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Article

A new infinite series identities involving modified Bessel functions and Hermite polynomials

Mikhail Trifonov

Sechenov Institute of Evolutionary Physiology and Biochemistry of the Russian Academy of Sciences, Saint-Petersburg 194223, Russia; mtrifonov@mail.ru

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Abstract: Some new infinite series identities and their analogues—integral representations involving the modified Bessel functions of the first kind and the Hermite polynomials were obtained using a probabilistic approach.

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1. Introduction

It is widely known that the probability density function (PDF) of a random variable is not always expressed in a closed form in terms of elementary functions, except for the special cases of Cauchy, Lévy, and Gaussian distributions [1]. It means that in practice we often deal with a PDF that can only be expressed as an infinite series of some functions. Sometimes the expression is not unique, i.e. we have different functional series with the same sum for the very same PDF. The one best-known example here is the PDF of the instantaneous value of a single sinusoidal signal additively combined with Gaussian noise. Rice [2] was the first who derived an analytical expression for the relevant PDF in terms of the derivatives of the error function $\varphi^{(2n)}(\cdot)$

$$p(x) = \frac{1}{\sigma}\sum_{n=0}^{\infty}\frac{\psi^n}{2^n(n!)^2}\varphi^{(2n)}\left(\frac{x}{\sigma}\right),$$

where

$$\varphi^{(m)}(y) = \frac{1}{\sqrt{2\pi}}\frac{d^m}{dy^m}\exp\left(-\frac{y^2}{2}\right).$$

Throughout the paper, $\psi = A^2/2\sigma^2$ means the signal-to-noise ratio (SNR), where A and σ^2 are the amplitude of the sinusoidal signal and the noise variance respectively. An equivalent representation for this PDF $p(x)$ in terms of infinite series of the confluent hypergeometric function of the first kind ${}_1F_1(\cdot;\cdot;\cdot)$

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma}\sum_{n=0}^{\infty}\frac{(-x^2/2\sigma^2)^n}{n!}{}_1F_1(n+1/2;1;-\psi)$$

was derived by Middleton [3]. The same PDF $p(x)$ can also be expressed in terms of the probabilist's Hermite polynomials $He_{2n}(x)$ [4]

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{\psi^n}{2^n (n!)^2} He_{2n}(x/\sigma) \exp\left(-\frac{x^2}{2\sigma^2}\right),$$

where

$$He_{2n}(x) = \exp(x^2/2) \frac{d^{2n} \exp(-x^2/2)}{dx^{2n}}.$$

Recently, we derived a new equivalent representation for this PDF $p(x)$ in terms of products of Gaussian and infinite series of the modified Bessel functions of the first kind [5]

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \sum_{n=0}^{\infty} (-1)^n \beta_n I_{2n}\left(\sqrt{2\psi}\frac{x}{\sigma}\right) \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad -\infty < x < \infty \tag{1}$$

$$\beta_n = k_n \exp(-\psi/2) I_n(\psi/2) > 0, \quad \sum_{n=0}^{\infty} \beta_n = 1 \tag{2}$$

$I_\nu(\cdot)$ is a modified Bessel function of the first kind of integer order ν, and k_n is the Neumann factor defined by

$$k_n = \begin{cases} 1, & n = 0, \\ 2, & n \geq 1. \end{cases}$$

It is interesting to note that the infinite series in Equation (1) represents a convex combination of $\xi_n(z, \psi) = (-1)^n I_{2n}(\sqrt{2\psi} r/\sigma)$ since all coefficients β_n are non-negative and sum to 1. From a statistical point of view, the coefficient β_n may be treated as the probability of some discrete variable Φ taking value $\xi_n(z, \psi)$. So, the series in Equation (1) is the expectation (or mean) of Φ. It is quite correctly here to generalize such statistical interpretation of infinite series involving $\varphi^{(2n)}(\cdot)$, $He_{2n}(\cdot)$, and ${}_1F_1(\cdot;\cdot;\cdot)$ as well, since all these functions can be appropriately scaled to get the sum of the new non-negative expansion coefficients to 1. It means that all infinite series mentioned above have deep statistical sense and theoretically can be converted one to another via a sequence of identity preserving transformations.

