

A comprehensive review article on fractional models involving ecology and eco-epidemiology

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ABSTRACT: This paper deals with the various definitions involved in the very old yet novel topic called fractional calculus. This survey intends to report some of the major works carried out in the arena of fractional calculus that took place since 2010. Fractional calculus is a prominent topic for research within the discipline of applied mathematics due to its usefulness in solving problems in several different branches of science, engineering, medicine, finance, economics and the likes. With the various definitions involved in this field, we explore the various models taken into consideration to study the effect and impact of fractional calculus to understand how the dynamics of such models change.

KEYWORDS: Caputo fractional derivative; Mittag-Leffler function; paradox of enrichment; Feigenbaums constants; generalized Hyers-Ulam stability

1. Introduction

Historical background

The onset of fractional calculus was marked by a question posed by L'Hopital in his letter to Leibniz in 1695. What might be a derivative of order $1/2$? With this question in mind, Leibniz foresees the genesis of the field which is popularly known as fractional calculus (FC). In the classical calculus, the derivative has an important geometric interpretation; namely, it is associated with the concept of tangent, in opposition to what occurs in the case of FC. This difference can be seen as a problem for the slow progress of FC up to 1900. After Leibniz, it was Euler that noticed the problem for a derivative of non-integer order as can be understood from the study of Miller and Ross^[1]. Fourier suggested an integral representation in order to define the derivative, and his version can be considered the first definition for the derivative of arbitrary (positive) order which is portrayed in the study of Miller and Ross^[1] together with Machado et al.^[2]. Abel solved an integral equation associated with the tautochrone problem, which is considered to be the first application of FC, portrayed in the study of Miller and Ross^[1] alongwith Machado et al.^[2]. Liouville, as can be understood in the study by Miller and Ross^[1] together with Machado et al.^[2], suggested a definition based on the formula for differentiating the exponential function. This expression is known as the first Liouville definition. The second definition formulated by Liouville is presented in terms of an integral and is now called the version by Liouville for the integration of non-integer order. Grunwald and Letnikov, independently, developed an

approach to non-integer order derivatives in terms of a convenient convergent series, conversely to the Riemann-Liouville approach, that is given by an integral. Letnikov showed that his definition coincides with the versions formulated by Liouville, for particular values of the order, and by Riemann, under a convenient interpretation of the so-called non-integer order difference. In the study of Machado et al.^[2], Hadamard published a paper where the non-integer order derivative of an analytical function must be done in terms of its Taylor series. After 1900, the FC experiences a fast development and, in an attempt to formulate particular problems, other definitions were proposed.

In the process of development of fractional calculus, the Caputo version is of utmost importance. The definition as proposed by Caputo inverts the order of integral and derivative operators with the non-integer order derivative of the Riemann-Liouville. The difference between these two formulations are summarized as follows:

In the Caputo: first derivative of integer order is calculated and then integral of non-integer order is calculated. In the Riemann-Liouville: first the integral of non-integer order is calculated and then the derivative of integer order is calculated. It is important to cite that the Caputo derivative is useful to affront problems where initial conditions are done in the function and in the respective derivatives of integer order. In recent decades, the field of fractional calculus has attracted interest of researchers in several areas including mathematics, physics, chemistry, engineering and even finance and social sciences.

Why is fractional calculus so significant?

Until recent times, fractional calculus was considered as a rather esoteric mathematical theory without applications, but in the last (few) decade(s) there has been an explosion of research activities on the application of fractional calculus to very diverse scientific fields ranging from the physics of diffusion and advection phenomenon, to control systems to finance and economics. Indeed, at present, applications and/or activities related to fractional calculus have appeared in at least the following fields:

- 1) Fractional control of engineering systems.
- 2) Advancement of calculus of variations and optimal control to fractional dynamic systems.
- 3) Fundamental explorations of the electrical and thermal constitutive relations and other properties of various engineering materials such as viscoelastic polymers, foams, gels, and animal tissues and their engineering and scientific applications.
- 4) Fundamental understanding of wave and diffusion phenomenon, their measurements and verifications, including applications to plasma physics.
- 5) Bioengineering and biomedical applications.
- 6) Thermal modeling of engineering systems.
- 7) Image and signal processing.

In this paper, Section 2 constitutes of definitions related to fractional calculus. Section 3 deals with literature review. Section 4 deals with the analytical and numerical techniques employed for analyzing the dynamic behaviour of the model systems. Conclusion is incorporated in Section 5 and the paper ends with the list of legitimate references.

2. Fractional Calculus: Definitions

The theory of fractional calculus is a novel topic and attracts a lot of attention to understand the critical dynamics of models. To develop concrete understanding of fractional models, the following definitions need to be perceived:

2.1. Riemann Liouville (RL) fractional integral

The RL integral provides the formula for fractional calculus in its traditional form. It is based on the Fourier series and calls for the disappearance of the constant Fourier coefficient (so, it is true for functions on the unit circle whose integrals are 0). The Weyl integral provides the theory of fractional integration for periodic functions, which includes the “border condition” of recurring after a period. There are two different versions of the R-L integral: upper and lower. The integrals are defined as for the range [a, b].

$${}_aD_t^{-\alpha} f(t) = {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau) d\tau \tag{1}$$

$${}_tD_b^{-\alpha} f(t) = {}_tI_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{\alpha-1} f(\tau) d\tau \tag{2}$$

where, the former is applicable for $t > a$ the latter being applicable for $t < b$ which can be seen in the study of Hermann^[3]. The Grunwald-Letnikov (G-L) derivative, in contrast, begins with the derivative rather than the integral.

2.2. Riemann Liouville fractional derivative

The Lagranges’s rule for differential operators is used to get the associative derivative. Calculating the n -th order derivative over the order integral ($n \alpha$), the α obtaining order derivative. The fact that n is the lowest integer bigger than α should be noted (i.e., $n = [\alpha]$). The definition of the derivative contains upper and lower variations, just as the R-L integral definitions which can be seen in the study of Hermann^[3].

$${}_aD_t^\alpha f(t) = \frac{d^n}{dt^n} {}_aD_t^{-(n-\alpha)} f(t) = \frac{d^n}{dt^n} {}_aI_t^{n-\alpha} f(t) \tag{3}$$

$${}_tD_b^\alpha f(t) = \frac{d^n}{dt^n} {}_tD_b^{-(n-\alpha)} f(t) = \frac{d^n}{dt^n} {}_tI_b^{n-\alpha} f(t) \tag{4}$$

2.3. Hadamard fractional integral (HFI)

Jacques Hadamard invented the HFI (see the study by Hadamard^[4]) and is determined by the subsequent formula,

$${}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\log \frac{t}{\tau})^{\alpha-1} f(\tau) \frac{d\tau}{\tau}, t > a \tag{5}$$

2.4. Atangana-Baleanu (A-B) fractional integral

A continuous function’s A-B fractional integral is defined as:

$${}^{AB}I_t^\alpha f(t) = \frac{1-\alpha}{AB(\alpha)} f(t) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau) d\tau \tag{6}$$

2.5. Caputo fractional derivative

An additional technique for calculating fractional derivatives is referred to as the Caputo fractional derivative which was first mentioned in the article from 1967, referred in the study of Caputo^[5]. When using Caputo’s concept to solve differential equations, it is not essential to define the fractional order initial conditions, in contrast to the R-L fractional derivative. Following is an illustration of Caputo’s definition, where once more $n = [\alpha]$:

$${}^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(\tau)^{\alpha+1-n}}{(t-\tau)} d\tau \tag{7}$$

The Caputo fractional derivative is described as follows:

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-u)^{n-\alpha-1} f^n(u) du, \quad (n-1) < \alpha < n \quad (8)$$

It benefits from being zero when $f(t)$ is constant and its Laplace transform is calculated using the function's initial values and derivative. Additionally, the Caputo fractional derivative of dispersed order is described as:

$${}_a^b D^\alpha f(t) = \int_a^b \phi(\alpha) [D^\alpha f(t)] d\alpha = \int_a^b \left[\frac{\phi(\alpha)}{\Gamma(1-\alpha)} \int_0^t (t-u)^{-\alpha} f'(u) du \right] d\alpha \quad (9)$$

where $\phi(\alpha)$ is a weight function that is used to formally express the existence of different memory formalisms.

2.6. Caputo-Fabrizio (C-F) fractional derivative

M. Caputo and M. Fabrizio provided a definition of the FD with a non-singular kernel for the following function $f(t)$ of the C^1 in a study that was published in 2015.

$${}^{CF}D_t^\alpha f(t) = \frac{1}{1-\alpha} \int_a^t f'(\tau) e^{(-\alpha \frac{t-\tau}{1-\alpha})} d\tau, \quad \alpha \in (0, 1] \quad (10)$$

2.7. Atangana-Baleanu (A-B) fractional derivative

Differential operators based on the generalized Mittag-Leffler function were introduced by Atangana and Baleanu in 2016. The intention was to present non-singular, nonlocal fractional differential operators. The following lists their Riemann-Liouville and Caputo sense fractional differential operators, respectively. For a C^1 function $f(t)$ provided in the study of Alghatani^[6] together with Atangana and Baleanu^[7].

$${}^{ABC}D_t^\alpha f(t) = \frac{AB(\alpha)}{1-\alpha} \int_a^t f'(\tau) E_\alpha \left(-\alpha \frac{(t-\alpha)^\alpha}{1-\alpha} \right) d\tau \quad (11)$$

$${}^{ABC}D_t^\alpha f(t) = \frac{AB(\alpha)}{1-\alpha} \frac{d}{dt} \int_a^t f(\tau) E_\alpha \left(-\alpha \frac{(t-\alpha)^\alpha}{1-\alpha} \right) d\tau \quad (12)$$

Atangana-Baleanu fractional derivative's kernel has certain characteristics of a cumulative distribution function. For instance, the function E_α is rising on the real line and converges to for every $\alpha \in (0, 1]$.

