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Article

A duality principle and an existence result for a non-linear model in elasticity and relaxation for related models in phase transition

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Abstract: This article develops duality principles applicable to originally non-convex primal variational formulations. More specifically, as a first application, we establish a convex dual variational formulation for a non-linear model in elasticity. The results are obtained through basic tools of functional analysis, calculus of variations, duality and optimization theory in infinite dimensional spaces. We emphasize such a convex dual formulation obtained may be applied to a large class of similar models in the calculus of variations. In a subsequent section, we present a global existence result for such a concerning model in elasticity. Finally, in the last sections, we develop duality principles and relaxation procedures for a related model in phase transition.

Keywords: duality principle; non-linear model in elasticity; global existence result
MSC Classification: 49N15; 74B20

1. Introduction

This article develops a duality principle applicable to a large class of models in the calculus of variations. Specifically in this text, in a first step, we present applications to a model in non-linear elasticity. In a second step, we develop in details an existence result for such a model.

We emphasize the results on duality theory here addressed and developed are inspired mainly in the approaches of J.J.Telega, W.R. Bielski and co-workers presented in the articles [1–4]. Other main reference is the article by Toland [5].

Moreover, details on the Sobolev spaces involved may be found in [6]. Similar results and models are addressed in [7–11].

Basic results on convex analysis and calculus of variations are addressed in [12–16]. Other similar results and approaches may be found in [17–20].

Now we start to describe the primal variational formulation for the model in non-linear elasticity in question.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Considering the model in elasticity found in [17], we define a functional $J : V \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\int_\Omega H_{ijkl}e_{ij}(u)e_{kl}(u)\,dx - \langle u, f\rangle_{L^2} \tag{1}$$

where $\{H_{ijkl}\}$ is a positive definite symmetric constant fourth order tensor and

$$u = (u_1, u_2, u_3) \in V = W_0^{1,2}(\Omega; \mathbb{R}^3)$$

is the field of displacements resulting from the action of an external load

$$f = (f_1, f_2, f_3) \in L^2(\Omega; \mathbb{R}^3).$$

We denote

$$Y = Y^* = L^2(\Omega; \mathbb{R}^3),$$

$$Y_1 = Y_1^* = L^2(\Omega; \mathbb{R}^{3\times 3})$$

and

$$\{\overline{H}_{ijkl}\} = \{H_{ijkl}\}^{-1}$$

in an appropriate tensorial sense.

Moreover, the strain tensor $e : V \to Y_1$, is defined by

$$e_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2}u_{m,i}u_{m,j},$$

$\forall i, j \in \{1, 2, 3\}$.

Here it is worth highlighting in this text we have adopted the Einstein's convention of summing up repeated indices.

We define also the functionals $F_1 : Y_1 \to \mathbb{R}$ and $F_2 : V \to \mathbb{R}$ by

$$F_1(e_{ij}(u)) = \frac{1}{2}\int_\Omega H_{ijkl}e_{ij}(u)e_{kl}(u)\, dx,$$

and

$$F_2(u) = \frac{K}{2}\langle u_i, u_i\rangle_{L^2},$$

respectively.

Furthermore, for $K \gg K_5 > 1$, we define

$$V_1 = \{u \in V \ : \ \|u_i\|_\infty \leq K_5, \ \forall i \in \{1, 2, 3\}\},$$

$$B^* = \{\sigma \in Y_1^* \ : \ \|\sigma_{ij}\|_\infty \leq K/16, \ \forall i, j \in \{1, 2, 3\}\},$$

$$D^* = \{z^* = (z_1^*, z_2^*, z_3^*) \in Y^* \ : \ \|z_i^*\|_\infty \leq K\ K_5, \ \forall i \in \{1, 2, 3\}\}$$

and $J_1 : V \times B^* \to \mathbb{R}$ by

$$J_1(u, \sigma) = J(u) + K_8 \sum_{i=1}^3 \|\sigma_{ij,j} + (\sigma_{im}u_{m,j})_{,j} + f_i\|^2_{H^{-1}(\Omega)} \tag{2}$$

for an appropriate $K_8 > 0$ to be specified.

Let $\alpha \in \mathbb{R}$ be such that

$$\alpha = \inf_{u \in V_1} J(u).$$

Hence,

$$\begin{aligned}
\alpha &= \inf_{u \in V_1} J(u) \\
&\leq J(u) \\
&\leq J_1(u,\sigma) \\
&= F_1(e_{ij}(u)) + K_8 \sum_{i=1}^{3} \|\sigma_{ij,j} + (\sigma_{im} u_{m,j})_{,j} + f_i\|^2_{H^{-1}(\Omega)} \\
&\quad + \frac{K}{2}\langle u_i, u_i\rangle_{L^2} - \langle u, f\rangle_{L^2} - \langle u_i, z_i^*\rangle_{L^2} \\
&\quad + \langle u_i, z_i^*\rangle_{L^2} - F_2(u) \\
&\leq F_1(e_{ij}(u)) + K_8 \sum_{i=1}^{3} \|\sigma_{ij,j} + (\sigma_{im} u_{m,j})_{,j} + f_i\|^2_{H^{-1}(\Omega)} \\
&\quad + \frac{K}{2}\langle u_i, u_i\rangle_{L^2} - \langle u, f\rangle_{L^2} - \langle u_i, z_i^*\rangle_{L^2} \\
&\quad + \sup_{v \in Y}\{\langle v_i, z_i^*\rangle_{L^2} - F_2(v)\} \\
&= F_1(e_{ij}(u)) + K_8 \sum_{i=1}^{3} \|\sigma_{ij,j} + (\sigma_{im} u_{m,j})_{,j} + f_i\|^2_{H^{-1}(\Omega)} \\
&\quad + \frac{K}{2}\langle u_i, u_i\rangle_{L^2} - \langle u, f\rangle_{L^2} - \langle u_i, z_i^*\rangle_{L^2} \\
&\quad + F_2^*(z^*)
\end{aligned} \tag{3}$$

$\forall u \in V_1,\ z^* \in D^*$.
Here, $F_2^* : Y^* \to \mathbb{R}$ stands for

$$\begin{aligned}
F_2^*(z^*) &= \sup_{v \in Y}\{\langle v_i, z_i^*\rangle_{L^2} - F_2(v)\} \\
&= \frac{1}{2K}\int_\Omega |z^*|^2\, dx
\end{aligned} \tag{4}$$

Define now $H_1 : V \times B^* \times D^* \to \mathbb{R}$ by

$$\begin{aligned}
H_1(u,\sigma,z^*) = F_1(e_{ij}(u)) &+ K_8 \sum_{i=1}^{3} \|\sigma_{ij,j} + (\sigma_{im} u_{m,j})_{,j} + f_i\|^2_{H^{-1}(\Omega)} \\
&+ \frac{K}{2}\langle u_i, u_i\rangle_{L^2} - \langle u, f\rangle_{L^2} - \langle u_i, z_i^*\rangle_{L^2}
\end{aligned} \tag{5}$$

Thus,

$$\begin{aligned}
H_1(u,\sigma,z^*) &= -\langle e_{ij}(u),\sigma_{ij}\rangle_{L^2} + F_1(e_{ij}(u)) \\
&\quad + \langle e_{ij}(u),\sigma_{ij}\rangle_{L^2} + K_8\sum_{i=1}^3 \|\sigma_{ij,j}+(\sigma_{im}u_{m,j})_{,j}+f_i\|^2_{H^{-1}(\Omega)} \\
&\quad + \frac{K}{2}\langle u_i,u_i\rangle_{L^2} - \langle u,f\rangle_{L^2} - \langle u_i,z_i^*\rangle_{L^2} \\
&\geq \inf_{w\in Y_1}\{-\langle w_{ij},\sigma_{ij}\rangle_{L^2}+F_1(w_{ij})\} \\
&\quad + \inf_{u\in V}\left\{\langle e_{ij}(u),\sigma_{ij}\rangle_{L^2} + K_8\sum_{i=1}^3 \|\sigma_{ij,j}+(\sigma_{im}u_{m,j})_{,j}+f_i\|^2_{H^{-1}(\Omega)}\right. \\
&\quad \left. +\frac{K}{2}\langle u_i,u_i\rangle_{L^2} - \langle u,f\rangle_{L^2} - \langle u_i,z_i^*\rangle_{L^2}\right\} \\
&= -F_1^*(\sigma) - F_3^*(\sigma,z^*) \\
&\quad \forall u\in V,\ \sigma\in B^*,\ z^*\in D^*.
\end{aligned} \tag{6}$$

Here we have denoted,

$$\begin{aligned}
F_3(u,\sigma) &= \langle e_{ij}(u),\sigma_{ij}\rangle_{L^2} + K_8\sum_{i=1}^3 \|\sigma_{ij,j}+(\sigma_{im}u_{m,j})_{,j}+f_i\|^2_{H^{-1}(\Omega)} \\
&\quad + \frac{K}{2}\langle u_i,u_i\rangle_{L^2} - \langle u,f\rangle_{L^2}
\end{aligned} \tag{7}$$

$F_3^*: B^*\times D^*\to\mathbb{R}$ by

$$F_3^*(\sigma,z^*) = \sup_{u\in V_1}\{\langle u_i,z_i^*\rangle_{L^2} - F_3(u,\sigma)\},$$

and $F_1^*: Y_1^*\to\mathbb{R}$ by

$$\begin{aligned}
F_1^*(\sigma) &= \sup_{w\in Y_1}\{\langle w_{ij},\sigma_{ij}\rangle_{L^2} - F_1(w_{ij})\} \\
&= \frac{1}{2}\int_\Omega \overline{H}_{ijkl}\sigma_{ij}\sigma_{kl}\,dx
\end{aligned} \tag{8}$$

Define now $J^*: B^*\times D^*\to\mathbb{R}$ by

$$J^*(\sigma,z^*) = -F_1^*(\sigma) - F_3^*(\sigma,z^*) + F_2^*(z^*).$$

Let $(\hat{\sigma},\hat{z}^*)\in B^*\times D^*$ be such that

$$\delta J^*(\hat{\sigma},\hat{z}^*) = \mathbf{0}.$$

Let $u_0\in V_1$ be such that

$$u_0 = \frac{\hat{z}^*}{K}.$$

From the Legendre transform proprieties, we may obtain

$$\delta J(u_0) = \mathbf{0},$$

$$\delta J_1(u_0,\hat{\sigma}) = \mathbf{0},$$

and

$$J(u_0) = J_1(u_0, \hat{\sigma}) = J^*(\hat{\sigma}, \hat{z}^*).$$

Observe that J^* is quadratic in z^* on D^*.