Since all PDFs mentioned above include an infinite series involving different functions, the primary concern in practical use of these PDF is that of convergence. In practice, the preferred choice of a concrete PDF representation depends on the standardized amplitude range and signal-to-noise ratio (SNR) [4]. As we pointed out earlier, the advantage of our representation Equation (1) for the PDF $p(x)$ is that it converges much faster than the series consisting of Hermite polynomials or the confluent hypergeometric functions (see [5] for details).

The lack of a unique representation for the same PDF $p(x)$ makes the optimal choice among alternatives difficult, but at the same time can provide deeper insights into the relations between different functions involved in a relevant infinite series defining the PDF. We shall restrict our consideration here to the study of various identities between infinite series involving the modified Bessel functions of the first kind and the

Hermite polynomials that may prove interesting at least in the field of mathematical analysis, probability theory and classical statistics.

2. One-Parameter infinite series identities

Here we start with the first non-trivial identity between infinite series involving the modified Bessel functions of the first kind and the Hermite polynomials. Originally, it was given in our previous work [5] and holds for any real variable $z \in (-\infty, \infty)$ and for any real parameter $\psi \geqslant 0$ which in the partial case of a single sine wave in the additive Gaussian noise has a clear physical sense presenting a signal-to-noise ratio. To study this identity in more detail we slightly transform the expression for the PDF $p(x)$ in terms of the probabilist's Hermite polynomials given above by changing $He_{2n}(x)$ to the physicist's Hermite polynomials $H_{2n}(y)$ of order $2n$ according to well-known relation

$$H_{2n}(y) = 2^n He_{2n}(\sqrt{2}y) = \exp(y^2)\frac{d^{2n}\exp(-y^2)}{dy^{2n}}.$$

As the we get an equivalent representation for the same PDF $p(x)$ rewritten in the form

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{\psi^n}{2^{2n}(n!)^2}H_{2n}(x/\sqrt{2}\sigma)\exp\left(-\frac{x^2}{2\sigma^2}\right) \tag{3}$$

Setting for simplicity $u = u(x) = x/(\sqrt{2}\sigma)$ in Equations (1) and (3), we are ready to state an explicit expression for the infinite series identity involving the modified Bessel functions of the first kind and the physicist's Hermite polynomials in Theorem 1 below and prove it directly via a sequence of identity preserving transformations.

Theorem 1. *For any real variable $u \in (-\infty, \infty)$ and for any nonnegative parameter ψ, the following identity holds*

$$\sum_{n=0}^{\infty}(-1)^n\beta_n I_{2n}(2\sqrt{\psi}u) = \sum_{n=0}^{\infty}\frac{\psi^n}{2^{2n}(n!)^2}H_{2n}(u) \tag{4}$$

Proof. Here, we use the following representation [6] (p. 252, Eq. 10.32.3) for the modified Bessel function of the first kind of integer order $2n$

$$I_{2n}(2\sqrt{\psi}u) = \frac{1}{\pi}\int_0^{\pi}\exp(2\sqrt{\psi}u\cos(\theta))\cos(2n\theta)d\theta.$$

to rewrite the infinite series of the modified Bessel functions in Equation (4) as

$$\sum_{n=0}^{\infty}(-1)^n\beta_n I_{2n}(2\sqrt{\psi}u)$$

$$= \frac{1}{\pi}\exp(-\psi/2)\int_0^{\pi}\exp(2\sqrt{\psi}u\cos(\theta))\sum_{n=0}^{\infty}(-1)^n k_n I_n(\psi/2)\cos(2n\theta)d\theta.$$

Using the Jacobi-Anger expansion (see e.g. [6] (p. 254, Eq. 10.35.2)) we have

$$\sum_{n=0}^{\infty}(-1)^n k_n I_n(\psi/2)\cos(2n\theta) = \exp(-(\psi/2)\cos(2\theta)) = \exp(\psi/2)\exp(-\psi\cos^2(\theta)).$$

Hence, we get

$$\sum_{n=0}^{\infty}(-1)^n \beta_n I_{2n}(2\sqrt{\psi}u) = \frac{1}{\pi}\int_0^{\pi}\exp(2\sqrt{\psi}u\cos(\theta) - \psi\cos^2(\theta))d\theta.$$

From the Hermite generating function [7] (p. 106, Eq. 5.5.7), we have

$$\exp(2\sqrt{\psi}u\cos(\theta) - \psi\cos^2(\theta)) = \sum_{n=0}^{\infty}H_n(u)\frac{\psi^{n/2}cos^n(\theta)}{n!}.$$