3. Literature review (Techniques employed: Analytical and numerical approach)

3.1. Approach on ecological models

Qian et al.,^[8] had made use of two-parameter Mittag-Leffler function and Gronwall inequality in their proposed paper. To deal with stability and asymptotic stability of fractional differential system of the form ${}_{RL}D_{a,t}^{\alpha-1} x(t) = Ax(t)$, with initial condition ${}_{RL}D_{a,t}^{\alpha-1} x(t)|_{t=a} = x_0 = (x_{10}, x_{20}, \dots, x_{n0})^T$, where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$, $\alpha \in (0, 1)$ and $A \in \mathbb{R}^{n \times n}$ we say that the mentioned system is said to be

- 1) stable if for any x_0 , there exists $\epsilon > 0$ s.t. $\|x(t)\| \leq \epsilon$ for $t \geq 0$;
- 2) asymptotically stable if $\lim_{t \rightarrow +\infty} \|x(t)\| = 0$.

Wei et al.,^[9] followed the definition of sequential fractional derivative presented by Miller and Ross

in the proposed manuscript. In the mentioned paper, the authors have defined classical, lower and upper solution of problem as defined in the paper. The paper also utilizes properties of Mittag-Leffler function. Monotone iterative method and Lipschitz criteria is employed by the authors. El-Sayed^[10] is concerned with the IVP of the non-linear fractional-order differential equation as follows:

$$D^\alpha x(t) = \sum_{k=0}^n a_k t f_k x(t) \tag{14}$$

with the initial data

$$x(0) = x_0 \tag{15}$$

The above IVP is considered with the following assumptions:

- 1) $a_k(t) \in C^1 [0, T], k = 0, 1, \dots$ where $C^1 [0, T]$ is the space of all continuously differential functions on $[0, T], a_k > \sup |a_{kt}|$ and $a'_k > \sup |\frac{d}{dt} a_k(t)|$.
- 2) $F: D \rightarrow R^+, \forall t \in I, D \subset R^+$ where $F(x(t)) = \sum_{k=0}^n a_k(t) f_k(x(t))$
- 3) Functions f_k satisfy Lipschitz condition.

With the help of the above assumptions, the existence and uniqueness of solutions of the IVP are established. Lyapunov uniform stability is employed to prove uniform stability of the IVP. Kansa incorporated the radial basis function (RBF) collocation method in order to solve the partial differential equations, which is now known as the Kansa method. Chen et al.,^[11] discusses time fractional diffusion equation and time fractional derivative discretization. With reference to analogizing the time fractional derivative term, the authors have employed the finite different scheme method. In the numerical simulation segment, MQ function is employed and the method of Tadjeran is introduced in one-dimensional case. In two-dimensional case, the authors had utilized TPS function. Li et al.,^[12] proposed the generalized Mittag-Leffler stability and the generalized fractional Lyapunov direct method in this manuscript. This paper discusses extension of the application of Riemann-Liouville fractional-order systems by using Caputo fractional-order systems. The authors had introduced class-K functions to the fractional Lyapunov direct method and provided the fractional comparison principle. Abbas et al.,^[13] establishes existence and uniqueness of solutions by defining an operator on the class of continuous functions. The operator having a fixed point establishes the uniqueness of solutions. For analyzing the proposed model system numerically, Adams-type predictor-corrector method is employed. Xiao^[14] employs Routh-Hurwitz criteria for dealing with local asymptotic stability. The authors have made use of Adams-Bashforth-Moulton algorithm for performing numerical simulations. Abdelouahab et al.,^[15] sets up fractional order Routh-Hurwitz (FR-H) conditions for verifying local asymptotic stability. The stability of the prescribed equilibrium points is carried out by calculating the variational matrix. In this manuscript, the authors have utilized the PECE (predict, evaluate, correct, evaluate) algorithm (time-domain method) which is related to the Caputo definition while exploring the stated model numerically. Delavari et al.,^[16] proves Bihari's inequality and Gronwall-Bellman integral inequality. The authors have defined class KL function. Moreover, the authors have proved comparison theorem for fractional order system. In order to establish uniform asymptotic stability, local Lipschitz condition is performed. The authors have also put in use the Mittag-Leffler function. Rana et al.,^[17] makes use of Lipschitz condition. Local stability of prescribed equilibrium points is performed with the aid of variational matrix. The authors also talk about local and non-local kernels. The authors have performed homotopy analysis method for finding approximate analytic solution. For estimation of the parameters, the authors have used the total least squares method, also known as orthogonal distance fitting (ODF). The numerical simulation segment uses the FDE12 package in MATLAB which is an implementation of

the Adams-Bashforth-Moulton predictor corrector method. The fmincon optimization algorithm in MATLAB is applied for data fitting of prescribed model. Agarwal et al.,^[18] applies Euler's discretization process in this paper. Having considered a variational matrix for one of the prescribed fixed points and calculating their characteristic roots, the authors have explored the stability of each fixed point. For identifying existence of chaos, the authors have computed Lyapunov characteristic exponents (LCEs) since the Lyapunov exponent is considered to be a good indicator of identifying existence of chaos. The local stability analysis has been done based on standard linearization technique and using the variational matrix in this paper presented by Javidi and Nyamoradi^[19]. In order to solve the proposed fractional model, the authors have used a numerical method introduced by Atanackovic and Stankovic. Choi et al.,^[20] in their paper, discusses Mittag-Leffler stability. The paper considers Volterra fractional differential equation. The authors have defined h-stable system and discusses h-stability. Zhou et al.,^[21] in their paper, applies globally Mittag-Leffler stability. This paper discusses class-K function. The existence and uniqueness of solution is established using Lipschitz condition. In this manuscript, Elsadany and Matouk^[22], achieves Euler-discretized fractional-order Lotka-Volterra system. The variational matrix is computed for verifying stability of equilibrium points. Matouk et al.,^[23] determine the equilibrium points analyze stability conditions by computing variational matrix at the stated equilibrium points. Matignon's results are employed for determining local asymptotic stability of equilibrium points. In the numerical simulation section, an effective method for solving fractional-order differential equation is applied which is PECE (predict, evaluate, correct, evaluate). For establishing global stability of equilibrium points, the system is linearized in this paper addressed by Rihan et al.,^[24]. To study the stability of the prescribed system, Laplace transform is applied. The authors discuss Lyapunov globally asymptotical stability provided the equilibrium points exist. Here, in FODDEs, implicit Euler's scheme is employed. The authors have deduced existence of unique solution with the aid of Lipschitz condition and Banach contraction principle. Caputo and Fabrizio^[25], in this paper, presents Laplace transform of the NFD_t. The authors have presented with a fractional gradient operator in order to describe non-local dependence in constitutive equations. The authors have also introduced fractional divergence in this paper. The authors have also presented Fourier transform of fractional gradient and divergence. Losada and Nieto^[26] proposes the condition when a solution of the fractional differential equation ${}^{CF}D^\alpha f(t) = 0$ becomes a constant function. This paper utilizes contraction map and Banach's fixed point theorem. One of the most fascinating features of this paper is the application to fractional falling body problem. Ghaziani et al.,^[27] discusses stability criterion for equilibrium points of the fractional order system taken into consideration by the aid of variational matrix. The authors have applied numerical technique put forward by Atanackovic and Stankovic in the segment of numerical analysis. Ji et al.,^[28] have mentioned uniform asymptotic stability theorem. To investigate the stability conditions of equilibrium points, variational matrix has been evaluated at those points and Routh-Hurwitz condition has also been employed. Uniform asymptotic stability of equilibrium points has been established using suitable positive definite Lyapunov function. Local stability of equilibrium points of a linearized fractional-order form of system proposed by Matouk and Elsadany^[29], is determined by the very familiar Matignon's conditions. The authors have found out the condition for existence and uniqueness of solutions with the assistance of contraction mapping. The numerical analysis deals with solving the fractional order differential equations using the predictor-correctors scheme or more precisely, predict, evaluate, correct, evaluate (PECE). A new control method has been introduced in this paper in order to stabilize the chaotic fractional-order GLV-system. Eulerian discretization has also been employed. Song et al.,^[30] evaluates asymptotic stability of equilibrium points. The stability of the model under consideration is investigated with the introduction of time delay. The effect of harvesting