Suppose

$$\left\{ \frac{\partial^2 J^*(\hat{\sigma}, z^*)}{\partial z_j^* \partial z_k^*} \right\} > \mathbf{0}.$$

Under such assumptions, we have

$$J^*(\hat{\sigma}, \hat{z}^*) = \inf_{z^* \in D^*} J^*(\hat{\sigma}, z^*).$$

Assume $K_8 > 0$ in under and close to the largest value possible such that J^* is concave in σ on B^*, for such a fixed $\hat{z}^*$.

Thus, we have

$$J^*(\hat{\sigma}, \hat{z}^*) = \sup_{\sigma \in B^*} J^*(\sigma, \hat{z}^*),$$

so that

$$\begin{aligned} J^*(\hat{\sigma}, \hat{z}^*) &= \inf_{z^* \in D^*} J^*(\hat{\sigma}, z^*) \\ &= \sup_{\sigma \in B^*} J^*(\sigma, \hat{z}^*) \end{aligned} \tag{9}$$

From such results and a standard Saddle Point Theorem we may infer that

$$\begin{aligned} J^*(\hat{\sigma}, \hat{z}^*) &= \inf_{z^* \in D^*} \left\{ \sup_{\sigma \in B^*} J^*(\sigma, z^*) \right\} \\ &= \sup_{\sigma \in B^*} \left\{ \inf_{z^* \in D^*} J^*(\sigma, z^*) \right\} \end{aligned} \tag{10}$$

Finally, observe that

$$\begin{aligned} \alpha \leq J(u_0) &= J_1(u_0, \hat{\sigma}) \\ &= J^*(\hat{\sigma}, \hat{z}^*) \\ &\leq J^*(\hat{\sigma}, z^*) \\ &\leq F_1(e_{ij}(u)) + F_3(u, \hat{\sigma}) - \langle u_i, z_i^* \rangle_{L^2} + F_2^*(z^*) \end{aligned} \tag{11}$$

$\forall z^* \in D^*$.

Consequently, for $z^* = Ku$, we obtain

$$\begin{aligned} \alpha \leq J(u_0) &= J_1(u_0, \hat{\sigma}) \\ &= J^*(\hat{\sigma}, \hat{z}^*) \\ &\leq J_1(u, \hat{\sigma}) \end{aligned} \tag{12}$$

$\forall u \in V_1$.

Joining the pieces, we have got

$$\begin{aligned} J(u_0) &= J_1(u_0, \hat{\sigma}) \\ &= \inf_{u \in V_1} J_1(u, \hat{\sigma}) \\ &= J^*(\hat{\sigma}, \hat{z}^*) \\ &= \inf_{z^* \in D^*} \left\{ \sup_{\sigma \in B^*} J^*(\sigma, z^*) \right\} \\ &= \sup_{\sigma \in B^*} \left\{ \inf_{z^* \in D^*} J^*(\sigma, z^*) \right\} \end{aligned} \tag{13}$$

The objective of this section is complete.

2. A global existence result for such a model in non-linear elasticity

In this section we obtain a global existence result for a model similar as that of the previous section.

It is worth emphasizing this result has been presented in similar format, with some slight changes in the notation, in the pre-prints [21,22].

We start with the following auxiliary result.

2.1. An auxiliary result

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

In this short subsection we prove that for a bounded sequence $\{u_n\} \subset W^{1,2}(\Omega)$ such that $\{\nabla^2 u_n\} \subset L^2(\Omega)$, up to a not relabeled subsequence, there exists $u_0 \in W^{1,2}(\Omega)$ such that

$$\nabla u_n \to \nabla u_0, \text{ almost everywhere in } \Omega.$$

About the references for this subsection, the main ones are [6, 20, 21]. Related results may be found in [7].

Here we state and prove the main theorem in such a subsection.

Theorem 1. *Suppose $\{u_n\} \subset W^{1,2}(\Omega)$ is such that there exists a real constant $K > 0$ such that*

$$\|u_n\|_{1,2} \leq K, \forall n \in \mathbb{N},$$

and

$$\nabla^2 u_n \in L^2(\Omega), \ \forall n \in \mathbb{N}.$$

Under such hypotheses, there exists $u_0 \in W^{1,2}(\Omega)$ such that, up to a not relabeled subsequence,

1)

$$u_n \rightharpoonup u_0, \text{ weakly in } W^{1,2}(\Omega);$$

2)

$$u_n \to u_0, \text{ strongly in } L^2(\Omega);$$

3)

$$\nabla u_n \to \nabla u_0, \text{ almost everywhere in } \Omega.$$

Proof. It is well a known result that from the hypotheses and from the Rellich-Kondrashov theorem there exists $u_0 \in W^{1,2}(\Omega)$ such that, up to a not relabeled subsequence,

1)

$$u_n \rightharpoonup u_0, \text{ weakly in } W^{1,2}(\Omega);$$

2)

$$u_n \to u_0, \text{ strongly in } L^2(\Omega).$$

The novelty here is that, up to a not relabeled subsequence, we may prove that

$$\nabla u_n \to \nabla u_0, \text{ almost everywhere in } \Omega.$$

Indeed, firstly for a measurable function u, we define the norm

$$\|u\|_{H^{-2}} = \sup\left\{|\langle u, \phi\rangle_{L^2}| \ : \ \phi \in W_0^{2,2} \text{ and } \|\phi\|_{2,2} \leq 1\right\},$$

and

$$H^{-2}(\Omega) = \{u \text{ measurable } \ : \ \|u\|_{H^{-2}} < +\infty\}.$$

Let $\phi \in W_0^{2,2}$.

Hence,

$$\begin{aligned} |\langle \nabla^2 u_n - \nabla^2 u_m, \phi\rangle_{L^2}| &= |\langle u_n - u_m, \nabla^2 \phi\rangle_{L^2}| \\ &\leq \|u_n - u_m\|_{0,2}\|\phi\|_{2,2} \end{aligned} \quad (14)$$

From such a result and considering that $\{u_n\}$ is a Cauchy sequence concerning the $L^2(\Omega)$ norm, we may infer that $\{\nabla^2 u_n\}$ is a Cauchy sequence in H^{-2}.

Therefore, for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that if $m, n \geq n_k$, then

$$\|\nabla^2 u_n - \nabla^2 u_m\|_{H^{-2}} < \frac{1}{k^2}.$$

Observe that $\{n_k\}$ may be chosen as an increasing sequence.

In particular,

$$\|\nabla^2 u_{n_{k+1}} - \nabla^2 u_{n_k}\|_{H^{-2}} < \frac{1}{k^2}.$$

Define

$$g_l = |\nabla^2 u_{n_1}| + \sum_{k=1}^{l-1} |\nabla^2 u_{n_{k+1}} - \nabla^2 u_{n_k}|,$$

and

$$g = |\nabla^2 u_{n_1}| + \sum_{k=1}^{\infty} |\nabla^2 u_{n_{k+1}} - \nabla^2 u_{n_k}|.$$

Observe that

$$\begin{aligned} \|g\|_{H^{-2}} &\leq \|\nabla^2 u_{n_1}\|_{H^{-2}} + \sum_{k=1}^{\infty} \|\nabla^2 u_{n_{k+1}} - \nabla^2 u_{n_k}\|_{H^{-2}} \\ &\leq \|\nabla^2 u_{n_1}\|_{H^{-2}} + \sum_{k=1}^{\infty} \frac{1}{k^2} \\ &< +\infty \end{aligned} \tag{15}$$

From such a result we may infer that

$$g(x) \in \mathbb{R}, \text{ almost everywhere in } \Omega.$$

Thus, there exists a measurable function h such that

$$g_l(x) = |\nabla^2 u_{n_1}| + \sum_{k=1}^{l-1} |\nabla^2 u_{n_{k+1}} - \nabla^2 u_{n_k}| \to h(x) \in \mathbb{R},$$

as $l \to \infty,$ almost everywhere in Ω.

From such a result, since an absolutely convergent sequence is also convergent, we may infer that there exists a measurable function h_1 such that

$$\nabla^2 u_{n_l} = \nabla^2 u_{n_1} + \sum_{k=1}^{l-1} (\nabla^2 u_{n_{k+1}} - \nabla^2 u_{n_k}) \to h_1(x) \in \mathbb{R},$$

as $l \to \infty,$ almost everywhere in Ω.