Now, by taking into account the following representation for the integral over $\theta \in [0,\pi]$ [8] (p. 395, Eq. 3.621.3) we obtain

$$\frac{1}{\pi}\sum_{n=0}^{\infty}H_n(u)\frac{\psi^{n/2}}{n!}\int_0^{\pi}cos^n(\theta)d\theta = \frac{2}{\pi}\sum_{n=0}^{\infty}H_{2n}(u)\frac{\psi^n}{(2n)!}\int_0^{\pi/2}cos^{2n}(\theta)d\theta$$

$$= \sum_{n=0}^{\infty}H_{2n}(u)\frac{\psi^n}{(2n)!}\frac{(2n-1)!!}{(2n)!!} = \sum_{n=0}^{\infty}\frac{\psi^n}{2^{2n}(n!)^2}H_{2n}(u).$$

The theorem is therefore proved. □

We will now propose and prove a new infinite series and integral-series identities.

Theorem 2. *For any real variable $u \in (-\infty,\infty)$ and for any nonnegative parameter ψ and coefficients β_n defined by Equation (2), we have the following integral representation of infinite series:*

$$\sum_{n=0}^{\infty}(-1)^n \beta_n I_{2n}(2\sqrt{\psi}u) = \frac{2}{\sqrt{\pi}}\exp(-\psi)\int_0^{\infty}\exp(-y^2)I_0\left(2\sqrt{\psi}\sqrt{u^2+y^2}\right)dy \tag{5}$$

Proof. Starting with the identity [9] (p. 306, Eq. 2.15.5.2)

$$\exp(\psi/2)I_n(\psi/2) = \frac{2}{\sqrt{\pi}}\int_0^{\infty}\exp(-y^2)I_{2n}(2\sqrt{\psi}y)dy,$$

we obtain that infinite series of the modified Bessel functions in Equation (4) can be expressed as follows:

$$\sum_{n=0}^{\infty}(-1)^n \beta_n I_{2n}(2\sqrt{\psi}u)$$

$$= \frac{2}{\sqrt{\pi}}\exp(-\psi)\int_0^{\infty}\exp(-y^2)\sum_{n=0}^{\infty}(-1)^n k_n I_{2n}(2\sqrt{\psi}u)I_{2n}(2\sqrt{\psi}y)dy.$$

By setting $a = \pi/2$ in the identity [9] (p. 696, Eq. 5.8.7.3), we obtain the equality

$$\sum_{n=0}^{\infty}(-1)^n k_n I_{2n}(2\sqrt{\psi}u)I_{2n}(2\sqrt{\psi}y) = I_0\left(2\sqrt{\psi}\sqrt{u^2+y^2}\right),$$

which proves the integral representation (5). □

The new identity (5) allows one to obtain the following integral representation for the PDF of the instantaneous value x of a single sinusoidal signal in additive white Gaussian noise given in [5]

$$p(x) = \frac{\sqrt{2}}{\pi\sigma}\exp(-\psi)\int_0^{\infty}\exp(-(u^2+y^2))I_0\left(2\sqrt{\psi}\sqrt{u^2+y^2}\right)dy \tag{6}$$

Note that the integral of Equation (6) is slightly similar in appearance with the Marcum Q-function [10].

We recall that the expressions (4) and (5) given above represent the one-parameter infinite series identity and integral-series identity, respectively. Below, we consider similar identities extended with two nonnegative parameters.

3. Two-Parameter infinite series identities

Theorem 3. *For any real variable $u \in (-\infty, \infty)$ and any nonnegative parameters ψ_1 and ψ_2 the following infinite series identity involving the Hermite polynomials is valid*

$$\sum_{n=0}^{\infty}\frac{\psi_1^n}{2^{2n}(n!)^2}\sum_{m=0}^{\infty}\frac{\psi_2^m}{2^{2m}(m!)^2}H_{2(n+m)}(u) = \sum_{m=0}^{\infty}\frac{|\psi_1-\psi_2|^m}{2^{2m}(m!)^2}P_m\Big(\frac{\psi_1+\psi_2}{|\psi_1-\psi_2|}\Big)H_{2m}(u) \tag{7}$$

where $P_m(y)$ is the Legendre polynomial of degree m is defined by the Rodrigues formula

$$P_m(y) = \frac{1}{2^m m!}\frac{d^m}{dy^m}(y^2-1)^m,$$

and $|\cdot|$ is an absolute value symbol.