is analyzed. The system model is thoroughly scrutinized numerically using MATLAB. The preliminary investigating tools are RL fractional integral operator using gamma function and Caputo fractional derivative. The manuscript addressed by Atangana and Koca^[31] uses Atangana-Baleanu fractional derivative in Caputo sense. Laplace transform operator is employed in this paper. Moreover, while exploring the relation with the integral transforms Laplace, Sumudu, Fourier and Mellin transforms are put to use. Model of Lorenz attractor with Atangana-Baleanu derivative is explored thoroughly. For finding existence of solutions, Picard Lindelöf method is employed. The authors Li et al.,^[32] have focused on the Feigenbaums constants in reverse bifurcation of fractional- order Rossler system. At first, the definition of Feigenbaums constants in reverse bifurcation has been investigated thoroughly. Secondly, the Feigenbaums constants in reverse bifurcation and the error percentage are obtained by analyzing a series of bifurcation diagrams of integer and fractional-order Rossler system. Khajanchi^[33] makes use of Nagumo's theorem to state that any solution with initial point shall remain positive throughout the prescribed region. The author establishes boundedness of solution. Also, in order to obtain global stability, Lyapunov function is employed. this paper derives the normal form theory to investigate the direction and stability of the limit cycle arising from Hopf bifurcation. To examine non-existence of periodic orbits, the authors have utilized Bendixson-Dulac criteria. MATLAB software is used for exploring the model numerically. Nosrati and Shafiee^[34] analyzes relation between fractional order derivatives and economic profits. The numerical analysis segment is established using Atanackovic and Stankovic. This paper introduces Grunwald-Letnikov (GL) fractional derivative operator for a continuous function. Moreover, the authors have considered the Fractional-order singular (FOS) model of predator-prey system. This manuscript tries to identify the conditions owing to which the FOS model can be regarded to be stable. This phenomenon uses linearization and variational matrix to achieve stability. In the segment dealing with local stability analysis, bifurcation and economic profit, the authors have defined AEP (admissible equilibrium point). In this paper, addressed by Owolabi and Atangana^[35], the authors implemented the basic concepts of fractional derivatives, presented the spectral method for approximating the Riemann-Liouville fractional derivative and later proposed the pseudo-spectral method that can be used in conjunction with any higher-order time solver. The discussion in this paper is focused on general space fractional reaction diffusion equation. In order to achieve the collocation equation, the Gauss-Lobatto nodes are considered at first. Deshpande et al.,^[36] proposes stability conditions due to Matignon's stability condition. The authors have studied and critically analyzed fractional Bhalekar-Gejji system. Existence of fractional Hopf bifurcation is performed with two singular conditions and one transversality condition. For detection of chaotic behaviour of the model under consideration, maximum Lyapunov characteristic coefficient is plotted for a parameter. For determining the long-term behaviour of the model system, the present paper employs fractional Adams method (FAM) along with Mathematica 10.0 software. Zou and He^[37] employs Lipschitz condition to establish uniqueness of continuous solution. Abdeljawad^[38] uses Banach contraction principle to proof uniqueness of solution for ABC and ABR initial value type problems. the author discusses Lyapunov inequality for the ABR boundary value problem. Properties of green function are employed in this paper. Liu et al.,^[39] have utilized Lyapunov-Krasovskii function while dealing with asymptotic stability of fractional neural systems. Li and Wang^[40] introduces a concept of delayed Mittag-Leffler matrix function similar to delayed exponential matrix. With the utilization of mathematical induction and mathematical properties the solution of fractional delay system has been set up. Finite time stability conditions are also achieved. Satriyantara et al.,^[41] investigates a discrete fractional-order predator-prey model and searches for the equilibrium points. Moustafa et al.,^[42], in this article, discusses existence and uniqueness of solution using the very well-known method of

implementation of Lipschitz condition. For ensuring non-negativity and boundedness of solutions of prescribed fractional order system, the standard comparison theorem using Mittag-Leffler function has been used. A suitable positive definite Lyapunov function is considered for establishing GAS of predator-extinction equilibrium point and coexistence equilibrium point. For the numerical simulation of R-M fractional-order system, the generalized Adams-Bashforth-Moulton type predictor-corrector scheme is employed. Baisad and Moonchai^[43] uses Mittag-Leffler function while performing Laplace transform. The authors have introduced local Lipschitz condition to establish existence and uniqueness of solution. Existence of equilibrium points is investigated and their stability is analyzed using the linearization method. In this manuscript, the authors have used the Adams method to solve model the proposed model by the MATLAB software. Li et al.,^[44] discusses existence of equilibrium points and employs variational matrix to analyze their stability. The PECE (predict, evaluate, correct, evaluate) scheme, regarded as a generalization of Adams-Bashforth-Moulton algorithm, is an effective technique to solve fractional-order differential equations. In the numerical analysis segment, the authors use the PECE scheme for finding the numerical solution of system under consideration. This paper addressed by Liang et al.,^[45] provides with the representation of a solution of fractional linear system with pure delay using the fractional delayed matrix sine and cosine and the variation of parameters method. Alidousti and Ghahfarokhi^[46] have incorporated final value theorem in this paper. Classical Routh-Hurwitz criteria comes into play where asymptotic stability of equilibrium points is concerned. The authors have performed numerical simulation based on the fractional Adams-Bashforth-Moulton method. Moustafa et al.,^[47] employs contraction mapping principle. Standard comparison theorem is applied in this paper in order to fetch uniform boundedness of solutions of fractional-order systems. The authors define a positive definite Lyapunov function to establish that the predator-free equilibrium point is GAS. In this process, inequalities involving arithmetic and geometric means are applied. For the numerical simulation of the prescribed fractional-order system, the generalized Adams-Bashforth-Moulton type predictor-corrector scheme is employed. Suryanto et al.,^[48] utilizes comparison theorem, Mittag-Leffler function and generalized Lasalle invariance principle. The authors implemented the predictor corrector scheme developed by Diethelm et al.,^[49] to solve proposed fractional-order model and to perform some numerical simulations.

Wang et al.,^[50] uses Laplace transform while verifying non-negativity conditions of solutions of proposed system. Lipschitz condition is as usual employed for establishing uniqueness of solutions. The authors talk about w -limit point. Routh-Hurwitz criteria is used for examining LAS of equilibrium points and a suitable Lyapunov function is constructed for examining GAS of non-trivial positive equilibrium point. The stability of the positive equilibrium point has been studied using the Lyapunov direct method in this manuscript addressed by Xie et al.,^[51]. The PECE (predict-evaluate and correct-evaluate) method for numerical simulation is applied to confirm the results of this paper. Panja et al.,^[52] utilizes convergence criteria of Mittag-Leffler function for verifying stability of equilibrium point. The author performs Hopf bifurcation analysis. While seeking sufficient condition for the existence and uniqueness of solutions of a fractional-order system, Lipschitz condition is employed. Panigoro et al.,^[53] evaluates variational matrix at the equilibrium points to evaluate stability conditions for equilibrium points. Existence of Hopf bifurcation is examined considering Caputo fractional differential equation. The authors have performed numerical simulation using the predictor-corrector method for fractional-order differential equations. Mondal et al.,^[54] considers Mittag-Leffler function and standard comparison theorem for establishing uniform boundedness and non-negativity conditions. Lipschitz condition is applied to find whether the solution fetched from fractional differential equations is unique. Population free equilibrium point is stated to be unstable using Matignon's conditions. For sensitivity of

parameters the approach of Latin hypercube sampling (LHS) and partial ranked correlation coefficient (PRCC). GAS is established using Lyapunov stability. While dealing with the numerical simulations, the authors have applied the PECE (predict, evaluate, correct, evaluate) method based on Adams-Bashforth-Moulton algorithm in order to analyze the qualitative behaviour of the proposed fractional order system. Panigoro et al.,^[55] applies local Lipschitz condition to determine uniqueness of solutions corresponding to the fractional order system. One-parameter Mittag-Leffler function is applied together with Standard Comparison theorem to determine non-negativity and boundedness of solutions of the model system under consideration. Matignon condition is stated to determine LAS. Existence of forward bifurcation is confirmed from stability conditions of respective equilibrium points. For establishment of GAS, generalized Lasalle invariance principle and construction of suitable positive definite Lyapunov function are employed. Moreover, the solution of the fractional-order model with ABC operator, under consideration, is explored by applying fixed-point theorem. The authors have applied the predictor corrector technique proposed by Diethelm et al.,^[49] to solve the Caputo fractional-order model and the predictor corrector scheme proposed by Baleanu et al.,^[56] to solve the Atangana-Baleanu in Caputo sense model (ABC). Ghanbari et al.,^[57] employs the Atangana-Baleanu derivative (AB) to analyze the effect of this derivative over the two prey and predator model. Mittag-Leffler function and Lipschitz condition are used in this paper. In search for numerical approximate solution, Newton's method has been used efficiently in this paper. Yildiz et al.,^[58] constructs optimality system by defining the modified performance index. To evaluate the time fractional term, the method of integration by parts is employed. The optimality condition acts as a catalyst in order to find solution of fractional optimal control problem. For discretizing state equation, the authors have applied forward Euler method. Again, for discretizing adjoint equation, the authors have used backward Euler method. Gronwall's lemma is applied in this paper. Singh et al.,^[59] analyzes fractional fish model by HATM utilizing inverse Laplace transform. The authors use Picard-Lindeloff scheme to discuss an ES of fractional fish farm model. In pursuing this agenda, the fixed point postulate of Banach space has been employed. Moreover, the paper talks about Picard's X-stability. The solution of fish farm model is determined by employing homotopy analysis transforms method (HATM). Existence and uniqueness of solution are studied through Picard Lindelöf approach. Mohammadi et al.,^[60] have defined a condensing map and have applied Sadovskii's fixed point theorem to determine a fixed point within a subset of a Banach space. The article utilizes endpoint theorem in order to determine unique endpoint. In the present manuscript, the authors are in search of at least one solution of proposed fractional Lio-Cap-Bvp model on $[0, 1]$ owing to which dominated convergence theorem of Lebesgue is put to use. To establish the inclusion version of the fractional Lio-Cap-Bvp, the notion of the approx-endpoint property is put to use. ul Rehman et al.,^[61] in this manuscript, uses local boundedness and Lipschitz condition to establish existence and uniqueness of solutions of proposed fractional order model. The authors have found means of locating the biologically feasible region where the dynamical transmission of the prescribe system shall be analyzed. While dealing with this, Mittag-Leffler function was put to use. The authors have found it very hard to establish stability of equilibrium points of the model under consideration as the fractional derivatives do not abide by Leibnitz rule. Hence, an alternative transformation had been used. Local asymptotic stability of disease-free equilibrium is determined with the use of Routh-Hurwitz criteria. Control parameter is introduced to regulate the spreading of ailment by constructing a suitable objective function. In the final segment, the authors have discussed the approximate solution via numerical methods such as Runge-Kutta method of fourth order (RK4) and Laplace Adomian decomposition method (LADM) for the system taken into account. Bantaojai and Borisut^[62], in this article, employs contraction mapping principle for fetching unique fixed point.

