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On the Halin Turán number of short cycles

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ABSTRACT: A Halin graph is a graph constructed by embedding a tree with no vertex of degree two in the plane and then adding a cycle to join the tree's leaves. The Halin Turán number of a graph F, denoted as $\mathrm{ex}_{\mathcal{H}}(n, F)$, is the maximum number of edges in an n-vertex Halin graph. In this paper, we give the exact value of $\mathrm{ex}_{\mathcal{H}}(n, C_4)$, where C_4 is a cycle of length 4. We also pose a conjecture for the Halin Turán number of longer cycles.

KEYWORDS: halin graphs; turán number; halin turán number

1. Introduction

Let F be a fixed graph. A graph G is called F*-free* if it contains no isomorphic copy of F as a subgraph. For the graph F and a positive integer n, the *Turán number* of F, denoted by $\mathrm{ex}(n, F)$, is the maximum number of edges in an n-vertex F-free graph, i.e.,

$$\mathrm{ex}(n, F) = \max\{e(G) : G \text{ is an } n\text{-vertex } F\text{-free graph}\}.$$

One of the classical results in extremal graph theory is the Turán's Theorem[1], which gives the exact value $\mathrm{ex}(n, K_r)$, where K_r is an r-vertex complete graph. This result is the generalization of the Mantel's Theorem[2] for the case of K_3. A major breakthrough in the study of Turán number of graphs came in 1966, with the proof of the famous theorem by Erdős, Stone and Simonovits[3, 4]. They determined an asymptotic value of the Turán number of any fixed non-bipartite graph F. In particular, they proved $\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F)-1}\right)\binom{n}{2} + o(n^2)$, where $\chi(F)$ is the chromatic number of F. Since these results, researchers have been interested in working on the Turán number of class of bipartite (degenerate) graphs and extremal problems in a particular family of graphs. In 2016, Dowden[5] initiated the study of Turán-type problems in the family of planar graphs.

Definition 1. *Let F be a fixed graph and n be a positive integer. The planar Turán number of F, denoted by $\mathrm{ex}_{\mathcal{P}}(n, F)$, is the maximum number of edges an n-vertex F-free planar graph contains, i.e.,*

$$\mathrm{ex}_{\mathcal{P}}(n, F) = \max\{e(G) : G \textit{ is an } n\textit{-vertex } F\textit{-free planar graph}\}.$$

Dowden[5] determined sharp upper bounds of $\mathrm{ex}_{\mathcal{P}}(n, C_4)$ and $\mathrm{ex}_{\mathcal{P}}(n, C_5)$, where C_k is a cycle of length k.

Theorem 1.[5] *(1) For $n \geq 4$,*

$$\mathrm{ex}_{\mathcal{P}}(n, C_4) \leq \frac{15(n-2)}{7}.$$

(2) For $n \geq 11$,

$$\mathrm{ex}_{\mathcal{P}}(n, C_5) \leq \frac{12n-33}{5}.$$

Extending the Dowden's result, Lan, Shi and Song[6] obtained an upper bound for $\mathrm{ex}_{\mathcal{P}}(n, C_6)$, and later Ghosh, Győri, Martin, Paulos and Xiao[7], improved the bound and gave a sharp upper bound with some interesting constructions realizing their bound. However $\mathrm{ex}_{\mathcal{P}}(n, C_k)$ is still open for general k. We refer[8–11] for a quick survey and conjectures on planar Turán numbers of graphs.

Theorem 1.[7] *For all $n \geq 18$,*

$$\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq \frac{5}{2}n - 7.$$

Recently, Fang and Zhai[12] initiated the study of Turán numbers in the family of outerplanar numbers. For a positive integer n and fixed graph F, the outerplanar Turán number of F, denoted by $\mathrm{ex}_{\mathcal{OP}}(n, F)$, is the maximum number of edges in an n-vertex outerplanar graph containing no isomorphic copy of F as a subgraph. They completely determined the outerplanar Turán numbers of cycles and paths.