Consequently, we may obtain a measurable vectorial function $\mathbf{h}_3$ such that

$$\nabla u_{n_l} \to \mathbf{h}_3, \text{ almost everywhere in } \Omega.$$

From such a result and

$$\nabla u_n \rightharpoonup \nabla u_0, \text{ weakly in } \Omega,$$

we obtain

$$\nabla u_0 = \mathbf{h}_3, \text{ almost everywhere in } \Omega,$$

so that, up to a not relabeled subsequence, we have

$$\nabla u_n \to \nabla u_0, \text{ almost everywhere in } \Omega.$$

The proof is complete. □

Now, let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega = S$.

Define a functional $J : V \to \mathbb{R}$ by

$$J(u) = \frac{1}{2} \int_\Omega H_{ijkl} e_{ij}(u) e_{kl}(u) \, dx - \langle u_i, f_i \rangle_{L^2},$$

where

$$e_{ij}(u) = \frac{u_{i,j} + u_{j,i}}{2} + \frac{1}{2} u_{m,i} u_{m,j},$$

$$V = \{u \in W^{1,2}(\Omega; \mathbb{R}^3) \ : \ u = \hat{v}_0 \text{ on } S_1 \subset \partial\Omega\}.$$

We also denote $Y = Y^* = L^2(\Omega; \mathbb{R}^3)$, so that $f = (f_1, f_2, f_3) \in Y$.

Here $\{H_{ijkl}\}$ is a fourth order constant, positive definite and symmetric tensor.

With such assumptions and statements in mind, we may prove the following theorem.

Theorem 2. *Assume $\{H_{ijkl}\}$ is such that*

$$\lim_{\|u\|_V \to \infty} J(u) = +\infty.$$

Under such hypothesis, there exists $u_0 \in V$ such that

$$J(u_0) = \min_{u \in V} J(u).$$

Proof. From the hypotheses, there exists $\alpha \in \mathbb{R}$ such that

$$\alpha = \inf_{u \in V} J(u).$$

Let $\{u_n\} \subset V$ be a sequence such that

$$\alpha \leq J(u_n) < \alpha + \frac{1}{n}, \forall n \in \mathbb{N}.$$

Suppose, to obtain contradiction, there exists a subsequence $\{n_k\} \subset \mathbb{N}$, such that

$$\|u_{n_k}\|_V \to \infty.$$

From the hypotheses, we have

$$J(u_{n_k}) \to +\infty, \text{ as } k \to \infty.$$

This contradicts

$$\lim_{k \to \infty} J(u_{n_k}) = \alpha \in \mathbb{R}.$$

From such results we may infer that there exists $K > 0$ such that

$$\|u_n\|_V \leq K, \ \forall n \in \mathbb{N}.$$

Consequently, from this, the Sobolev Embedding and Rellich Kondrashov theorems, there exists $u_0 \in V \cap L^\infty(\Omega; \mathbb{R}^3)$ for which, up to a not relabelled subsequence, we have

$$u_n \rightharpoonup u_0, \text{ weakly in } W^{1,4}(\Omega; \mathbb{R}^3),$$

$$u_n \to u_0, \ \text{ strongly in } L^4(\Omega),$$

$$u_n \to u_0, \text{ strongly in } L^\infty(\Omega;\mathbb{R}^3).$$

From an application of the Ekeland variational principle, we may assume that such a subsequence is such that

$$\|\delta J(u_n)\|_{0,2} < 1/n,$$

so that

$$\begin{aligned}&\|H_{ijkl}(u_n)_{k,lj} + H_{ijkl}((u_n)_{m,k}(u_n)_{m,l})_{,j}/2\\&+(H_{imkl}(((u_n)_{k,l}+(u_n)_{l,k})/2+(u_n)_{s,k}(u_n)_{s,l}/2)(u_n)_{m,j})_{,j}+f_i\|_{0,2}\\&<1/n\end{aligned} \tag{16}$$

$\forall i \in \{1,2,3\},\ \forall n \in \mathbb{N}$, so that

$$\|H_{ijkl}(u_n)_{k,lj}\|_{0,2} < +\infty.$$

Since $\{H_{ijkl}\}$ is positive definite, we may obtain a positive real constant $C = C(n)$ such that

$$\sum_{i,j,k=1}^{3} \|(u_n)_{i,jk}\|_{0,2} \leq C \sum_{i=1}^{3} \|H_{ijkl}(u_n)_{k,lj}\|_{0,2} < +\infty.$$

From such results, we may infer that

$$u_n \in W^{2,2}(\Omega),\ \forall n \in \mathbb{N}.$$

Consequently, considering also the last subsection results, we have got, up to not relabeled subsequences,

$$\frac{\partial (u_{n_l})_i}{\partial x_j} \to \frac{\partial (u_0)_i}{\partial x_j}, \text{ a.e. in } \Omega.$$

Now fix $i, j, m \in \{1,2,3\}$.

Observe that from the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\int_\Omega \left(\frac{\partial (u_{n_l})_m}{\partial x_j}\frac{\partial (u_{n_l})_m}{\partial x_j}\right)^2 dx\\&\leq \left\|\frac{\partial (u_{n_l})_m}{\partial x_i}\right\|_4^2 \left\|\frac{\partial (u_{n_l})_m}{\partial x_j}\right\|_4^2\\&\leq K_1,\ \forall l \in \mathbb{N}\end{aligned} \tag{17}$$

for some appropriate real constant $K_1 > 0$.

Therefore, up to a not relabeled subsequence there exists $v_0 \in L^2(\Omega)$ such that

$$\frac{\partial (u_{n_l})_m}{\partial x_i}\frac{\partial (u_{n_l})_m}{\partial x_j} \rightharpoonup v_0, \text{ weakly in } L^2(\Omega),$$

Since

$$\frac{\partial (u_{n_l})_m}{\partial x_i}\frac{\partial (u_{n_l})_m}{\partial x_j} \to \frac{\partial (u_0)_m}{\partial x_i}\frac{\partial (u_0)_m}{\partial x_j}, \text{ a.e. in } \Omega,$$

we obtain

$$v_0 = \frac{\partial (u_0)_m}{\partial x_i}\frac{\partial (u_0)_m}{\partial x_j}, \text{ a.e. in } \Omega,$$

so that

$$\frac{\partial (u_{n_l})_m}{\partial x_i}\frac{\partial (u_{n_l})_m}{\partial x_j} \rightharpoonup \frac{\partial (u_0)_m}{\partial x_i}\frac{\partial (u_0)_m}{\partial x_j}, \text{ weakly in } L^2(\Omega),$$

$\forall i, j, m \in \{1, 2, 3\}$.

Therefore, from such results we may infer that

$$e_{ij}(u_{n_l}) \rightharpoonup e_{ij}(u_0), \text{ weakly in } L^2(\Omega), \forall i, j \in \{1, 2, 3\}.$$

Moreover, since J is convex in $\{e_{ij}\}$ we finally obtain

$$\alpha = \liminf_{l\to\infty} J(u_{n_l}) \geq J(u_0),$$

so that

$$J(u_0) = \min_{u\in V} J(u).$$

The proof is complete. □

3. Duality principles and relaxation for a related model

In this section we present a relaxation procedure suitable for a large class of vectorial models in the calculus of variations.

It is worth highlighting such a procedure has been presented in similar format in the pre-print [22].

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Consider a functional $J : V \to \mathbb{R}$ defined by

$$J(u) = F(\nabla u) - \langle u, f\rangle_{L^2},$$

where

$$F(\nabla u) = \int_\Omega g(\nabla u)\, dx,$$

$V = W_0^{1,2}(\Omega; \mathbb{R}^N)$, and $g : \mathbb{R}^{Nn} \to \mathbb{R}$ is a continuous and piecewise C^1 class function such that

$$\lim_{|y|\to\infty} \frac{g(y)}{|y|} \to +\infty.$$

Here $f \in L^2(\Omega; \mathbb{R}^N)$ and we denote $Y = Y^* = L^2(\Omega; \mathbb{R}^N)$ and $Y_1 = Y_1^* = L^2(\Omega; \mathbb{R}^{Nn})$.

Assume $\alpha \in \mathbb{R}$ is such that

$$\alpha = \inf_{u\in V} J(u).$$

From the convex analysis theory, it is well known that

$$\inf_{u\in V} J(u) = \inf_{u\in V} J^{**}(u).$$

Fixing $m \in \mathbb{N}$, define $J_1 : V \to \mathbb{R}$ by

$$J_1(u) = \inf_{(v,\lambda)\in D_u} \left\{ \sum_{j=1}^m \lambda_j J(v_j) \right\},$$

where

$$D_u = \{(v,\lambda) = ((v_1, \cdots, v_m), (\lambda_1, \cdots, \lambda_m)) \in V^m \times [0,1]^m \; : \\ \left. \sum_{j=1}^m \lambda_j v_j = u, \text{ in } \Omega \text{ and } \sum_{j=1}^m \lambda_j = 1 \right\} \tag{18}$$

Observe that

$$\begin{aligned} J^{**}(u) &\leq \inf_{(v,\lambda)\in D_u} \left\{ \sum_{j=1}^m \lambda_j J^{**}(v_j) \right\} \\ &\leq J_1(u) \\ &\leq J(u) \end{aligned} \tag{19}$$

From such results, we may infer that

$$\alpha = \inf_{u\in V} J^{**}(u) = \inf_{u\in V} J_1(u) = \inf_{u\in V} J(u).$$

Observe also that for $(v,\lambda) \in D_u$, we have

$$\sum_{j=1}^m \lambda_j v_j = u,$$

so that

$$\sum_{j=1}^{m-1} \lambda_j v_j + \left(1 - \sum_{j=1}^{m-1} \lambda_j\right) v_m = u,$$

that is,

$$\sum_{j=1}^{m-1} \lambda_j (v_j - v_m) + v_m = u.$$

Define now

$$\phi_j = v_m - v_j, \; \forall j \in \{1, \cdots, m-1\}.$$

Thus, we have obtained

$$-\sum_{j=1}^{m-1} \lambda_j \phi_j + v_m = u,$$

so that

$$v_m = u + \sum_{j=1}^{m-1} \lambda_j \phi_j.$$

Recall that

$$\phi_j = v_m - v_j,$$

so that

$$v_j = u + \sum_{k=1}^{m-1} \lambda_k \phi_k - \phi_j, \ \forall j \in \{1, \cdots, m-1\}.$$

Moreover clearly we have $\phi_j \in V_0$, where $V_0 = W_0^{1,2}(\Omega; \mathbb{R}^N)$.