Moreover, Krasnoselskii's fixed point theorem is applied to fetch at least one solution from the proposed BVP. The authors have applied Boyd and Wong fixed point theorem on a Banach space E to determine a unique fixed point in E . In this quest, application of Arzela-Ascoli theorem is worth mentioning. The authors Rahmi et al.,^[63] uses Matignon condition to establish that an equilibrium point is LAS. Existence of Hopf bifurcation is determined analytically. The present article applies the predictor-corrector scheme for fractional-order system developed by Diethelm et al.,^[49]. Ghosh et al.,^[64] have modified the classical Rosenzweig-MacArthur model of prey-predator engagement to Bazykin prey-predator model to indulge with intra-specific competition among the predators. The proposed fractional differential system has a unique solution for any non-negative initial condition which has been proved by providing Lipschitz constant. GAS of axial and interior equilibrium point is established by constructing a positive Lyapunov function. For numerical analysis, the authors have solved the proposed model with the help of two-step Adams-Bashforth-Moulton algorithm for the system of two FODE. Panigoro et al.,^[65] have achieved the discrete-time model by applying the piecewise constant arguments (PWCA) scheme. Moreover, a fixed point is defined and their stability conditions are studied by computing the variational matrix. The proposed article shows that the integral step-size(h) plays an important role in establishing the dynamical behaviour of the prescribed model. Existence of Neimark-Sacker bifurcation has been exhibited numerically in this article. Barman et al.,^[66] investigated Hopf-bifurcation analysis with the discovery of existence of two complex-conjugate eigen values. In the numerical scheme, we have used modified predictor-corrector method based on Adams-Bashforth-Moulton formula through a special module coding FDE12 in the mathematical software MATLAB (2020a version). Yousef et al.,^[67] uses generalized mean value theorem, Mittag-Leffler function for determining non-negativity, boundedness, existence and uniqueness of the solution obtained from the proposed model. Hopf bifurcation is established by evaluating variational matrix. GAS corresponding to predator-free equilibrium point is set up by constructing a positive definite Lyapunov function. In order to carry out numerical analysis to scrutinize the qualitative behaviour of fractional order system taken into account, the authors have employed Adams type predictor-corrector method. Panigoro et al.,^[68] have stated the standard comparison theorem for Caputo fractional-order derivative and generalized LaSalle invariance principle. GAS of predator extinction point is examined using Lyapunov stability. The authors have explored the dynamics of proposed model numerically by performing some numerical simulations using a Caputo fractional-order predictor-corrector scheme developed by Diethelm et al.,^[49]. The influence of conversion efficiency rate of predation and order of the derivative have been investigated. Rahmi et al.,^[69] have examined how biologically well behaved their proposed model is by determining the conditions for existence and uniqueness of solutions using Lipschitz condition. Equilibrium points are determined and their stability conditions are evaluated under strong and weak Allee effect utilizing Routh-Hurwitz criterion. GAS of interior equilibrium point and predator-extinction point are established by defining a suitable Lyapunov function. In the paper addressed by Abbas et al.,^[70] Lipschitz condition helps find conditions for existence and uniqueness of solutions. With the help of limit superior and limit inferior, permanence of the fractional model has been tackled. For convenience, the stated model has been discretized utilizing fractional Adams-Bashforth-Moulton numerical methods.

3.2. Approach on eco-epidemiological models

The local stability of the equilibrium point have been determined by the absolute value of the argument eigen values of the Jacobian matrix equilibrium in this article addressed by Nugraheni et al.,^[71]. In order to evaluate the numerical solutions of a fractional order eco-epidemiological model, the

authors use the Grunwald-Letnikov approximation method. Mondal et al.,^[72] in this present paper, have stated generalized mean value theorem. To investigate how biologically well behaved the proposed system is, the authors have determined conditions for non-negativity and boundedness of the proposed fractional-order system. For performing extensive numerical investigations, both Adams-type predictor corrector method and PECE (predict, evaluate, correct, evaluate) method have been employed. The paper addressed by Moustafa et al.,^[73] investigates the local and global asymptotic stability of all equilibrium points of the prescribed fractional order model by using Matignon's condition and constructing suitable Lyapunov functions respectively. The generalized Adams-Bashforth-Moulton type predictor-corrector scheme is applied in order to determine an approximate solution for the proposed fractional-order system in this manuscript. Kumar et al.,^[74] promotes an iterative scheme to find exact solution of non-linear fractional-order eco-epidemiological system by utilizing Sumudu transform and its inverse property. In the present article, the authors have ensured usage of three kinds of fractional operators: first of all, Caputo fractional operator based on the power-law kernel, second Caputo-Fabrizio fractional operator based on exponential decay law, and lastly Atangana-Baleanu fractional operator based on Mittag-Leffler kernel. The authors have obtained the maximal bifurcation graph of the eco-epidemiological system which is numerically solved by adopting Atangana-Seda (AS) numerical method. Moreover, Newtons polynomial has been put to use. The Adams-Bashforth predictor corrector and a mathematical numerical technique dependent on the Lagrange polynomial has been employed in this manuscript. Mondal et al.,^[75] have discretized the proposed fractional-order model with piecewise constant argument. This paper used the Jury criterion for determining local stability of the discrete fractional-order system. It is noticed that stability of the system depends on both the step-size and fractional order. With the aid of numerous examples, the authors have illustrated the stability of predator free, infection-free, and coexistence equilibrium points. Qi et al.,^[76] analyzes Hopf bifurcation due to feedback delay. The L1 formula is established by a piecewise linear interpolation approximation for the integrand function on each small interval. At the same time, modified Adams-Bashforth-Moulton predictor corrector scheme is applied to find out how the dynamics of the prescribed model works. Ghosh et al.,^[77] uses one-parameter Mittag-Leffler function for establishing non-negativity and boundedness of solutions. As usual, Matignon's condition is utilized to establish stability criterion for fractional-order model system. Moustafa et al.,^[78] provides with the proof of existence of transcritical bifurcation by utilizing Sotomayor's theorem. Threshold parameters (R_{01} and R_{02}) are used to determine the existence conditions of the equilibrium points. Rahmi et al.,^[79] states comparison lemma and Volterra-type Lyapunov function. LAS of concerned equilibrium point is established using Matignon's condition. The present article also mentions and uses generalized LaSalle invariance principle. The authors have demonstrated numerical simulations using Adams-Bashforth-Moulton predictor-corrector method provided by Diethelm et al.,^[49].

4. Literature review: Recent advancements

4.1. Survey on ecological models

Qian et al.,^[8] focused on establishing stability theorems for fractional differential system with Riemann-Liouville derivative, in particular our analysis covers the linear system, the perturbed system and the time-delayed system. Wei et al.,^[9] discussed the properties of the well-known Mittag-Leffler function, and consider the existence and uniqueness of solution of the initial value problem for fractional differential equation involving Riemann-Liouville sequential fractional derivative by using monotone iterative method. El-Sayed^[10] are concerned here with a class of nonlinear fractional-order