In this paper, we initiate the study of Turán number of cycles in the family of Halin graphs. A *Halin graph* H is constructed as follows: Start with a tree T in which each non-leaf has degree at least 3, i.e., every non-leaf of T is with degree at least 3. Embed the tree in the plane in a planar fashion and then add new edges to form a cycle C containing all the leaves of T in such a way that the resulting graph H is planar. We write $H = T \cup C$, and we call T and C respectively as *characteristic tree* and *outer cycle* of the Halin graph H.

Halin graphs were studied by Halin[13]. A Halin graph has at least four vertices. The wheel graph, W_n, is an example of Halin graph with the characteristic tree being a star on n leaves. Halin graphs are, edge-minimal and 3-connected[14]. Every edge of a Halin graph is part of some Hamiltonian cycle[15].

Definition 2. *Let n be a positive integer and F be a fixed graph. The Halin Turán number of F, denoted by $\mathrm{ex}_{\mathcal{H}}(n, F)$, is the maximum number of edges in an n-vertex F-free Halin graph, i.e.,*

$$\mathrm{ex}_{\mathcal{H}}(n, F) = \max\{e(H) : H \textit{ is an } n\textit{-vertex } F\textit{-free Halin graph}\}.$$

Bondy and Lovasz[16] have shown that Halin graphs are almost pancyclic. More precisely, they showed that if a Halin graph H on n vertices does not have any vertex of degree three in its characteristic tree, then it has all cycles of length ℓ, where, $3 \leq \ell \leq n$. If the characteristic tree contains a vertex of degree three, then cycles of all lengths will still be there with a possible exception of an even-length cycle. In a different study, He and Liu explored the maximum count of short paths in a Halin graph, as discussed in[17].

The almost pancyclic property of Halin graphs makes them interesting from a theoretical perspective, as it implies that these graphs are highly connected and can be used to model a wide variety of complex systems and phenomena. As a result, much research in this area focuses on developing efficient algorithms and techniques for analyzing the structure and properties of Halin graphs.

Concerning cycles, it is still interesting to study and distinguish the extremal graph structures and the Halin Turán number of cycles of even length. In this paper, we determine the exact value of the Halin Turán number of the 4-cycle, and later we pose our conjecture for longer cycles. The following theorem states our main result.

Theorem 3. *For $n \geq 16$,*

$$\mathrm{ex}_{\mathcal{H}}(n, C_4) = \begin{cases} \frac{5}{3}(n-1), & 3|(n-1), \\ \frac{5}{3}(n-2)+1, & 3|(n-2), \\ \frac{5}{3}(n-3)+3, & 3|(n-3). \end{cases}$$

The following notations and terminologies are needed. Let G be a graph. We denote the vertex and the edge sets of G by $V(G)$ and $E(G)$ respectively. The number of vertices and edges in G respectively are denoted by $v(G)$ and $e(G)$. For a vertex v in G, the degree of v is denoted by $d_G(v)$. We may omit the subscript if the underlying graph is clear. The set of all vertices in G which are adjacent to v is denoted as $N_G(v)$ or simply $N(v)$ when the underlying graph is clear. For the sake of simplicity, we use the terms k-cycle and k-path to mean a cycle of length k and a path of length k respectively. We denote a k-cycle with vertices $v_1, v_2, \ldots, v_k$ in sequential order by $(v_1, v_2, \ldots, v_k, v_1)$. We denote a k-path with vertices $v_0, v_2, \ldots, v_k$ in sequential order by $(v_0, v_1, \ldots, v_k)$. A (u, v)-path is a path with end vertices u and v. Given a k-path $(v_0, v_1, \ldots, v_k)$, we may describe v_1 and v_{k-1} as *semi-pendant* vertices of the path. For a plane graph G, the length of a cycle C in G is denoted by $|C|$. Similarly, the size of a face F in G is denoted by $|F|$.