Proof. First, we show that infinite series in the left side of identity (7) is associated with the PDF $p(x)$ of the instantaneous amplitude x of the sum of two sinusoidal signals and Gaussian noise, i.e. has in some sense a probabilistic origin. We start with the following standard representation of $p(x)$ (see, e.g. [11])

$$p(x) = \frac{1}{\pi}\int_0^{\infty}J_0(A_1\omega)J_0(A_2\omega)\exp(-\sigma^2\omega^2/2)\cos(x\omega)d\omega \tag{8}$$

where $J_0(\cdot)$ is the zeroth-order Bessel function of the first kind, A_1, A_2 are the amplitudes of the sinusoidal signals respectively, and σ^2 is the noise variance. Using the well-known series representation formula for the $J_0(x)$ [12] (p. 40, Eq. 8)

$$J_0(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\Big(\frac{x}{2}\Big)^{2n},$$

we obtain

$$p(x)=\frac{\sqrt{2}}{\pi\sigma}\sum_{n=0}^{\infty}\frac{(-1)^n\psi_1^n}{(n!)^2}\sum_{m=0}^{\infty}\frac{(-1)^m\psi_2^m}{(m!)^2}\int_0^{\infty}\rho^{2(n+m)}\exp(-\rho^2)\cos(\sqrt{2}x\rho/\sigma)d\rho \tag{9}$$

where $\psi_1=A_1^2/2\sigma^2$ and $\psi_2=A_2^2/2\sigma^2$. Using the integral representation [8] (p. 503, Eq. 3.952.9) this equation reduces to

$$p(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp(-u^2)\sum_{n=0}^{\infty}\frac{\psi_1^n}{2^{2n}(n!)^2}\sum_{m=0}^{\infty}\frac{\psi_2^m}{2^{2m}(m!)^2}H_{2(n+m)}(u) \tag{10}$$

Suppose now that $p(x)$ can also be represented as follows

$$p(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp(-u^2)\sum_{p=0}^{\infty}c_pH_{2p}(u) \tag{11}$$

Multiplying the right sides of Equations (10) and (11) by $H_{2k}(u)$ and integrating over the variable $x\in(-\infty,\infty)$, using the orthogonality of the Hermite polynomials [7] (p. 105, Eq. 5.5.1), we find that expansion coefficient c_k is obtained from the following sequence of transformations

$$\begin{aligned}c_k&=\frac{1}{2^{2k}(2k)!}\sum_{n=0}^{\infty}\frac{\psi_1^n}{2^{2n}(n!)^2}\sum_{m=0}^{\infty}\frac{\psi_2^m}{2^{2m}(m!)^2}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-t^2)H_{2(n+m)}(t)H_{2k}(t)dx\\&=\sum_{n=0}^{\infty}\frac{\psi_1^n}{2^{2n}(n!)^2}\sum_{m=0}^{\infty}\frac{\psi_2^m}{2^{2m}(m!)^2}\delta_{k,m+n}=\sum_{n=0}^{k}\frac{\psi_1^n}{2^{2n}(n!)^2}\frac{\psi_2^{k-n}}{2^{2(k-n)}((k-n)!)^2}\\&=\frac{\psi_2^k}{2^{2k}(k!)^2}\sum_{n=0}^{k}\frac{\psi_1^n}{\psi_2^n}\frac{(k!)^2}{(n!)^2((k-n)!)^2}=\frac{|\psi_1-\psi_2|^k}{2^{2k}(k!)^2}P_k\Big(\frac{\psi_1+\psi_2}{|\psi_1-\psi_2|}\Big)\end{aligned} \tag{12}$$