differential equations. We study the existence of a unique positive solution, its uniform stability and its global stability at the equilibrium points. The fractional-order logistic equation, replicator (hawk-dove (HD) game) equation, law of mass actions and some other examples will be considered as applications. This study addressed by Chen et al.,^[11] makes the first attempt to apply the Kansa method in the solution of the time fractional diffusion equations, in which the multiquadrics and thin plate spline serve as the radial basis function. In the discretization formulation, the finite difference scheme and the Kansa method are respectively used to discretize time fractional derivative and spatial derivative terms. The numerical solutions of one- and two-dimensional cases are presented and discussed, which agree well with the corresponding analytical solution. Stability of fractional-order nonlinear dynamic systems is studied using Lyapunov direct method with the introductions of Mittag-Leffler stability and generalized Mittag-Leffler stability notions in the paper presented by Li et al.,^[12]. With the definitions of Mittag-Leffler stability and generalized Mittag-Leffler stability proposed, the decaying speed of the Lyapunov function can be more generally characterized which include the exponential stability and power-law stability as special cases. Finally, four worked out examples are provided to illustrate the concepts. Abbas et al.,^[13] studied a fractional differential equation model of the single species multiplicative Allee effect. First the stability of equilibrium points is studied. Further, some sufficient conditions are established ensuring the existence and uniqueness of integral solution. In the last section, several numerical simulations are performed to validate the analytical findings. Xiao^[14] studies the dynamical behaviors of a fractional order Hindmarsh-rose neuronal model. First, based on the stability theory of fractional order systems, some sufficient conditions for the stability and Hopf-type bifurcation are given for such fractional order system. Then, the frequency and amplitude of periodic oscillations are determined by numerical simulations. It has been shown that the frequency of oscillations incurs a small variation with respect to different values of the order, while the amplitude of oscillations gets larger as the order is increased. Abdelouahab et al.,^[15] presents a chaotic fractional-order modified hybrid optical system. Some basic dynamical properties are further investigated by means of Poincare mapping, parameter phase portraits, and the largest Lyapunov exponents. Fractional Hopf bifurcation conditions are proposed; it is found that Hopf bifurcation occurs on the proposed system when the fractional-order varies and passes a sequence of critical values. The chaotic motion is validated by the positive Lyapunov exponent. In this paper stability analysis of fractional-order nonlinear systems is studied by Delavari et al.,^[16]. An extension of Lyapunov direct method for fractional-order systems using Biharis and Bellman-Gronwalls inequality and a proof of comparison theorem for fractional-order systems are proposed. Rana et al.,^[17] formulated the fractional counterpart of the Rosenzweig model and analyze the stability behaviour of a system. It has been concluded that there is a threshold for the memory effect parameter beyond which the Rosenzweig model is stable and may be used as a potential agent to resolve PoE from a new perspective via fractional differential equations. Agarwal et al.,^[18] are interested in the fractional-order form of Chua's system. A discretization process will be applied to obtain its discrete version. Fixed points and their asymptotic stability are investigated. Chaotic attractor, bifurcation and chaos for different values of the fractional-order parameter are discussed. It has been shown that the proposed discretization method is different from other discretization methods, such as predictor-corrector and Euler methods, in the sense that our method is an approximation for the right-hand side of the system under study. Javidi and Nyamoradi^[19] introduced a fractional-order prey-predator model and deals with the mathematical behaviors of the model. The dynamical behavior of the system is investigated from the point of view of local stability. A detailed analysis on the stability of equilibrium has been studied thoroughly. Numerical simulations are presented to illustrate the results. Choi et al.,^[20] introduced the notion of h-stability for fractional differential equations. Then, the

boundedness and h-stability of solutions of Caputo fractional differential systems are investigated by using fractional comparison principle and fractional Lyapunov direct method. Stability analysis of nonlinear fractional differential systems has been an open problem since the 1990s of the last century. Apparently, Lyapunov's second method seems to be invalid for nonlinear fractional differential systems (equations). In this paper, Zhou et al.,^[21] are concerned with this open problem and have solved it partly. Based on Lyapunov's second method, a novel stability criterion for a class of nonlinear fractional differential system is derived. Our result is simple, global and theoretically rigorous. The conditions to guarantee the stability of the nonlinear fractional differential system are convenient for testing. Compared with the stability criteria in the literature, our criterion is straightforward and suitable for application. Several examples are provided to illustrate the applications of our result. Elsadany and Matouk^[22] have studied the dynamical behaviour of fractional-order Lotka-Volterra predator-prey system and its discretized counterpart. It is shown that the discretized system exhibits much richer dynamical behaviors than its corresponding fractional-order form; in the discretized system, many types of bifurcations (transcritical, flip, Neimark-Sacker) and chaos are obtained however the dynamics of fractional-order counterpart is included only stable (unstable) equilibria. The dynamical behaviour of fractional-order Hastings-Powell food chain model is investigated by Matouk et al.,^[23] and a new discretization method of the fractional-order system is introduced. A sufficient condition for existence and uniqueness of the solution of the proposed system is obtained. Local stability of the equilibrium points of the fractional-order system is studied. Furthermore, the necessary and sufficient conditions of stability of the discretized system are also studied. It is shown that the systems fractional parameter has effect on the stability of the discretized system which shows rich variety of dynamical behaviors such as Hopf bifurcation, an attractor crisis and chaotic attractors. Numerical simulations show the tea-cup chaotic attractor of the fractional-order system and the richer dynamical behavior of the corresponding discretized system. In this paper by Rihan et al.,^[24] a fractional dynamical system of predator-prey with Holling type-II functional response and time delay is studied. Local and global stability of existence steady states and Hopf bifurcation with respect to the delay is investigated, with fractional order $0 < \alpha \leq 1$. It is found that Hopf bifurcation occurs when the delay passes through a sequence of critical values. Unconditionally, stable implicit scheme for the numerical simulations of the fractional-order delay differential model is introduced. The numerical simulations show the effectiveness of the numerical method and confirm the theoretical results. The presence of fractional order in the delayed differential model improves the stability of the solutions and enrich the dynamics of the model. In the paper, Caputo and Fabrizio^[25] presented a new definition of fractional derivative with a smooth kernel which takes on two different representations for the temporal and spatial variable. The first works on the time variables; thus, it is suitable to use the Laplace transform. The second definition is related to the spatial variables, by a non-local fractional derivative, for which it is more convenient to work with the Fourier transform. The interest for this new approach with a regular kernel was born from the prospect that there is a class of non-local systems, which have the ability to describe the material heterogeneities and the fluctuations of different scales, which cannot be well described by classical local theories or by fractional models with singular kernel. Losada and Nieto^[26] introduced the fractional integral corresponding to the new concept of fractional derivative recently introduced by Caputo and Fabrizio and we study some related fractional differential equations. Ghaziani et al.,^[27] introduced a fractional order Leslie-Gower prey-predator model, which describes interaction between two populations of prey and predator. Stability and dynamical behaviors of the equilibria of this system are determined. The dynamical behaviour consists of quasi-periodic and limit cycles. Further by numerical solution of the fractional system and numerical simulations, more dynamical behaviour of the model are explored and

established. Ji et al.,^[28] introduced a fractional order two-species cooperative systems with harvesting. By using the Routh-Hurwitz conditions and the Lyapunov method, we provide several sufficient conditions to ensure the stability of the equilibriums for the system. Finally, a numerical example is presented in the paper to demonstrate the validity and feasibility of the theoretical result. Matouk and Elsadany^[29] investigates quantitatively the fractional-order generalized Lotka-Volterra (GLV) model and its discretization. A sufficient condition for existence and uniqueness of the solution of the proposed system is shown. Analytical conditions of the stability of the systems three non-negative steady states are proved. The conditions of the existence of Hopf bifurcation in the fractional-order GLV system are discussed. The necessary conditions for this system to remain chaotic are obtained. Based on the stability theory of fractional-order differential systems, a new control scheme is introduced to stabilize the fractional order GLV system to its steady states. Furthermore, the analytical conditions of stability of the discretized system are also studied. It is shown that the systems fractional parameter has effect on the stability of the discretized system which shows rich variety of dynamical analysis such as bifurcations, an attractor crisis and chaotic attractors. Song et al.,^[30] considered a fractional order delayed predator-prey system with harvesting terms. The discussion is divided into two cases in this paper. Without harvesting, the stability of the model is investigated, as well as some criteria by analyzing the associated characteristic equation is derived. With harvesting, the dynamics of the system is examined from the aspect of local stability and the influence of harvesting to prey and predator is analyzed. Finally, numerical simulations are presented to verify our theoretical results. In addition, using numerical simulations, the effects of fractional order and harvesting terms are explored on dynamic behaviour. The numerical results show that fractional order can affect not only the stability of the system without harvesting terms, but also the switching times from stability to instability and to stability. The harvesting can convert the equilibrium point, the stability and the stability switching times. Recently, Atangana and Baleanu proposed a derivative with fractional order to answer some outstanding questions that were posed by many researchers within the field of fractional calculus. Their derivative has a non-singular and nonlocal kernel. In this paper, Atangana and Koca^[31] presented further relationship of their derivatives with some integral transform operators. New results are presented. This derivative has been applied to a simple nonlinear system. It has been shown in detail, the existence and uniqueness of the system solutions of the fractional system. A chaotic behaviour was obtained which was not obtained by local derivative. Li et al.,^[32] demonstrates the existence of Feigenbaums constants in reverse bifurcation for fractional order Rossler system. First, the numerical algorithm of fractional-order Rossler system is presented. Then, the definition of Feigenbaums constants in reverse bifurcation is provided. Third, in order to observe the effect of fractional-order to Feigenbaums constants in reverse bifurcation, a series of bifurcation diagrams are computed. The Feigenbaums constants in reverse bifurcation are measured and the error percentage in fractional-order Rossler system is presented. The simulation results show that Feigenbaums constants exist in reverse bifurcation for fractional-order Rossler system. Especially, the Feigenbaums constants still exist in the periodic windows. A summary on previous others works about Feigenbaums constants is proposed. This paper draws a conclusion that the constants are universal in both period-doubling bifurcation and reverse bifurcation for both integer and fractional-order system. A stage-structure predator prey model is proposed and analyzed in this paper by Khajanchi^[33] in which predators are divided into juvenile and mature predators using Monod-Haldane-type response function. The dynamical behavior of this system both analytically and numerically is investigated from the view point of stability and bifurcation. We investigate global stability around the interior equilibrium point E by constructing suitable Lyapunov function. Our model simulation indicates that the conversion of prey population to juvenile predators