Let H be a Halin graph and T be its characteristic tree. A non-leaf $v \in V(T)$ is an *interior vertex* if every vertex in $N_T(v)$ is not a leaf. A non-leaf $u \in V(T)$ is a *branching vertex* if it has at most one non-leaf in $N_T(u)$. A *semi-branching vertex* $w \in V(T)$ is a non-leaf that is neither an interior nor a branching vertex. Sometimes we may call a leaf in T a *pendant vertex*.

2. Proof of Theorem 3

The following lemmas and observations are important to complete the proof of the theorem.

Lemma 1. *Let H be a C_4-free Halin graph and T be its characteristic tree. For a longest path L in T, each semi-pendant vertex of L is a branching vertex and is adjacent to only two leaves.*

Proof. Let $L = (v_0, v_1, v_2, \ldots, v_k)$. Since L is a longest path, v_{k-1} can not be adjacent to a non-leaf vertex except v_{k-2}. Moreover, from the definition of a Halin graph, $d_T(v_{k-1}) \geq 3$. Thus $N_T(v_{k-1}) \backslash \{v_{k-2}\}$ contains leaves. If $N_T(v_{k-1}) \backslash \{v_{k-2}\}$ contains three leaves, say u_1, u_2, and u_3 in sequential order in counterclockwise direction, then H contains a 4-cycle, namely $(v_{k-1}, u_1, u_2, u_3, v_{k-1})$, and hence a contradiction. □

Lemma 2. *Let $H = T \cup C$ be a Halin graph, and u_1 and u_2 be leaves in T such that $u_1u_2 \in E(C)$. Let F be the bounded face incident to u_1u_2. If $\mathcal{C}$ is a cycle in H containing u_1u_2 we have, $|\mathcal{C}| \geq |F|$.*

Proof. Let the boundary cycle of F be $(u_1, u_2, u_3, \ldots, u_k, u_1)$. Denote $R = \{u_3, u_4, \ldots, u_k\}$. Each vertex in R is not a leaf in T, since H is a Halin graph and the degree of each vertex is at least three. For each vertex $u \in R$, there is a unique leaf u' in T such that we have a (u, u')-path with the set of interior vertices disjoint from R. We may call u' s as *child-pendant* vertices of u. Any (u_1, u_2)-path other than the edge u_1u_2 must contain either u or some child-pendant vertex u' for each $u \in R$. This implies $|\mathcal{C}| \geq |F|$. □

Lemma 3. *Let H be a Halin graph with a characteristic tree T. Let $e = uv \in E(T)$ such that both u and v are non-leaf in T. If F_1 and F_2 are the two bounded faces incident to e, then for a cycle $\mathcal{C}$ in T containing e, then $|\mathcal{C}| \geq \min\{|F_1|, |F_2|\}$.*

Proof. Since u and v are non-leaf and H is a Halin graph, then $d_T(u), d_T(v) \geq 3$. Therefore, we have vertices $u_1, u_2 \in N(u)$ and $v_1, v_2 \in N(v)$ such that (u_1, u, v, v_1) and (u_2, u, v, v_2) are paths incident to F_1 and

F_2 respectively. Moreover, we have vertices $u_1',\ v_1'$ and $u_2',\ v_2'$, which are leaves in T such that $e_1 = u_1'v_1'$ and $e_2 = u_2'v_2'$ are edges incident to F_1 and F_2 respectively. Notice that, u_1' can be u_1 and v_1' can be v_1, and similarly for u_2' and v_2' with u_2 and v_2. Clearly, $\mathcal{C}$ contains either e_1 or e_2, but not both. If $\mathcal{C}$ contains e_1, then by Lemma 2, $|\mathcal{C}| \geq |F_1|$. Moreover if $\mathcal{C}$ contains e_2, the $|\mathcal{C}| \geq |F_2|$. Therefore, $|C| \geq \min\{|F_1|,\ |F_2|\}$. □