From such results, we may infer that

$$\begin{aligned} J_1(u) &= \inf_{(v,\lambda) \in D_u} \left\{ \sum_{j=1}^{m} \lambda_j J(v_j) \right\} \\ &= \inf_{(\phi,\lambda) \in E_u} \left\{ \sum_{j=1}^{m-1} \lambda_j J\left(u + \sum_{k=1}^{m-1} \lambda_k \phi_k - \phi_j\right) \right. \\ &\quad \left. + \lambda_m J\left(u + \sum_{k=1}^{m-1} \lambda_k \phi_k\right) \right\} \end{aligned} \tag{20}$$

where

$$\begin{aligned} E_u = \{ (\phi, \lambda) = ((\phi_1, \cdots, \phi_m), (\lambda_1, \cdots, \lambda_m)) \in V_0^m \times [0,1]^m \\ \text{such that } \sum_{j=1}^{m} \lambda_j = 1 \right\} \end{aligned} \tag{21}$$

3.1. A numerical example for a closely related model

In this subsection we develop duality principles for some models in phase transition. Similar results and models are addressed in [22].

Let $\Omega = [0,1] \subset \mathbb{R}$, and consider a functional $J : V \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2} \int_\Omega \min\{(u'-1)^2, (u'+1)^2\} \, dx + \frac{1}{2} \int_\Omega (u-f)^2 \, dx,$$

where

$$V = \{ u \in W^{1,2}(\Omega) \ : \ u(0) = 0 \text{ and } u(1) = 1/2 \}.$$

In this subsection, we also denote

$$Y = Y^* = L^2(\Omega).$$

Observe that

$$\begin{aligned} J(u) = & -\langle u', v^* \rangle_{L^2} + \frac{1}{2} \int_\Omega \min\{(u'-1)^2, (u'+1)^2\}\, dx \\ & + \langle u', v^* \rangle_{L^2} + \frac{1}{2} \int_\Omega (u-f)^2\, dx \\ \geq & \inf_{v \in Y} \left\{ -\langle v, v^* \rangle_{L^2} + \frac{1}{2} \int_\Omega \min\{(v-1)^2, (v+1)^2\}\, dx \right\} \\ & + \inf_{u \in V} \left\{ \langle u', v^* \rangle_{L^2} + \frac{1}{2} \int_\Omega (u-f)^2\, dx \right\} \\ = & -\int_\Omega \max \left\{ \frac{1}{2}(v^*)^2 + v^*, \frac{1}{2}(v^*)^2 - v^* \right\}\, dx \\ & - \frac{1}{2} \int_\Omega ((v^*)' + f)^2\, dx - v^*(1)u(1) \\ = & -\frac{1}{2} \int_\Omega (v^*)^2\, dx - \int_\Omega |v^*|\, dx \\ & - \frac{1}{2} \int_\Omega ((v^*)' + f)^2\, dx + v^*(1)u(1) \end{aligned} \tag{22}$$

Defining

$$\begin{aligned} J^*(v^*) = & -\frac{1}{2} \int_\Omega (v^*)^2\, dx - \int_\Omega |v^*|\, dx \\ & - \frac{1}{2} \int_\Omega ((v^*)' + f)^2\, dx + v^*(1)u(1) \end{aligned} \tag{23}$$

we have obtained

$$\inf_{u \in V} J(u) \geq \sup_{v^* \in Y^*} J^*(v^*).$$

Indeed, from standard results on basic convex analysis for such a scalar case, we have

$$\inf_{u \in V} J(u) = \sup_{v^* \in Y^*} J^*(v^*).$$

We have obtained numerical results concerning the maximization of J^* for the cases A and B, where

Case A: $f(x) = \sin(\pi x)/2$,

Case B: $f(x) \equiv 0,$ on $[0,1]$.

For the corresponding graphs for the optimal solutions

$$u_0 = (v_0^*)' + f$$

obtained, please see **Figures 1** and **2**, respectively.

The results were obtained through the first part of software presented at the end of this subsection.

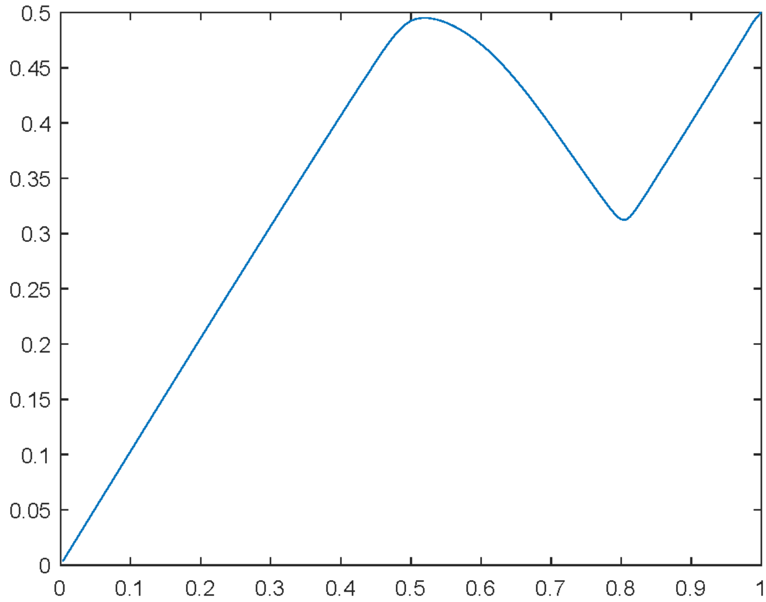

Figure 1. Solution $u_0(x)$ on the interval $[0, 1]$ for the Case A.

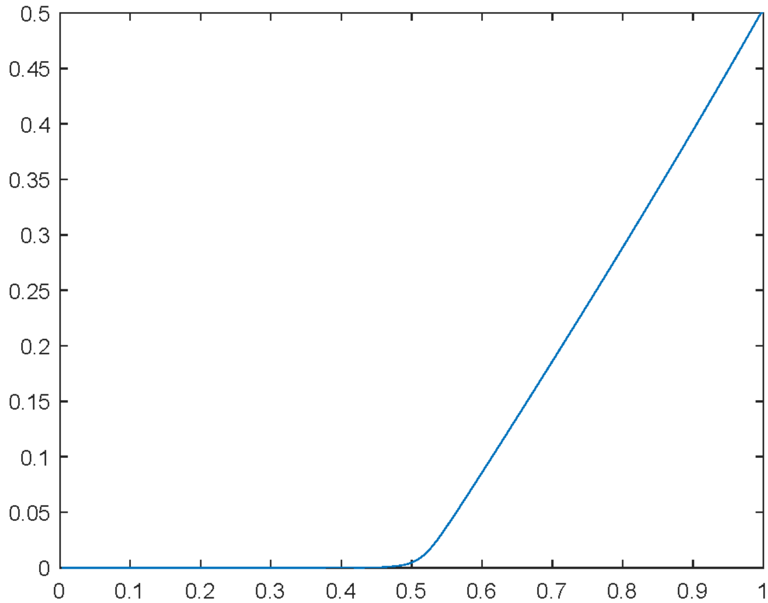

Figure 2. Solution $u_0(x)$ on the interval $[0, 1]$ for the Case B.

Consider now a closely related functional $J_2 : V \to \mathbb{R}$, where

$$J_2(u) = \frac{1}{2}\int_\Omega ((u')^2 - 1)^2 \, dx + \frac{1}{2}\int_\Omega (u - f)^2 \, dx.$$

Define also $J_7 : V \to \mathbb{R}$ by

$$J_7(u) = \frac{1}{2}\int_\Omega ((u')^2 - 1)^2 \, dx.$$

It is well known from the current literature that a global minimum for J_2 may not be attained.

Having obtained such a previous solution $u_0 \in V$ of the closely related functional J, which has critical points close to those of J_2, we shall present a procedure to obtain a critical point for J_2 intended to be approximately optimal.