The theorem is thus proved. □

Corollary 1. *An alternative identity for Equation (7) involving the modified Bessel functions of the first kind is as follows:*

$$\sum_{m=0}^{\infty}\frac{|\psi_1-\psi_2|^m}{2^{2m}(m!)^2}P_m\Big(\frac{\psi_1+\psi_2}{|\psi_1-\psi_2|}\Big)H_{2m}(u)=\frac{1}{\pi}\sum_{m=0}^{\infty}(-1)^m\int_0^{\pi}\gamma_m(t)I_{2m}\big(2u\sqrt{\Psi(t)}\big)dt, \tag{13}$$

where

$$\Psi(t)=\psi_1+\psi_2+2\sqrt{\psi_1\psi_2}\cos(t),$$

and

$$\gamma_m(t)=k_m\exp(-\Psi(t)/2)I_m(\Psi(t)/2).$$

Proof. Note that according to the identity [8] (p. 405, Eq. 3.661.3) we have

$$|\psi_1-\psi_2|^mP_m\Big(\frac{\psi_1+\psi_2}{|\psi_1-\psi_2|}\Big)=\frac{1}{\pi}\int_0^{\pi}(\psi_1+\psi_2+2\sqrt{\psi_1\psi_2}\cos(t))^mdt.$$

As a result, the infinite series in the left side of Equation (13) takes the following form

$$\frac{1}{\pi}\int_0^\pi \sum_{m=0}^{\infty} \frac{(\Psi(t))^m}{2^{2m}(m!)^2} H_{2m}(u)dt.$$

Applying the identity (4) to this result yields Equation (13). The corollary is thus proved. □

4. Concluding remarks

In the present work, we have derived a new set of identities involving the infinite series of the modified Bessel functions of the first kind and the Hermite polynomials, and provided their integral representations. All proposed identities were analytically proved and then verified using MATLAB software.

The new identities show that there are non-trivial relations between the modified Bessel functions and the Hermite polynomials being unknown before. And this fact needs future analysis.

Taking into account that the Hermite polynomials $H_{2n}(u)$ are connected to the generalized Laguerre polynomials $L_n^{(-1/2)}(u^2)$ as following [7] (p. 106, Eq. 5.6.1):

$$H_{2n}(u) = (-1)^n 2^{2n} n! L_n^{(-1/2)}(u^2),$$

one can represent all the above identities involving the $H_{2n}(u)$ in terms of $L_n^{(-1/2)}(u^2)$

$$\sum_{n=0}^{\infty}(-1)^n \beta_n I_{2n}(2\sqrt{\psi}u) = \sum_{n=0}^{\infty} \frac{(-1)^n \psi^n}{n!} L_n^{(-1/2)}(u^2), \tag{14}$$

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(-1)^n \psi_1^n}{n!} \sum_{m=0}^{\infty} \frac{(-1)^m \psi_2^m}{m!} L_{n+m}^{(-1/2)}(u^2) \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m |\psi_1 - \psi_2|^m}{m!} P_m\Big(\frac{\psi_1+\psi_2}{|\psi_1-\psi_2|}\Big) L_m^{(-1/2)}(u^2) \\ &= \frac{1}{\pi}\sum_{m=0}^{\infty}(-1)^m \int_0^\pi \gamma_m(t) I_{2m}\big(2u\sqrt{\Psi(t)}\big)dt. \end{aligned} \tag{15}$$

According to [11] it is more convenient, principally for numerical calculations, to use the Laguerre polynomials rather than the Hermite polynomials.

By analogy with the orthonormal Hermite functions $h_n(x)$ [7] (p. 190) we suggest to define the normalized modified Bessel-Gauss (mBG) function in one-dimension as follows:

$$mBG_n(x) = \frac{1}{\sqrt{2\pi}\sigma} \frac{\exp(-\psi/2)}{I_n(\psi/2)} I_{2n}(2\sqrt{\psi}u) \exp(-u^2), \tag{16}$$

where $u = u(x) = x/(\sqrt{2}\sigma)$, as before. As opposed to $h_n(x)$ all $mBG_n(x)$ functions are not mutually orthogonal as it follows from [9] (p. 322, Eq. 2.15.20.10). Note that all $mBG_n(x)$ functions are nonnegative everywhere, and their integrals from $-\infty$ to ∞ are equal to 1 (see, e.g. [9] (p. 306, Eq. 2.15.5.2)). Therefore, they form an overcomplete family of individual probability densities which are not always strongly localized near the origin. As a result, the PDF given in [5] has the form of alternating infinite series of non-negative $mBG_n(x)$ functions

$$p(x) = \sum_{n=0}^{\infty}(-1)^n k_n I_n^2(\psi/2) mBG_n(x). \tag{17}$$

It is worth noting that the summation [9] (p. 696, Eq. 5.8.7.3) gives

$$\sum_{n=0}^{\infty}(-1)^n k_n I_n^2(\psi/2) = 1.$$

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