can destabilize the model system which lead to limit cycle oscillations. We also investigate that the rate of juvenile predators becoming mature predators play an important role to destabilize the model system for the stable coexistence of both the populations. In this paper, presented by Nosrati and Shafiee^[34] a fractional-order singular (FOS) predator-prey model with Holling type II functional response has been introduced, and the mathematical behavior of the model from the aspect of local stability is investigated. Through the fractional calculus and economic theory, a new and more realistic predator-prey model has been extended, and the solvability condition is presented. Besides, numerical simulations are considered to illustrate the effectiveness of the numerical method and confirm the theoretical results to explore the impacts of fractional-order and economic interest on the presented system in biological context. It is found that the presence of fractional-order in the differential model can improve the stability of the solutions and enrich the dynamics of system. In addition, singular models exhibit more complicated dynamics rather than standard models, especially the bifurcation phenomena, which can reveal the instability mechanism of systems. In this paper, addressed by Owolabi and Atangana^[35], pseudo-spectral method have been proposed as an efficient and easy to adapt method for solving the space fractional reaction-diffusion system. A fractional predator-prey system has been studied where the predator has a life history that takes through the immature and mature stages. Sufficient feasible conditions are obtained for the global asymptotic of the equilibrium state of the system. The main advantage of this approach is that it gives a full diagonal representation of the fractional operator, being able to achieve spectral convergence regardless of the fractional power in the problem. Additional advantage is that the application of the proposed method to two and three spatial dimensions requires a straightforward extension to the one-dimensional case. Numerical simulation results of the space fractional reaction-diffusion system, especially in two and three dimensions provide some amazing dynamics when compared to the classical reaction-diffusion equation, and as such consider as a powerful modelling approach for understanding the various aspects of heterogeneity in excitable media. Numerical experiments justify that the results obtained by the proposed method agree well with the theoretical findings. Fractional order dynamical systems admit chaotic solutions and the chaos disappears when the fractional order is reduced below a threshold value which is exhibited in the study of Grigorenko and Grigorenko. Thus, the order of the dynamical system acts as a chaos controlling parameter. Hence it is important to study the fractional order dynamical systems and chaos. Study of fractional order dynamical systems is still in its infancy and many aspects are yet to be explored. In pursuance to this in the present paper, Deshpande et al.,^[36] proved the existence of fractional Hopf bifurcation in case of fractional version of a chaotic system introduced by Bhalekar and Daftardar-Gejji. We numerically explore the (A, B) parameter space and identify the regions in which the system is chaotic. Further we find (global) threshold value of fractional order below which the chaos in the system disappears regardless of values of system parameters A and B. Zoua and He^[37] are concerned with the uniqueness of solutions for the following nonlinear fractional boundary value problem:

$$\begin{aligned}
 D^p x(t) + f(t, x(t)) &= 0, & p \in (2, 3], t \in (0, 1) \\
 x(0) = x'(0) = 0, x(1) &= 0
 \end{aligned}
 \tag{13}$$

where D^p denotes the standard Riemann-Liouville fractional derivative. Our analysis relies on the theory of linear operators and the $\| \cdot \|_e$ norm.

Abdeljawad^[38], in this paper, extended fractional operators with nonsingular Mittag-Leffler kernels, a study initiated recently by Atangana and Baleanu, from order $\alpha \in [0, 1]$ to higher arbitrary order and we formulate their correspondent integral operators. We prove existence and uniqueness theorems for the Caputo (ABC) and Riemann (ABR) type initial value problems by using the Banach contraction

theorem. Then we prove a Lyapunov type inequality for the Riemann type fractional boundary value problems of order $2 < \alpha \leq 3$ in the frame of Mittag-Leffler kernels. Illustrative examples are analyzed and an application as regards the Sturm-Liouville eigenvalue problem in the sense of this fractional calculus is given as well. Liu et al.,^[39] investigated the asymptotical stability of Riemann-Liouville fractional neutral systems. Applying Lyapunov direct method, new sufficient conditions on asymptotical stability are presented in terms of linear matrix inequality (LMI) which can be easily solved. The advantage of our employed method is that one may directly calculate integer-order derivatives of the Lyapunov functions. Finally, two simple examples are given to show that the proposed method is computationally flexible and efficient. Li and Wang^[40] firstly introduced a concept of delayed Mittag-Leffler type matrix function, an extension of Mittag-Leffler matrix function for linear fractional ODEs, which shall help to seek explicit formula of solutions to fractional delay differential equations by using the variation of constants method. Secondly, the finite time stability results are presented by virtue of delayed Mittag-Leffler type matrix. In this paper, a fractional-order predator-prey model with prey refuge and additional food for predator is solved numerically by Satriyantara et al.,^[41]. For that aim, the model is discretized using a piecewise constant argument. The equilibrium points of the discrete fractional-order model are investigated. Numerical simulations are conducted to see the stability of each equilibrium point. The numerical simulations show that stability of the equilibrium points is dependent on the time step. Moustafa et al.,^[42] considered a fractional order Rosenzweig-MacArthur (R-M) model incorporating a prey refuge. The model is constructed and analyzed in detail. The existence, uniqueness, non-negativity and boundedness of the solutions as well as the local and global asymptotic stability of the equilibrium points are studied. Sufficient conditions for the stability and the occurrence of Hopf bifurcation for the fractional order R-M model are demonstrated. The resolution of the paradox of enrichment is investigated. The impact of fractional order and the prey refuge effects on the stability of the system are also studied both theoretically and by using numerical simulations. The Kolmogorov model has been applied to many biological and environmental problems. In this paper, the authors Baisad and Moonchai^[43] are particularly interested in one of its variants, that is, a Gauss-type predator-prey model that includes the Allee effect and Holling type-III functional response. Instead of using classic first order differential equations to formulate the model, fractional order differential equations are utilized. The existence and uniqueness of a nonnegative solution as well as the conditions for the existence of equilibrium points are provided. We then investigate the local stability of the three types of equilibrium points by using the linearization method. The conditions for the existence of a Hopf bifurcation at the positive equilibrium are also presented. To further affirm the theoretical results, numerical simulations for the coexistence equilibrium point are carried out. In this paper, a kind of fractional-order predator-prey (FOPP) model with a constant prey refuge and feedback control is considered by Li et al.,^[44]. By analyzing characteristic equations, detailed discussion with respect to stability of equilibrium points of the considered FOPP model is carried out. Besides, the effects of prey refuge and feedback control are also studied by numerical analysis. The present study reveals that prey refuge and feedback control can be used to adjust the biomass of prey species and predator species such that prey species and predator species finally reach a better state level. Liang et al.,^[45] gave a representation of a solution to the Cauchy problem for a fractional linear system with pure delay. The fractional delayed matrices cosine and sine of a polynomial of degree are determined and some properties are established. Then, the variation of constants method is used to obtain the solution and our results extend those for second order linear system with pure delay. As an application, the representation of a solution is used to obtain a finite time stability result. Alidousti and Ghahfarokhi^[46] considered a fractional delayed predator-prey model with Holling type II functional response which

incorporates prey refuge and diffusion. The conditions of the Hopf bifurcation existence are obtained by analyzing the associated characteristic equation. The influence of fractional order and time delay to control the system is considered. By applying analytic and numerical method, in order to locate all unstable poles and determine the locus crosses the imaginary axis, the conditions under which the positive equilibrium becomes asymptotically stable are derived. Furthermore, the impulsive perturbation of the fractional system is introduced and dynamics of this system is revealed using a numerical scheme. Numerical simulation of the fractional system indicates that the system experiences the process of cycles, period-doubling bifurcation, period-halving bifurcation. Finally, it concludes that the fractional system exhibits periodic solution with shorter period comparing to that of the classical case and the stability domain can be extended under the fractional order. A fractional order prey-predator model with stage structure incorporating a prey refuge is established and analyzed by Moustafa et al.,^[47]. The predation is modelled using a Hollings type II functional response. The existence, uniqueness, non-negativity and boundedness of the solutions of the model is established. In addition to investigating the stability of the equilibrium points, conditions for the stability and Hopf bifurcation are obtained. The impact of fractional order, prey refuge and conversion coefficient on the stability of the fractional-order system are theoretically and numerically investigated. Suryanto et al.,^[48] considered a model of predator-prey interaction at fractional-order where the predation obeys the ratio-dependent functional response and the prey is linearly harvested. For the proposed model, the existence, uniqueness, non-negativity and boundedness of the solutions are shown. Conditions for the existence of all possible equilibrium points and their stability criteria, both locally and globally, are also investigated. The local stability conditions are derived using the Matignon's theorem, while the global stability is proven by formulating an appropriate Lyapunov function. The occurrence of Hopf bifurcation around the interior point is also shown analytically. At the end, the predictor corrector scheme is implemented to perform some numerical simulations. Wang et al.,^[50] considered a delayed generalized fractional-order prey-predator model with interspecific competition. The existence of the nontrivial positive equilibrium is discussed, and some sufficient conditions for global asymptotic stability of the equilibrium are developed. Meanwhile, the existence of Hopf bifurcation is discussed by choosing time delay as the bifurcation parameter. A fractional-order diffused prey-predator model with prey refuges is considered by Xie et al.,^[51]. The existence, uniqueness, non-negativity and boundedness of the solutions for the model are proved. Moreover, some sufficient conditions are given to ensure the existence and uniform asymptotic stability of the equilibrium point of the studied system by using Lyapunov method and graph theoretic approach.