Define, for $m = 3$ the functionals $F_1 : V \to \mathbb{R}$ and $F_2 : V \to \mathbb{R}$ by

$$F_1(u) = \inf_{(\phi,\lambda)\in E_u} \left\{ \sum_{j=1}^{m-1} \lambda_j J_7 \left(u + \sum_{k=1}^{m-1} \lambda_k \phi_k - \phi_j \right) + \lambda_m J_7 \left(u + \sum_{k=1}^{m-1} \lambda_k \phi_k \right) \right\} + \frac{K}{2} \int_\Omega (u - u_0)^2 \, dx + \frac{1}{2} \int_\Omega (u - f)^2 \, dx \tag{24}$$

and

$$F_2(u) = \frac{K}{2} \int_\Omega (u - u_0)^2 \, dx,$$

so that, denoting

$$J_5(u) = F_1(u) - F_2(u),$$

and

$$J_3(u, z^*) = F_1(u) - \langle u - u_0, z^* \rangle_{L^2} + F_2^*(z^*),$$

we have

$$\begin{aligned} J_5(u) &= F_1(u) - F_2(u) \\ &= -\langle u - u_0, z^* \rangle_{L^2} + F_1(u) \\ &\quad + \langle u - u_0, z^* \rangle_{L^2} - F_2(u) \\ &\leq -\langle u - u_0, z^* \rangle_{L^2} + F_1(u) \\ &\quad + \sup_{v \in Y} \{ \langle v - u_0, z^* \rangle_{L^2} - F_2(v) \} \\ &= F_1(u) - \langle u - u_0, z^* \rangle_{L^2} + F_2^*(z^*) \\ &= J_3(u, z^*) \end{aligned} \tag{25}$$

Observe that from the results of the previous section, we may obtain

$$\inf_{u \in V} J_2(u) = \inf_{u \in V} J_5(u) = \inf_{(u,z^*) \in V \times Y^*} J_3(u, z^*).$$

Therefore, for a large value of $K > 0$, we shall obtain an appropriate critical point of J_2, by obtaining a critical point of J_3, intended to be approximately optimal for J_2, through the following algorithm.

1) Set $n = 1$, $K = 500$, $0 < \varepsilon \ll 1$ (small) and $z_n^* \equiv 0$.

2) Find $u_n \in V$ such that
$$\frac{\partial J_3(u_n, z_n^*)}{\partial u} = \mathbf{0}.$$

3) Find $z_{n+1}^* \in Y^*$ such that
$$\frac{\partial J_3(u_n, z_{n+1}^*)}{\partial z^*} = \mathbf{0},$$
so that
$$z_{n+1}^* = K(u_n - u_0).$$

4) If $\|z_{n+1}^* - z_n^*\|_\infty \leq \varepsilon$, then stop. Otherwise $n := n + 1$ and go to item 2).

Remark 1. *We have obtained solutions for the cases* C *and* D.

Case C*, for* $f(x) = \sin(\pi x)/2$ *and* u_0 *of Case A of the previous section.*

Case D*, for* $f(x) \equiv 0$ *and* u_0 *obtained in Case B of the previous section.*

For the corresponding graphs obtained for the solutions $\hat{u}$*, please see* ***Figures 3*** *and* ***4****, respectively.*

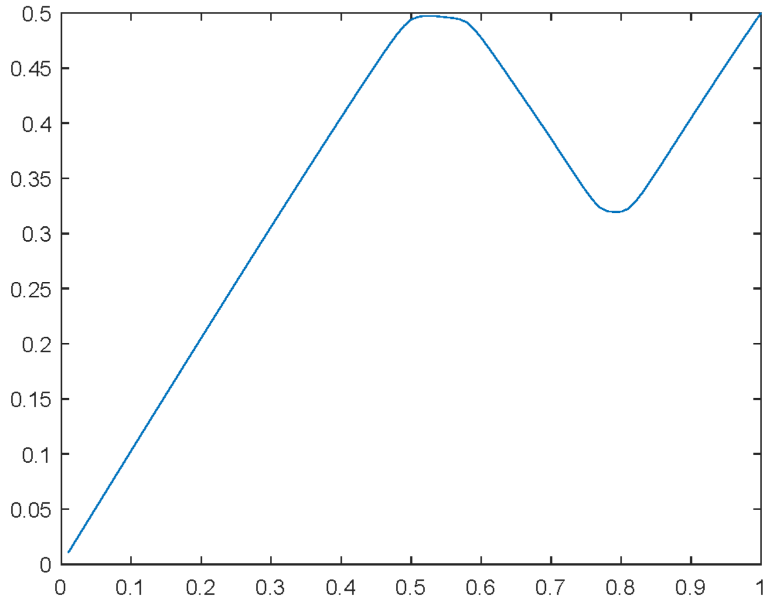

Figure 3. Solution $\hat{u}(x)$ on the interval $[0, 1]$ for the Case C.

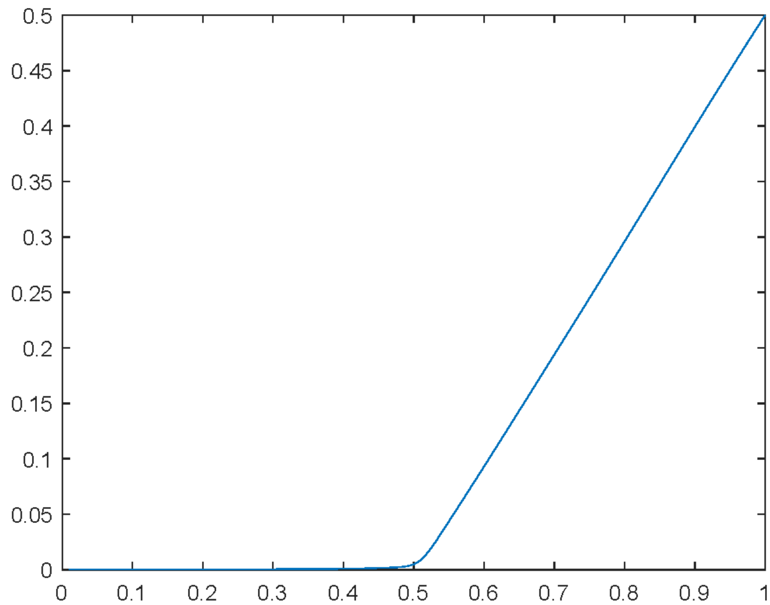

Figure 4. Solution $\hat{u}(x)$ on the interval $[0, 1]$ for the Case D.

Observe that figures for the Cases A *and* C *and,* B *and* D *respectively, are very similar, as expected since it is well known in the literature the functionals* J *and* J_2 *have in general very close critical points.*

In the final section, we present the software phase transition-1 in MAT-LAB, through which the numerical results in this section have been obtained.

3.2. The previous results through a D.C. approach

In the section we shall write a dual functional as a difference between two convex functionals, a so-called D.C. approach.

Consider again the functional $J_2 : V \to \mathbb{R}$, where

$$J_2(u) = \frac{1}{2}\int_\Omega ((u')^2 - 1)^2 \, dx + \frac{1}{2}\int_\Omega (u - f)^2 \, dx.$$

Having obtained such a previous solution $u_0 \in V$ of the closely related functional J, which has critical points close to those of J_2, through a D.C. approach, we shall present a procedure to obtain a critical point for J_2 intended to be approximately optimal.

Define the not relabeled functionals $F_1 : V \to \mathbb{R}$ and $F_2 : V \to \mathbb{R}$ by

$$F_1(u) = J_2(u) + \frac{K}{2}\int_\Omega (u - u_0)^2 \, dx,$$

and

$$F_2(u) = \frac{K}{2}\int_\Omega (u - u_0)^2 \, dx,$$

so that

$$\begin{aligned} \alpha = \inf_{u \in V} J_2(u) &\leq J_2(u) \\ &= F_1(u) - F_2(u) \\ &= -\langle u - u_0, z^* \rangle_{L^2} + F_1(u) \\ &\quad + \langle u - u_0, z^* \rangle_{L^2} - F_2(u) \\ &\leq -\langle u - u_0, z^* \rangle_{L^2} + F_1(u) \\ &\quad + \sup_{v \in Y}\{\langle v - u_0, z^* \rangle_{L^2} - F_2(v)\} \\ &= F_1(u) - \langle u - u_0, z^* \rangle_{L^2} + F_2^*(z^*) \end{aligned} \tag{26}$$

$\forall u \in V, \; z^* \in Y^*$.

Here we have denoted,

$$F_2^*(z^*) = \sup_{u \in V}\{\langle u - u_0, z^* \rangle_{L^2} - F_2(u)\}.$$

From such results, we may infer that

$$\begin{aligned} \alpha &= \inf_{u \in V} J_2(u) \\ &\leq \inf_{u \in V}\{F_1(u) - \langle u - u_0, z^* \rangle_{L^2}\} + F_2^*(z^*) \\ &= -F_1^*(z^*) + F_2^*(z^*) \end{aligned} \tag{27}$$

$\forall z^* \in Y^*$.

Consequently, we have got

$$\alpha = \inf_{u \in V} J_2(u) \leq \inf_{z^* \in Y^*}\{-F_1^*(z^*) + F_2^*(z^*)\}.$$

Defining $J^* : Y^* \to \mathbb{R}$ by

$$J^*(z^*) = -F_1^*(z^*) + F_2^*(z^*),$$

we have obtained its critical points and corresponding solutions $\hat{u} \in V$ for the cases E and F, where:

Case E, for $f(x) = \sin(\pi x)/2$ and u_0 of Case A of the previous section.

Case F, for $f(x) \equiv 0$ and u_0 obtained in Case B of the previous section.

For the corresponding graphs obtained for the solutions $\hat{u}$, please see **Figures 5** and **6**, respectively.