Lemma 4. *Let H be an n-vertex C_4-free Halin graph with the characteristic tree T. If T contains a semi-branching vertex of degree at least 4, then there is an $(n-1)$-vertex C_4-free Halin graph H' such that $e(H) = e(H') + 2$.*

Proof. Let C be the outer cycle of H. Let $v \in V(T)$, with $d_H(v) \geq 4$, be a semi-branching vertex and $u \in N_T(v)$ be a leaf. Let the path $(u_1,\ u,\ u_2)$ be the portion of C in the clockwise direction and denote F_1 and F_2 as faces in H incident to the paths $(v,\ u,\ u_1)$ and $(v,\ u,\ u_2)$ respectively. It can be seen that either $|F_1|$ and $|F_2|$ is at least 5. Indeed, since H is a C_4-free graph, no face is of size 4. On the other hand if $|F_1| = |F_2| = 3$, then $(v,\ u_1,\ u,\ u_2,\ v)$ is in H and this contradicts the C_4-free assumption of H. Now obtain the graph H' by deleting u and joining the vertices u_1 and u_2 with an edge. H' is a Halin graph with characteristic tree $T' = T - u$, as $d_{T'}(v) \geq 3$ and $d_{T'}(w) = d_T(w)$ for every $w \in V(T)\backslash\{v\}$. Let C' be the characteristic tree and the outer cycle of H'. H' is C_4-free as the bounded face, say F, incident to the edge u_1u_2 is of size at least 5, and by Lemma 2 the boundary cycle of F is the smallest cycle containing u_1u_2. □

Lemma 5. *Let H be an n-vertex C_4-free Halin graph with characteristic tree T. Let $(u,\ v,\ w)$ be a path in T such that v is a semi-branching vertex with $d_T(v) = 3$. If the bounded face incident to the path is with size at least 6, then there is an $(n-2)$-vertex C_4-free Halin graph H' such that $e(H) = e(H') + 3$.*

Proof. Let $v' \in N(v)$ and $F_1,\ F_2$ and F_3 as the faces incident to the paths $(u,\ v,\ w)$, $(u,\ v,\ v')$ and $(v',\ v,\ w)$ respectively. By assumption $|F_1| \geq 6$. Since H is C_4-free and v is a semi-branching vertex, then $|F_2|,\ |F_3| \geq 5$. Denote u' and w' as the leaves in T such that $v'u'$ is incident to F_2 and $v'w'$ is incident to the face F_3. Let H' be a graph obtained from H by deleting v and adding the edges uw and $u'w'$. It can be checked that H' is an $(n-2)$-vertex Halin graphs, with the two faces incident to uw with size at least 5 and at least 6, and hence by Lemma 2 H' contains no 4-cycle and $e(H) = e(H') + 3$. □

Lemma 6. *Let H be an n-vertex C_4-free Halin graph with characteristic tree T. Let $e \in E(T)$ such that its end vertices are non-leaf in T. If the two faces incident to e are with size at least 6, then there is an $(n-1)$-vertex C_4-free Halin graph, H', such that $e(H) = e(H') + 1$.*

Proof. Denote $H = T \cup C$, where C is the outer cycle of H. Let $e = vu$ and F_1 and F_2 be the two bounded faces in H incident to e. u and v by assumption are non-leaf, and hence $d_T(u) \geq 3$ and $d_T(v) \geq 3$. Let T' be the graph obtained after contracting e in T. Clearly, T' an $(n-1)$-vertex tree and a leaf in T' is a leaf in T. Moreover, for each non-leaf vertex $w \in V(T'),\ d_{T'}(w) \geq 3$. Therefore by contracting e in H we get a Halin graph $H' = T' \cup C$.