In this paper, a fractional order predator-prey mathematical model has been developed by Panja^[52]. In this model, the concept of intraguild predation has been introduced. It is assumed that intermediate predator consumes only prey and intraguild predator consumes prey as well as intermediate predator. Also, intraspecific competition of prey, intermediate predator and intraguild predator has been considered. Uniqueness, boundedness and non-negativity of solutions of our proposed model have been discussed. Different possible equilibrium points are determined and the stability of our proposed model around these equilibrium points has been studied. It is found that fractional order system can show some interesting dynamics of our proposed model. Panigoro et al.,^[53] focused on studying the effects of continuous predator threshold harvesting policy on the dynamical behavior of a fractional-order Gause-type predator-prey system. This policy is applied to ensure that harvesting does not occur when the population density is less than a specified threshold. The dynamical analysis is done to study the local stability of equilibrium points and the existence of Hopf bifurcation. Plants send signals through releasing volatile organic compounds (VOCs), to attract beneficial natural carnivorous insects as

reinforcement against harmful herbivorous insects which are responsible for hampering the growth of plants. In this work, Mondal et al.,^[54] explored the dynamical behavior of a volatile mediated plant-herbivore-carnivore system with fractional order differential equations. Basic results on the existence, uniqueness, non-negativity and boundedness of the solutions, local and global stability of coexistence equilibrium points and limit cycles emerging through Hopf bifurcation are investigated. Stability behavior around coexistence equilibrium point changes with varying fractional order (α). Also, the existence of Hopf bifurcation is established by considering the fractional order α as a bifurcation parameter. Moreover, the attraction factor of plant volatile to carnivore and predation rate for plant-herbivore are responsible for changing the system dynamics. The harvesting management is developed to protect the biological resources from over-exploitation such as harvesting and trapping. Panigoro et al.,^[55] considered a predator-prey interaction that follows the fractional-order Rosenzweig-MacArthur model where the predator is harvested obeying a threshold harvesting policy (THP). The THP is applied to maintain the existence of the population in the prey-predator mechanism. We first consider the Rosenzweig-MacArthur model using the Caputo fractional-order derivative (that is, the operator with the power-law kernel) and perform some dynamical analysis such as the existence and uniqueness, non-negativity, boundedness, local stability, global stability, and the existence of Hopf bifurcation. The same model is reconsidered involving the Atangana-Baleanu fractional derivative with the Mittag-Leffler kernel in the Caputo sense (ABC). The existence and uniqueness of the solution of the model with ABC operator are established. The dynamics of the model is explored with both fractional derivative operators numerically and confirm the theoretical findings. In particular, it is shown that models with both Caputo operator and ABC operator undergo a Hopf bifurcation that can be controlled by the conversion rate of consumed prey into the predator birth rate or by the order of fractional derivative. However, the bifurcation point of the model with the Caputo operator is different from that of the model with the ABC operator. Ghanbari et al.,^[57] studied a dynamic system that models the interactions between two densities of immature and mature prey and predator populations. In the model, prey population is divided into two populations, including mature prey and immature prey. Another feature of the model is that predator depends on mature prey only and it followed by Crowley-Martin type functional response. Moreover, the fractional operator used in this model as derivative is of the Atangana-Baleanu AB type. Using this kind of fractional derivative causes the results to depend on the fractional order of the derivative. The addition of the concept of memory to the model is another highlight of using this type of derivative for the biological model. This helps the model to apply all the essential information of the phenomenon from the beginning to the desired time in the calculations. Existence and uniqueness of solutions to the fractional model are also investigated in this manuscript. The numerical method used in the article is also one of the most efficient patterns in solving problems with fractional derivatives. Using this effective method makes the results very consistent with what we actually expect to happen. Many simulations have been carried out to investigate the effect of parameters in the model on its overall behavior. Numerical results show the impressive performance of the fractional operator on the dynamic behavior of the considered predator-prey model. This efficient fractional operator can also be tested in the structure of other existing biological models. This paper addressed by Yildiz et al.,^[58] deals with a new formulation of time fractional optimal control problems governed by Caputo-Fabrizio (CF) fractional derivative. The optimality system for this problem is derived, which contains the forward and backward fractional differential equations in the sense of CF. These equations are then expressed in terms of Volterra integrals and also solved by a new numerical scheme based on approximating the Volterra integrals. The linear rate of convergence for this method is also justified theoretically. Three illustrative examples are presented to show the performance of this

method. These examples also test the contribution of using CF derivative for dynamical constraints and we observe the efficiency of this new approach compared to the classical version of fractional operators. Singh et al.,^[59] analyzed the dynamical behavior of fish farm model related to Atangana-Baleanu derivative of arbitrary order. The model is constituted with the group of non-linear differential equations having nutrients, fish and mussel. We have included discrete kind gestational delay of fish. The solution of fish farm model is determined by employing homotopy analysis transforms method (HATM). Existence of and uniqueness of solution are studied through Picard Lindelöf approach. The influence of order of new non-integer order derivative on nutrients, fish and mussel is discussed. The complete study reveals that the outer food supplies manage the behavior of the model. Moreover, to show the outcomes of the study, some numerical results are demonstrated through graphs. Mohammadi et al.,^[60] first investigated the existence of solutions for a new fractional boundary value problem in the Liouville Caputo setting with mixed integro-derivative boundary conditions. To do this, Krakowski's measure of noncompactness and Sadoski's fixed point theorem are the proposed tools to reach this aim. In the sequel, the continuous dependence of solutions on parameters are discussed by means of the generalized Gronwall inequality. Moreover, an inclusion version of the given boundary problem is considered in which its existence are studied by means of the endpoint theory. ul Rehman et al.,^[61] presented a paper that deals with a fractional-order mathematical epidemic model of malaria transmission accompanied by temporary immunity and relapse. The model is revised by using Caputo fractional operator for the index of memory. The utilization of temporary immunity and the possibility of relapse is also recommended. The theory of locally bounded and Lipschitz is employed to inspect the existence and uniqueness of the solution of the malaria model. It is shown that temporary immunity has a great effect on the dynamical transmission of host and vector populations. The stability analysis of these equilibrium points for fractional-order derivative α and basic reproduction number R_0 is discussed. The model will exhibit a Hopf-type bifurcation. The two control variables are introduced in this model to decrease the number of populations. Mandatory conditions for the control problem are produced. Two types of numerical method via Laplace Adomian decomposition and Runge-Kutta of fourth order for simulating the proposed model with fractional-order derivative are presented. Bantaojai and Borisut^[62] studied and investigated the following implicit Caputo fractional derivative and non-local fractional integral conditions of the form:

$$\begin{aligned} {}^cD_{0+}^q u(t) &= f(t, u(t)), {}^cD_{0+}^q u(t), t \in [0, T] \\ u(0) &= \eta, u(T) = {}_{RL}I_{0+}^p u(\kappa), \kappa \in (0, T) \end{aligned}$$

where $1 < q \leq 2$, $0 < p \leq 1$, $\eta \in R$, $D_{0+}^q u(t)$ is the Caputo fractional derivative of order; q , ${}_{RL}I_{0+}^p$ is the Riemann-Liouville fractional integral of order p and $f: [0, T] \times R \times R \rightarrow R$ is continuous function by using Krasnoselskii's fixed point theorem and Boyd-Wong non-linear contraction. Also, the existence and uniqueness of this problem has been studied. Rahmi et al.,^[63] investigates the dynamics of a fractional-order Leslie-Gower model with Allee effect in predator. Firstly, the existing condition and local stability of all possible equilibrium points are determined. The model has four equilibrium points, namely both prey and predator extinction point, the prey extinction point, the predator extinction point, and the interior point. Furthermore, it has been shown that dynamic change around the interior point due to the changing of the order of the fractional derivative, namely the Hopf bifurcation. At the end, some numerical simulations are demonstrated to illustrate the dynamics of the model. Heere, the local stability, the occurrence of Hopf bifurcation, and the impact of the Allee effect to the prey and predator densities are shown numerically. Ghosh et al.,^[64] deals with a system of two fractional order differential equations for prey-predator interaction with intra-specific competition among predators. The fractional