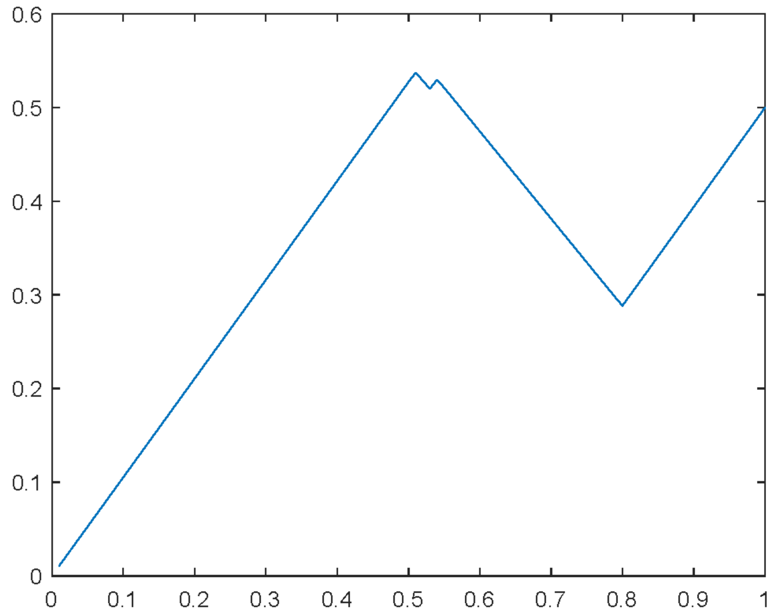

Figure 5. Solution $\hat{u}(x)$ on the interval $[0, 1]$ for the Case E.

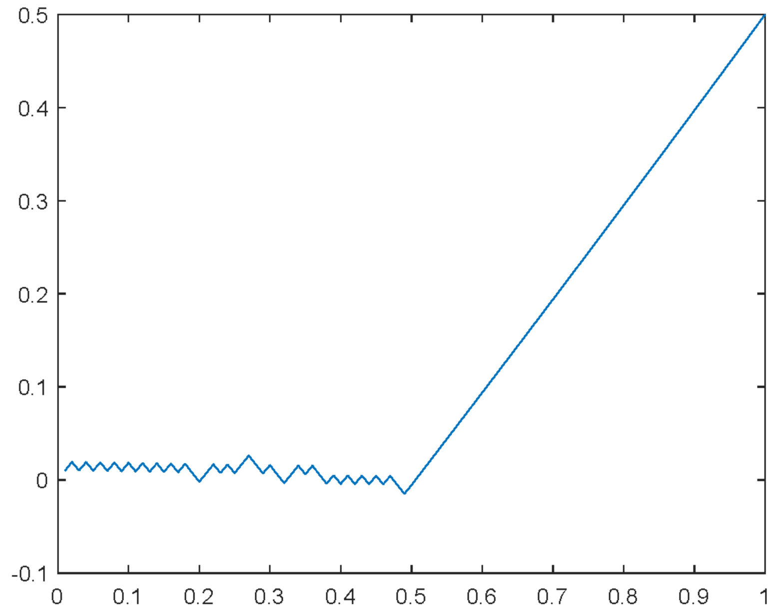

Figure 6. Solution $\hat{u}(x)$ on the interval $[0, 1]$ for the Case F.

Remark 2. *Observe that figures for the Cases A and E and, B and F respectively, are very similar, as expected since it is well known in the literature the functionals J and J_2 have in general very close critical points.*

In the final section, we present the software phase transition-2 in MAT-LAB, through which the numerical results in this subsection have been obtained.

It is worth mentioning in this software we have made the change of variables

$$v = u - x/2,$$

so that $v \in W_0^{1,2}(\Omega)$.

Finally, we would also mention the concerning numerical algorithm is developed through a linearization procedure similar, in some sense, to the Newton's method.

3.3. Another duality principle for a relaxed formulation concerning the previous model

Let $\Omega = [0,1] \subset \mathbb{R}$ and consider a functional $J_2 : V \to \mathbb{R}$ defined by

$$J_2(u) = \frac{1}{2}\int_\Omega ((u')^2 - 1)^2 \, dx + \frac{1}{2}\int_\Omega (u - f)^2 \, dx,$$

where

$$V = \{u \in W^{1,2}(\Omega) \; : \; u(0) = 0 \text{ and } u(1) = 1/2\},$$

and $f \in L^2(\Omega)$.

Denoting $Y = Y^* = L^2(\Omega)$, we define $F : Y \to \mathbb{R}$ by

$$\begin{aligned} F((u')^2 - 1) &= \frac{1}{2}\int_\Omega ((u')^2 - 1)^2 \, dx \\ &= \frac{1}{2}\int_\Omega g(u') \, dx \end{aligned} \tag{28}$$

where $g : \mathbb{R} \to \mathbb{R}$ is such that

$$g(x) = \frac{1}{2}(x^2 - 1)^2.$$

Define also $F_1 : V \to \mathbb{R}$ by

$$F_1(u) = \inf_{(v,\lambda) \in B_u} \left\{ \int_\Omega (\lambda g(v_1') + (1-\lambda) g(v_2')) \, dx \right\},$$

where

$$\begin{aligned} B_u = \{(v,\lambda) = (v_1, v_2, \lambda) \in V^2 \times [0,1] \; : \\ \text{and } \lambda v_1' + (1-\lambda) v_2' = u', \;\; \text{in } \Omega\}. \end{aligned} \tag{29}$$

Observe that

$$\lambda v_1' + (1-\lambda) v_2' = u', \text{ in } \Omega,$$

so that

$$\lambda(v_1' - v_2') + v_2' = u'.$$

Define $\phi = v_2 - v_1$.

Thus,

$$-\lambda \phi' + v_2' = u',$$

so that

$$v_2' = u' + \lambda \phi'.$$

Hence

$$\begin{aligned} v_1' &= v_2' - \phi' \\ &= u' + \lambda\phi' - \phi' \\ &= u' - (1-\lambda)\phi' \end{aligned} \tag{30}$$

With such results in mind, denoting $V_0 = W_0^{1,2}(\Omega)$, we define $F_3 : V \times V_0 \times [0,1] \to \mathbb{R}$ by

$$F_3(u,\phi,\lambda) = \frac{1}{2}\int_\Omega (\lambda g(u' - (1-\lambda)\phi') + (1-\lambda)g(u' + \lambda\phi'))\, dx,$$

so that

$$F_1(u) = \inf_{(\phi,\lambda)\in V_0\times[0,1]} F_3(u,\phi,\lambda).$$

Here we define the relaxed functional $J_1 : V \times V_0 \times [0,1] \to \mathbb{R}$ by

$$J_1(u,\phi,\lambda) = F_3(u,\phi,\lambda) + \frac{1}{2}\int_\Omega (u-f)^2\, dx,$$

Observe that

$$\begin{aligned} J_1(u,\phi,\lambda) &= F_3(u,\phi,\lambda) + \frac{1}{2}\int_\Omega (u-f)^2\, dx \\ &= -\langle((u' - (1-\lambda)\phi')^2 - 1), v_1^*\rangle_{L^2} + \frac{1}{2}\int_\Omega \lambda g(u' - (1-\lambda)\phi')\, dx \\ &- \langle((u' + \lambda\phi')^2 - 1), v_2^*\rangle_{L^2} + \frac{1}{2}\int_\Omega (1-\lambda) g(u' + \lambda\phi')\, dx \\ &+ \langle((u' - (1-\lambda)\phi')^2 - 1), v_1^*\rangle_{L^2} + \langle((u' + \lambda\phi')^2 - 1), v_2^*\rangle_{L^2} \\ &- \langle u', v_3^*\rangle_{L^2} - \langle \phi', v_4^*\rangle_{L^2} \\ &+ \langle u', v_3^*\rangle_{L^2} + \langle \phi', v_4^*\rangle_{L^2} \\ &+ \frac{1}{2}\int_\Omega (u-f)^2\, dx \\ &\geq \inf_{w_1\in Y}\left\{\langle w_1, v_1^*\rangle_{L^2} + \frac{1}{2}\int_\Omega \lambda w_1^2\, dx\right\} \\ &+ \inf_{w_2\in Y}\left\{\langle w_2, v_2^*\rangle_{L^2} + \frac{1}{2}\int_\Omega (1-\lambda) w_2^2\, dx\right\} \\ &+ \inf_{(w_3,w_4)\in Y\times Y}\{\langle (w_3 - (1-\lambda)w_4)^2 - 1, v_1^*\rangle_{L^2} \\ &+ \langle (w_3 + \lambda w_4)^2 - 1, v_2^*\rangle_{L^2} \\ &- \langle w_3, v_3^*\rangle_{L^2} - \langle w_4, v_4^*\rangle_{L^2}\} \\ &+ \inf_{(u,\phi)\in V\times V_0}\left\{\langle u', v_3^*\rangle_{L^2} + \langle \phi', v_4^*\rangle_{L^2} + \frac{1}{2}\int_\Omega (u-f)^2\, dx\right\} \\ &= -\frac{1}{2}\int_\Omega \frac{1}{\lambda}(v_1^*)^2\, dx - \frac{1}{2}\int_\Omega \frac{1}{(1-\lambda)}(v_2^*)^2\, dx \\ &- F_5^*(v_1^*, v_2^*, v_3^*, v_4^*, \lambda) \\ &- \frac{1}{2}\int_\Omega ((v_3^*)' + f)^2\, dx + v_3^*(1)u(1) \\ &\forall v^* = (v_1^*, v_2^*, v_3^*, v_4^*) \in A^*,\ \lambda \in [0,1],\ u \in V,\ \phi \in V_0. \end{aligned} \tag{31}$$

Here

$$A^* = \{v^* \in [Y^*]^4 \ : \ (v_4^*)' = 0, \ v_1^* > 0 \text{ and } v_2^* > 0 \text{ in } \Omega\}.$$