Since $|F_1|,\ |F_2| \geq 6$, then by Lemma 3, for each cycle $\mathcal{C}$ containing e we have $|\mathcal{C}| \geq 6$. Hence, by contracting e, every cycle in H' is with no 4-cycle. Therefor, H' is an $(n-1)$-vertex C_4-free Halin graph. This completes the proof of Lemma 6. □

Lemma 7. *For $n \geq 16$, we have*

$$\mathrm{ex}_{\mathcal{H}}(n, C_4) \geq \begin{cases} \frac{5}{3}(n-1), & 3|(n-1), \\ \frac{5}{3}(n-2)+1, & 3|(n-2), \\ \frac{5}{3}(n-3)+3, & 3|(n-3). \end{cases}$$

Proof. We give extremal constructions to verify the bounds. First, we give constructions of the characteristic tree of the Halin graph, when $n = 16,\ 17$, and $n = 18$. For the sake of simplicity, we may call the trees as *base-tree* and denote them by $T_{16},\ T_{17}$ and T_{18}. Denote also the corresponding Halin graphs by $H_{16},\ H_{17}$ and H_{18} respectively. It is easy to see the Halin graphs are C_4-free (see **Figure 1**).

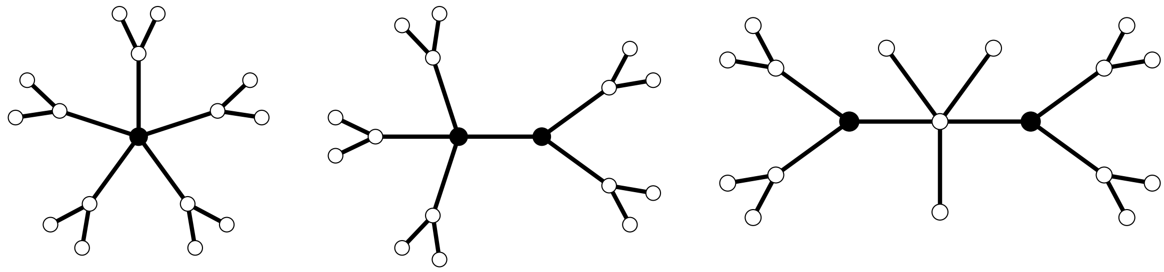

Figure 1. Characteristic trees of Halin graphs on 16, 17, and 18 vertices.

Now let $n \geq 19$. We define an n-vertex Halin graphs $H^n_{16},\ H^n_{17}$ and H^n_{18} based on the base-trees $T_{16},\ T_{17}$ and T_{18} as follows. The star $K_{1,3}$, which is shown in **Figure 2**, is an important component in describing the constructions. For simplicity reasons, we call it *star*. Notice the dark-spotted vertices in both the base-trees and the star.

For $n \equiv 0 \pmod 3$, the Halin graph H^n_{18} is obtained by having $\frac{n-18}{3}$ copies of the star and identifying any of the dark-spotted vertices of T_{18} and the dark-spotted vertex of the star. Similarly, when $n \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$, we respectively get H^n_{16} and H^n_{17} by having $\frac{n-16}{3}$ and $\frac{n-17}{3}$ copies of the star and identifying the dark-spotted vertices of the corresponding base-trees and the star.

It is easy to see that the Halin graphs, $H^n_{16},\ H^n_{17}$ and H^n_{18} are C_4-free. Moreover it is easy to calculate and check that $e(H^n_{16}) = \frac{5}{3}(n-1),\ e(H^n_{17}) = \frac{5}{3}(n-2)+1$ and $e(H^n_{18}) = \frac{5}{3}(n-3)+3$. Therefore, for $n \equiv 0 \pmod 3$, $n \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$, we have $\mathrm{ex}_{\mathcal{H}}(n, C_4) \geq e(H^n_{18}),\ \mathrm{ex}_{\mathcal{H}}(n, C_4) \geq e(H^n_{16})$, and $\mathrm{ex}_{\mathcal{H}}(n, C_4) \geq e(H^n_{17})$ respectively. □

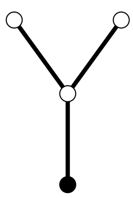

Figure 2. Star $K_{1,3}$.