order differential equation is considered in the sense of Caputo derivative and the derivation of the fractional order model is explained in terms of memory effect on population growth. Detailed mathematical results are provided to establish the positiveness, existence uniqueness and boundedness of the solutions. The conditions required for local asymptotic stability of various equilibrium points and global stability of coexistence equilibrium are derived along with the Hopf-bifurcation condition for coexistence equilibrium. The effect of memory on the system dynamics through the shift of Hopf-bifurcation threshold is demonstrated with the help of exhaustive numerical simulations. This study also reveals the effect of memory-based growth on global bifurcation threshold. Panigoro et al.,^[65] studied the dynamical behaviors of a discrete-time fractional-order Rosenzweig-MacArthur model with prey refuge. The piecewise constant arguments scheme is applied to obtain the discrete time model. All possible fixed points and their existence conditions are investigated as well as the local behaviour of nearby solutions in various contingencies. Numerical simulations such as the time series, phase portraits, and bifurcation diagrams are portrayed. Three types of bifurcations are shown numerically namely the forward, the period-doubling, and Neimark-Sacker bifurcations. Some phase portraits are depicted to justify the occurrence of those bifurcations. In this article, addressed by Barman et al.,^[66] a predator-prey model has been evolved in the form of a system of fractional order differential equations incorporating two important factors, namely, fear factor and prey refuge factor. Here, the fractional calculus has been taken into consideration to investigate the dynamical behaviour of the solutions of the proposed model system as the changes in life cycle of prey species are of memory bound. Biological validation and well-posedness such as positivity and boundedness of solutions of the model system have been proved analytically. Stability analysis of all the feasible equilibrium points of the model system has been performed in a systematic way. Some important dynamical features of the model system (such as transition of stability of the system) have been demonstrated through rigorous numerical simulation. It is observed that our proposed model system experiences Hopf-bifurcation around the interior equilibrium point with respect to both the parameters f and m_1 , which are linked with amount of predator induced fear and rate of prey refuge, respectively. The system dynamics is more likely to be stable in the framework of fractional order derivative in comparison to integer-order derivative. The high amount of predator induced fear f and prey refuge rate m_1 are independently capable to make the system dynamics to be stable in integer order model system. On the other hand, the dynamics of the model system shifts towards the stability from its unstable behaviour when we continuously reduce the order of the model system; especially under the scenario of low level of predator induced fear and prey refuge rate. Thus, our comprehensive mathematical findings reveal the fact that fading memory can play a contributory role towards stable coexistence of the predator-prey system whereas strong memory of the species deteriorates the stable coexistence of the model system. Yousef et al.,^[67] have formulated a fractional-order predator-prey system with fear effect, where the death rate of the prey population is predator density-dependent. Generally, most of the ecological study considers the direct killing of prey in a predator's presence, but they ignore the effect of predator's presence on prey. Some experimental studies confirmed that fear affects the reproduction rate of the prey population, but a few studies are there, which conclude that fear also affects the death rate of the prey population. Thus, the main aim in this work is to investigate the influence of the fear effect produced by a predator on the reproduction rate and death rate of the prey population. At first, the existence, uniqueness, non-negativity, and boundedness of the considered model solutions are proved. After that, detailed analysis of different equilibria and their stability criteria based on some conditions are shown where the global stability of the interior equilibrium point has also been investigated. Besides that, the existence of Hopf bifurcation and the persistence of the system is derived. It has been also observed how fractional-order derivative

also has an impact on the proposed system. Finally, some numerical simulation is performed to validate the findings. Lyapunov function gives a major contribution in studying the dynamics of biological models. Panigoro et al.,^[68] in this paper, studied the global stability of a fractional-order Gause-type predator-prey model with threshold harvesting policy in predator by using Lyapunov function. The present work is initiated by investigating the existence and uniqueness of solution, and then the non-negativity and boundedness of solution is proved. Furthermore, it has been shown that the model has four equilibrium points, where the non-trivial equilibrium points are conditionally globally asymptotically stable. At the end, some numerical simulations are demonstrated by using the generalized Adam-Basforth-Moulton method to support theoretical results. Numerically it has been shown that the conversion efficiency rate of predation and the order of the derivative influence the dynamics of the model. Also, the existence of forward and Hopf bifurcation are presented numerically driven by conversion efficiency rate of predation and the order of the derivative respectively. Rahmi et al.,^[69] proposed a modified Leslie-Gower predator-prey model with Beddington-DeAngelis functional response and double Allee effect in the growth rate of a predator population. In order to consider memory effect on the proposed model, we employ the Caputo fractional-order derivative. We investigate the dynamic behaviors of the proposed model for both strong and weak Allee effect cases. The existence, uniqueness, non-negativity, and boundedness of the solution are discussed. Then, the existing condition and local stability analysis of all possible equilibrium points are determined. Necessary conditions for the existence of the Hopf bifurcation driven by the order of the fractional derivative are also determined analytically. Furthermore, by choosing a suitable Lyapunov function, the sufficient conditions to ensure the global asymptotic stability for the predator extinction point for the strong Allee effect case as well as for the prey extinction point and the interior point for the weak Allee effect case are derived. Finally, numerical simulations are shown to confirm the theoretical results and can explore more dynamical behaviour of the system, such as the bi-stability and forward bifurcation. Abbas et al.,^[70] presented a fractional model of interacting phytoplankton species in which one species produces chemical which is stimulatory in nature to the other species. We study existence, uniqueness, permanence, persistence and stability of the solution. A new method has been introduced to prove permanence and persistence, which may be applicable to several ecological models of fractional order. At the end we propose a discretization method and perform some numerical simulations to validate our analytical findings.

4.2. Survey on eco-epidemiological models

Nugraheni et al.,^[71] discussed a fractional order eco-epidemiological model. The aim of considering the fractional order is to describe effect of time memory in the growth rate of the three populations. Analytically the dynamical behaviour of the model is investigated and then simulated using the Grunwald-Letnikov approximation to support our analytical results. It is found that the model has five equilibrium points, namely the origin, the survival of susceptible prey, the predator-free equilibrium, the infected prey free equilibrium and the interior equilibrium. Numerical simulations show that the order of fractional derivative affects the behavior of solutions. Mondal et al.,^[72] introduced fractional order into an eco-epidemiological model, where predator consumes disproportionately large number of infected preys following type II response function. We prove different mathematical results like existence, uniqueness, non-negativity and boundedness of the solutions of fractional order system. We also prove the local and global stability of different equilibrium points of the system. A fractional-order eco-epidemiological model with disease in the prey population is formulated and analyzed by Moustafa et al.,^[73]. Mathematical analysis and numerical simulations are

performed to clarify the characteristics of the proposed fractional-order model. The existence, uniqueness, non-negativity and boundedness of the solutions are proved. The local and global asymptotic stability of all equilibrium points are investigated. Finally, numerical simulations are conducted to illustrate the analytical results. The occurrence of Hopf bifurcations and transcritical bifurcations for the fractional-order eco-epidemiological model are demonstrated. It is observed that the fractional order has a stabilization effect and it may help to control the coexistence between susceptible prey, infected prey and predator populations. Kumar et al.,^[74] explored the dynamical aspects of the arbitrary order eco-epidemiological model. The authors have employed power law kernel, exponential decay kernel, and generalized Mittag-Leffler kernel functions for treatment of arbitrary order eco-epidemiological model where the considered eco-epidemiological model is a non-linear dynamical system with three population species. The uniqueness and existence of the solutions by adopting the fixed point theory. The authors have examined the possibility for finding new dynamical phase portraits with singular and non-singular arbitrary order operator and demonstrate the dynamical phase portraits at various values of arbitrary order. Mondal et al.,^[75] have taken initiative to understand the dynamics of a three-dimensional discrete fractional-order eco-epidemiological model with Holling type II functional response. At first, a fractional-order predator-prey-parasite system with piecewise constant arguments has been discretized. Analytical conditions for the local stability of different fixed points have been determined. the critical value of the step-size, where the switching of stability occurs, decreases as the order of the fractional derivative decreases. Numerical simulation results claim that the discrete fractional-order system may also exhibit complex dynamics, like chaos, for higher step-size. Qi et al.,^[76] investigates a delayed fractional eco-epidemiological model with extended feedback controller. The stability and Hopf bifurcation of the system in the controller-free and controller states are discussed with regarding the digestion delay as a parameter, respectively, and the existence conditions of periodic solution are given. Ghosh et al.,^[77] have scrutinized a fractional-order eco-epidemiological model incorporating fear, treatment, and hunting cooperation effects to examine memory effect in the ecological system by means of Caputo-type fractional-order derivative. The present paper studies the behaviour of different equilibrium points with memory effect. The proposed system undergoes through Hopf bifurcation with respect to the memory parameter as the bi- furcation parameter. In the numerical results, it appears that the system is exhibiting a stable behavior from a period or chaotic nature with the increase in the memory effect. The system also exhibits two transcritical bifurcations with respect to the growth rate of the prey. At low values of prey growth, all species go to extinction, at moderate values of prey's growth, only preys (susceptible and infected) can survive, and at higher values of preys growth, all species survive simultaneously. Moustafa et al.,^[78] examines global stability criterion of a fractional-order eco-epidemiological model with infected predator and harvesting. Rahmi et al.,^[79] proposes a fractional-order modified Leslie-Gower predator-prey model with ailment and the double Allee effect in predator population. Numerically, it has been interpreted that there exists risk of extinction for the predator with a strong Allee effect which is higher when the spread of ailment is relatively high.

5. Conclusion

Fractional models are a widely studied topic in biomathematics. In this study, we presented novel fractional derivatives to the literature that are closely connected to the predator-prey models in history. We have reviewed some recent works focusing on the implementation of fractional order in Caputo sense, Riemann-Liouville sense, Caputo-Fabrizio sense, Atangana-Baleanu sense and the likes. Fractional models are regarded as realistic models from which rich dynamics of systems can be studied

and analyzed.

Conflict of interest

The authors declare no conflict of interest.

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