Moreover,

$$\begin{aligned} F_5^*(v_1^*, v_2^*, v_3^*, v_4^*, \lambda) = \sup_{(w_3, w_4) \in Y \times Y} & \{-\langle (w_3 - (1-\lambda) w_4)^2 - 1, v_1^* \rangle_{L^2} - \langle (w_3 + \lambda w_4)^2 - 1, v_2^* \rangle_{L^2} \\ & + \langle w_3, v_3^* \rangle_{L^2} + \langle w_4, v_4^* \rangle_{L^2} \} \end{aligned} \tag{32}$$

Defining now $J^* : A^* \times [0,1] \to \mathbb{R}$ by

$$\begin{aligned} J^*(v^*, \lambda) = & -\frac{1}{2} \int_\Omega \frac{1}{\lambda} (v_1^*)^2 \, dx - \frac{1}{2} \int_\Omega \frac{1}{(1-\lambda)} (v_2^*)^2 \, dx \\ & - F_5^*(v_1^*, v_2^*, v_3^*, v_4^*, \lambda) \\ & - \frac{1}{2} \int_\Omega ((v_3^*)' + f)^2 \, dx + v_3^*(1) u(1) \end{aligned} \tag{33}$$

we have got

$$\begin{aligned} & \inf_{(u,\phi,\lambda) \in V \times V_0 \times [0,1]} J_1(u, \phi, \lambda) \\ & \geq \inf_{\lambda \in [0,1]} \left\{ \sup_{v^* \in A^*} J^*(v^*, \lambda) \right\} \\ & \geq \sup_{v^* \in A^*} \left\{ \inf_{\lambda \in [0,1]} J^*(v^*, \lambda) \right\} \end{aligned} \tag{34}$$

We have obtained critical points for J^* for the cases G and H, where,
Case G: For $f(x) = \sin(\pi x)/2$,
Case H: For $f(x) \equiv 0, \ \text{in } [0,1]$.
For the corresponding solutions

$$\hat{u} = (v_3^*)' + f,$$

please see **Figures 7** and **8**, respectively

In the final section, we present the software phase transition-3 in MAT-LAB, through which the numerical results in this subsection have been obtained.

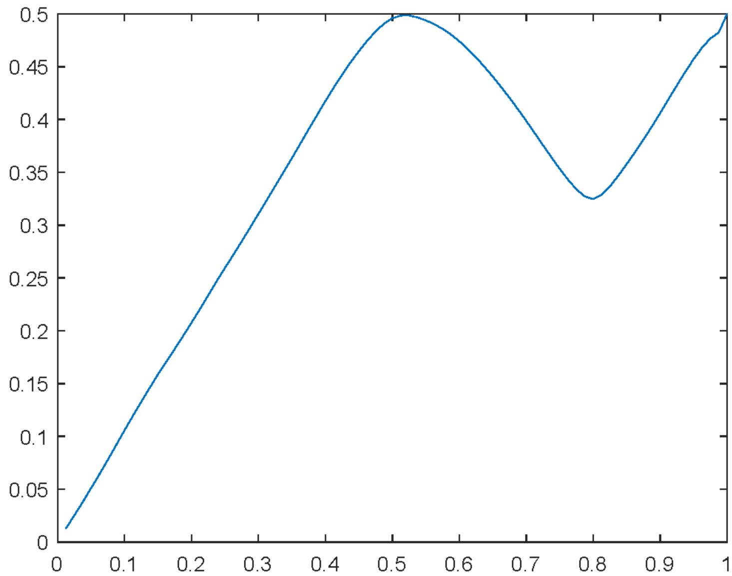

Figure 7. Solution $\hat{u}(x)$ on the interval $[0,1]$ for the Case G.

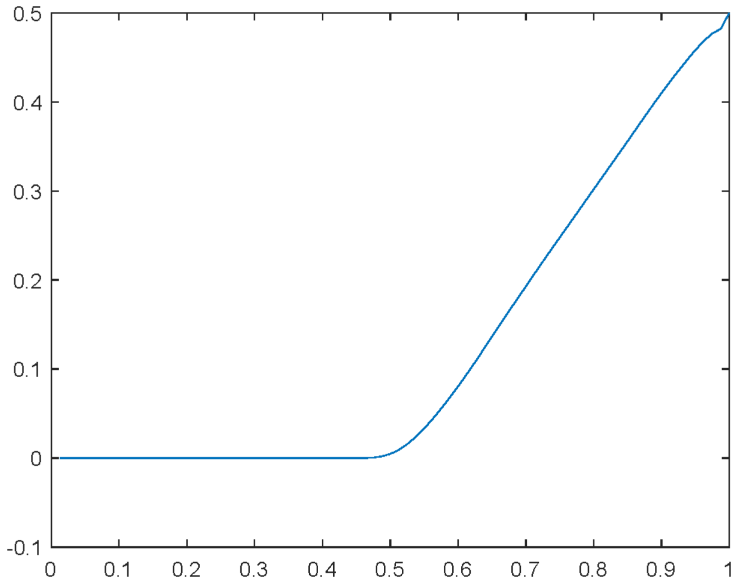

Figure 8. Solution $\hat{u}(x)$ on the interval $[0, 1]$ for the Case H.

4. Conclusion

In this article, in a first step, we have developed a duality principle and related convex dual variational formulation for a non-linear model in elasticity.

It is worth emphasizing, in the last sections we have presented duality principles and relaxation procedures suitable for a large class of vectorial models in the calculus of variations. Applications, including numerical examples and respective software, have been developed for models in phase transition.

We highlight the results here obtained are applicable to a large class of models in the calculus of variations, including some plate and shell non-linear theories, such as those found in references [23, 24] models in superconductivity and micro-magnetism, among many others.

In a near future research we intend to apply such results to some of these mentioned related models.

Institutional review board statement: Not applicable.

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Conflict of interest: The author declares no conflict of interest.

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Appendix

I. Softwares in MAT-LAB through which the numerical results in the subsections 3.1, 3.2 and 3.3 have been obtained

i. Software Phase-Transition-1, through which the results of subsection 3.1 have been obtained

```
********************************
1.  clear all
global m8 d yo u z K e3 u3
m8=100;
d=1/m8;
e3=0.0000001;
K=500;
for i=1:m8
yo(i,1)=sin(i*d*pi)/2*0;
end;
for i=1:m8-1
y1(i,1)=(yo(i+1,1)-yo(i,1))/d;
end;
vo(:,1)=0.1*ones(m8,1);
b12=1.0;
k=1;
while (b12 > 10^-4) && (k < 100)
k
k=k+1;
for i=1:m8
wo(i,1) = sqrt(vo(i,1)^2 + e3);
end;
i=1;
m12 = 1 + d^2/wo(i,1) + d^2;
m50(i)=1/m12;
z(i) = m50(i) * (y1(i,1) * d^2 + yo(i,1) * d);
for i=2:m8-1
m12 = 2 + d^2/wo(i,1) + d^2 - m50(i - 1);
m50(i)=1/m12;
z(i) = m50(i) * (y1(i,1) * d^2 + z(i - 1));
end;
v(m8,1)=1/(1-m50(m8-1))*(d/2-d*yo(m8,1)+z(m8-1));
for i=1:m8-1
v(m8-i,1)=m50(m8-i)*v(m8-i+1,1)+z(m8-i);
end;
b12=max(abs(v-vo));
vo=v;
for i=1:m8-1
```

```
u(i,1)=(vo(i+1,1)-vo(i,1))/d+yo(i+1,1);
end;
u(m8,1)=1/2;
u(m8/2,1)
end;
u3=u;
xo=0.1*ones(5*m8+3,1);
b12=1;
k=1;
while (b12 > 10^-6) && (k < 50)
k
k=k+1;
X=fminunc('funOct202414',xo);
b12=max(abs(xo-X))
xo=X;
u(m8/2,1)
end;
for i=1:m8
x3(i,1)=i*d;
end;
plot(x3,u)
```

****

with the auxiliary function "funOct202414"

****

```
1.  function S=funOct202414(x)
global m8 d yo u K u3
e1=0.0005;
for i=1:m8
u(i,1)=x(i,1);
z(i,1)=x(i+4*m8,1);
v1(i,1)=x(i+m8,1);
v2(i,1)=x(i+2*m8,1);
v3(i,1)=x(i+3*m8,1);
end;
u(m8,1)=1/2;
v1(m8,1)=0;
v2(m8,1)=0;
v3(m8,1)=0;
L1=(sin(x(5*m8+1,1))+1)/2;
L2=min((sin(x(5*m8+2,1))+1)/2,1-L1);
L3=min((sin(x(5*m8+3,1))+1)/2,1-L1-L2);
L4=1-L1-L2-L3;
du(1,1)=u(1,1)/d;
```

dv1(1,1)=v1(1,1)/d;
dv2(1,1)=v2(1,1)/d;
dv3(1,1)=v3(1,1)/d;
for i=2:m8
du(i,1)=(u(i,1)-u(i-1,1))/d;
dv1(i,1)=(v1(i,1)-v1(i-1,1))/d;
dv2(i,1)=(v2(i,1)-v2(i-1,1))/d;
dv3(i,1)=(v3(i,1)-v3(i-1,1))/d;
end;
$d2u(1,1) = (-2 * u(1,1) + u(2,1))/d^2;$
$d2v1(1,1) = (-2 * v1(1,1) + v1(2,1))/d^2;$
$d2v2(1,1) = (-2 * v2(1,1) + v2(2,1))/d^2;$
$d2v3(1,1) = (-2 * v3(1,1) + v3(2,1))/d^2;$
for i=2:m8-1
$d2u(i,1) = (u(i+1,1) - 2 * u(i,1) + u(i-1,1))/d^2;$
end;
S=0;
for i=1:m8
$S = S + L1/2 * ((du(i,1) - (1 - L1) * dv1(i,1) + L2 * dv2(i,1) + L3 * dv3(i,1))^2 - 1)^2;$
$S = S + L2/2 * ((du(i,1) + L1 * dv1(i,1) - (1 - L2) * dv2(i,1) + L3 * dv3(i,1))^2 - 1)^2;$
$S = S + L3/2 * ((du(i,1) + L1 * dv1(i,1) + L2 * dv2(i,1) - (1 - L3) * dv3(i,1))^2 - 1)^2;$
$S = S + L4/2 * ((du(i,1) + L1 * dv1(i,1) + L2 * dv2(i,1) + L3 * dv3(i,1))^2 - 1)^2;$
$S = S + (u(i,1) - yo(i,1))^2/2;$
$S = S + K * (u(i,1) - u3(i,1))^2/2 - z(i,1) * (u(i,1) - u3(i,1)) + z(i,1)^2/2/K;$
end;
for i=1:m8-1
$S = S + e1 * d2u(i,1)^2/2;$
end;
****

ii. Software Phase-Transition-2, through which the results of subsection 3.2 have been obtained

