1-1}\right)^{k_1} \left(\frac{1+b_2}{b_2-1}\right)^{k_2} \dots \left(\frac{1+b_n}{b_n-1}\right)^{k_n}. \quad (7)$$

Let $v(w) = -\frac{1+w}{w-1}$ for $w \in (0, 1)$. Then $v(w) > 0$ for all $w \in (0, 1)$. Since $v'(w) = 2(w-1)^{-2}$ and $v''(w) = -4(w-1)^{-3}$, we see that v is increasing and convex on $(0, 1)$. By Proposition 1, $c_n(\mathbf{b}, t)$ is **S-Conv** and increasing on \mathbb{R}_+^n . Using Lemma 4 (i), we have that

$$P_n(\mathbf{b}, t) := \sum_{k_1+k_2+\dots+k_n=t} \left(-\frac{1+b_1}{b_1-1}\right)^{k_1} \left(-\frac{1+b_2}{b_2-1}\right)^{k_2} \dots \left(-\frac{1+b_n}{b_n-1}\right)^{k_n}$$

is **S-Conv** and increasing on $(0, 1)^n$.

Noting that

$$P_n(\mathbf{b}, t) = (-1)^{k_1+k_2+\dots+k_n} S_n\left(\frac{1}{\mathbf{b}}, t\right) = (-1)^t S_n\left(\frac{1}{\mathbf{b}}, t\right).$$

If t is even integer, then $S_n\left(\frac{1}{\mathbf{b}}, t\right) = P_n(\mathbf{b}, t)$ is **S-Conv** and increasing on $(0, 1)^n$.

If t is odd integer, then $S_n\left(\frac{1}{\mathbf{b}}, t\right) = -P_n(\mathbf{b}, t)$ is **S-Conc** and decreasing on $(0, 1)^n$.

For $D = (1, +\infty)^n$, $\frac{1}{D} = (0, 1)^n$, by Proposition 5, it follows that if t is even integer (or odd integer, resp.), then $S_n(\mathbf{b}, t)$ is Schur-harmonically convex (or Schur-harmonically concave, resp.) on $(1, +\infty)^n$. This shows that the conclusion (ii) of Theorem 3 holds. Therefore we complete the proof. \square

4. Conclusions

In this work, we studied theorems of Sun et al. [4]. By applying the properties of Schur-geometrically convex functions, Schur-harmonically convex functions and Schur-convex functions, we have provided alternative proofs of Theorems 2, 3 and 1.

The study of determining Schur-convexity, Schur-geometrically convexity and Schur-harmonically convexity of various types of multivariate symmetric functions is an important aspect of majorization theory research. In recent years, a large number of literature has been generated, most of which use decision Theorems 1–3 to make judgments, which are often complex. However, flexibly using the relevant properties of these convexities to prove them may be innovative and achieve concise and direct results. This article provides a typical example.

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References

1. Guan KZ. Schur-convexity of the complete symmetric function. *Mathematical Inequalities & Applications*. 2006; 9(4): 567–576.
2. Chu YM, Wang GD, Zhang XH. The Schur multiplicative and harmonic convexities of the complete symmetric function. *Mathematische Nachrichten*. 2011; 284(5–6): 653–663.
3. Shi HN, Zhang J, Ma QH. Schur-convexity, Schur-geometric and Schur-harmonic convexity for a composite function of complete symmetric function. *SpringerPlus*. 2016; 5: 296.
4. Sun MB, Zhang YG, Zhang ZY, Chen NB. Schur convexity of a class of symmetric functions with its applications (Chinese). *Chinese Journal of Contemporary Mathematics*. 2017; 38A(2): 177–190.
5. Marshall AW, Olkin I, Arnold BC. *Inequalities: Theory of Majorization and Its Application*, 2nd ed. Springer; 2011.
6. Wang BY. *Foundations of Majorization Inequalities (Chinese)*. Beijing Normal University Press; 1990.
7. Zhang XM. *Geometrically Convex Functions (Chinese)*. Anhui University Press; 2004.
8. Chu YM, Zhang X, Wang G. The Schur geometrical convexity of the extended mean values. *Journal of Convex Analysis*. 2008; 15(4): 707–718.
9. Chu YM, Lv YP. The Schur harmonic convexity of the Hamy symmetric function and its applications. *Journal of Inequalities and Applications*. 2009. doi: 10.1155/2009/838529
10. Chu M, Sun TC. The Schur harmonic convexity for a class of symmetric functions. *Acta Mathematica Scientia*. 2010; 30(5): 1501–1506.
11. Shi HN. *Theory of Majorization and Analytic Inequality (Chinese)*. Harbin Institute of Technology Press; 2012.
12. Shi HN, Du WS. Schur-power convexity of a completely symmetric function dual. *Symmetry*. 2019; 11(7): 897.
13. Shi HN, Du WS. New inequalities and generalizations for symmetric means induced by majorization theory. *Axioms*. 2022; 11(6): 279.
14. Arnold BC. Majorization: Here, There and Everywhere. *Statistical Science*. 2007; 22(3): 407–413.