****

1. clear all

```
global m8 d yo K m2 vo v w1 u u3
m8=100;
d=1/m8;
for i=1:m8
yo(i,1)=sin(i*d*pi)/2*0;
w1(i,1)=i*d/2;
end;
K=1000000;
e3=0.0000001;
for i=1:m8-1
y1(i,1)=(yo(i+1,1)-yo(i,1))/d;
```

```
end;
vo(:,1)=0.1*ones(m8,1);
b12=1.0;
k=1;
while (b12 > 10^-4) && (k < 100)
k
k=k+1;
for i=1:m8
wo(i,1) = sqrt(vo(i,1)^2 + e3);
end;
i=1;
m12 = 1 + d^2/wo(i,1) + d^2;
m50(i)=1/m12;
z(i) = m50(i) * (y1(i,1) * d^2 + yo(i,1) * d);
for i=2:m8-1
m12 = 2 + d^2/wo(i,1) + d^2 - m50(i - 1);
m50(i)=1/m12;
z(i) = m50(i) * (y1(i,1) * d^2 + z(i - 1));
end;
v(m8,1)=1/(1-m50(m8-1))*(d/2-d*yo(m8,1)+z(m8-1));
for i=1:m8-1
v(m8-i,1)=m50(m8-i)*v(m8-i+1,1)+z(m8-i);
end;
b12=max(abs(v-vo));
vo=v;
for i=1:m8-1
u(i,1)=(vo(i+1,1)-vo(i,1))/d+yo(i+1,1);
end;
u(m8,1)=1/2;
u(m8/2,1)
end;
u3=u;
m2=zeros(m8,m8);
for i=2:m8-1
m2(i,i)=-2.0;
m2(i,i+1)=1.0;
m2(i,i-1)=1.0;
end;
m2(1,1)=-2.0;
m2(1,2)=1.0;
m2(m8,m8)=-2.0;
m2(m8,m8-1)=1.0;
vo(:,1)=0.1*ones(m8,1);
xo(:,1)=0.1*ones(m8,1);
```

```
x1=xo;
b14=1;
k1=1;
while (b14 > 10^-4) && (k1 < 20)
k1
k1=k1+1;
b12=1;
k=1;
while (b12 > 10^-4) && (k < 120)
k
k=k+1;
X=fminunc('funOct202415',xo);
b12=max(abs(X-xo))/K
xo=X;
u(m8/2,1)
end;
b14=max(abs(xo-x1))/K
x1=xo;
vo=v;
end;
for i=1:m8
x3(i,1)=i*d;
end;
plot(x3,u);
```

****

With the auxiliary function "funOct202415"

****

```
1.      function S=funOct202415(x)
global m8 d yo K m2 vo v w1 u u3
for i=1:m8
z(i,1)=x(i,1);
end;
Id=eye(m8);
vo(m8,1)=0;
dvo(1,1)=vo(1,1)/d;
for i=2:m8
dvo(i,1)=(vo(i,1)-vo(i-1,1))/d;
end;
for i=1:m8
w(i,1) = -6 * (dvo(i,1) + 1/2)^2 + 2;
end;
v(:,1) = inv(diag(w(:,1)) * m2/d^2 + K * Id + Id) * (z(:,1) - w1(:,1) + K * (u3(:,1) - w1(:,1)) + yo(:,1));
v(m8,1)=0;
```

```
u=v+w1;
du(1,1)=u(1,1)/d;
for i=2:m8
du(i,1)=(u(i,1)-u(i-1,1))/d;
end;
z(m8,1)=0;
dz(1,1)=z(1,1)/d;
for i=2:m8
dz(i,1)=(z(i,1)-z(i-1,1))/d;
end;
S=0;
for i=1:m8
S=S+(du(i,1)^2-1)^2/2+(u(i,1)-yo(i,1))^2/2;
S=S-z(i,1)*(v(i,1)-(u3(i,1)-w1(i,1)))+K*(v(i,1)-(u3(i,1)-w1(i,1)))^2/2;
S=S+z(i,1)^2/2/K;
end;
```

****

iii. Software Phase-Transition-3, through which the results of subsection 3.3 have been obtained

****

1. clear all

```
global m8 d yo e1 u L v1 v2 v3 v4 L1 e5
m8=80;
d=1/m8;
e5=0.0005;
e1=e5;
for i=1:m8
yo(i,1)=sin(i*d*pi)/2*0;
end;
L=1/2;
L1=1/2;
xo(:,1)=0.1*ones(4*m8,1);
x1(1,1)=1/2;
b14=1;
k1=1;
while (b14 > 10^-4) && (k1 < 80)
k1
k1=k1+1;
b12=1;
k=1;
while (b12 > 10^-4) && (k < 80)
k
k=k+1;
X=fminunc('funOct202456',xo);
```

```
b12=max(abs(X-xo))
xo=X;
u(m8/2,1)
end;
X1=fminunc('funOct202458',x1);
b14=max(abs(L-L1));
L1=L;
L
x1=X1;
end;
for i=1:m8
x3(i,1)=i*d;
end;
plot(x3,u);
```

****

With the auxiliary function "funOct202456"

****

1. function S=funOct202456(x)

```
global m8 d yo e1 u L v1 v2 v3 v4 L1 e5
for i=1: m8
```

$v1(i,1) = x(i,1)^2;$

$v2(i,1) = x(i+m8,1)^2;$

```
v3(i,1)=x(i+2*m8,1);
end;
v4(1,1)=x(3*m8+1,1);
S=0;
for i=1:m8-1
dv3(i,1)=(v3(i+1,1)-v3(i,1))/d;
end;
S=0;
for i=1:m8-1
D=4*v1(i,1)*v2(i,1)+0.000000001;
A=(2*v2(i,1)*v3(i,1)*L-2*v2(i,1)*v4(1,1))/D;
B=(2*v1(i,1)*((1-L))*v3(i,1)+2*v1(i,1)*v4(1,1))/D;
x1=L*A+(1-L)*B;
y1=B-A;
```

$S = S - v1(i,1) * (A^2 - 1) - v2(i,1) * (B^2 - 1) + v3(i,1) * x1 + v4(1,1) * y1;$

$S = S + (dv3(i,1) + yo(i,1))^2/2;$

$S = S + v1(i,1)^2/2/(L + e5) + v2(i,1)^2/2/((1 - L) + e5);$

```
end;
S=S-v3(m8,1)*1/2/d;
for i=1:m8-1
u(i,1)=dv3(i,1)+yo(i,1);
```

end;
u(m8,1)=1/2;
du(1,1)=u(1,1)/d;
for i=2:m8
du(i,1)=(u(i,1)-u(i-1,1))/d;
end;
$d2u(1,1) = (-2 * u(1,1) + u(2,1))/d^2$;
for i=2:m8-1 $d2u(i,1) = (u(i+1,1) - 2 * u(i,1) + u(i-1,1))/d^2$;
end;
for i=1:m8-1
$S = S + 0.0005 * du(i,1)^2/2$;
end;
****

And the auxiliary function "funOct202458"

****

1. function S=funOct202458(x)
global m8 d yo e1 u L v1 v2 v3 v4 L1 e5
L=(sin(x(1,1))+1)/2;
for i=1:m8-1
dv3(i,1)=(v3(i+1,1)-v3(i,1))/d;
end;
S=0;
for i=1:m8-1
D=4*v1(i,1)*v2(i,1)+0.000000001;
A=(2*v2(i,1)*v3(i,1)*L-2*v2(i,1)*v4(1,1))/D;
B=(2*v1(i,1)*((1-L))*v3(i,1)+2*v1(i,1)*v4(1,1))/D;
x1=L*A+(1-L)*B;
y1=B-A;
$S = S - v1(i,1) * (A^2 - 1) - v2(i,1) * (B^2 - 1) + v3(i,1) * x1 + v4(1,1) * y1$;
$S = S + (dv3(i,1) + yo(i,1))^2/2$;
$S = S + v1(i,1)^2/2/(L + e5) + v2(i,1)^2/2/((1 - L) + e5)$;
end;
S=S-v3(m8,1)*1/2/d;
S=-S;
for i=1:m8-1
u(i,1)=dv3(i,1)+yo(i,1);
end;
u(m8,1)=1/2;
****