Observation 1. *Let H be an n-vertex C_4-fee Halin graph with a characteristic tree T. Let $L = (v_0,\ v_1,\ v_2,\ \ldots,\ v_{k-2},\ v_{k-1},\ v_k)$ be a longest path in T.* From Lemma 1, both v_1 and v_{k-1} are branching vertices, and each of them is adjacent to two leaves. Denote the leaf, other than v_0, adjacent to v_1 by v'_0. Denote also the leaf, other than v_k, adjacent to v_{k-1} by v'_k. Since each non-leaf in T is with a degree at least 3, there must be a vertex, say u, adjacent to v_2 such that either both v_0v_1 and v_2u or both $v_1v'_0$ and v_2u are incidents to the same bounded face in H. Without loss of generality assume the latter case holds. It can be seen that u can not be a leaf in T. Otherwise, $(v_1,\ v_2,\ u,\ v'_0,\ v_1)$ is a 4-cycle in H, which is a contradiction. Hence u is non-leaf in the characteristic tree. Therefore, $d_T(u) \geq 3$. From the assumption that L is of maximum length in T, u is adjacent to exactly two leaves, and say u_1 and u_2. For a similar argument, v_{k-2} is adjacent to a non-leaf w, which is adjacent to two leaves w_1 and w_2, see **Figure 3**. When $k \geq 5$,

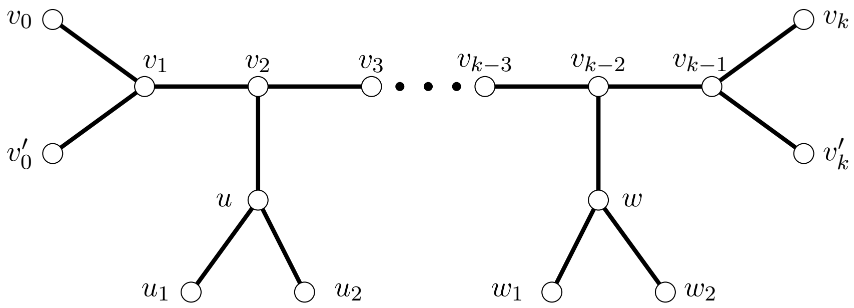

Figure 3. Distribution of vertices on a longest path of a Halin graph.

$S = \{v_0, v'_0, v_1, v_2, u, u_1, u_2, v_k, v'_k, v_{k-1}, v_{k-2}, w, w_1, w_2\}$ gives the 14 labeled vertices in T. If $k = 4$, v_2 and v_{k-2} are identical vertices and the stars attached at v_2 and v_{k-2} could be identical.

Lemma 8. *Let H be an n-vertex C_4-free Halin graph, where $n \geq 19$. Then*

$$\mathrm{ex}_{\mathcal{H}}(n, C_4) \leq \begin{cases} \frac{5}{3}(n-1), & 3|(n-1), \\ \frac{5}{3}(n-2)+1, & 3|(n-2), \\ \frac{5}{3}(n-3)+3, & 3|(n-3). \end{cases}$$

Proof. Our proof relies on induction on the number of vertices. The base cases are shown in the upcoming section. Let $L = (v_0,\ v_1,\ v_2,\ \ldots,\ v_k)$ be a longest path in T. It is easy to check that $k \geq 4$. Next, we prove the following sequence of lemmas as part of the proof. □

Claim 1. *If $k = 4$, then $3|(n-1)$ and $e(H) = \frac{5}{3}(n-1)$.*

Proof. From observation 1, $S = \{v_0, v'_0, v_1, v_2, u, u_1, u_2, v_3, v_4, v'_4\}$. For each vertex $v \in V(T)\backslash S$ and is incident to L, $v \in N(v_2)$. Moreover, v is not a leaf in T. Indeed, suppose for contradiction v is a leaf. Since L is the longest path, then the two faces incident to the edge v_2v are with size either 3 or 4. The latter, can not happen as H is C_4-free. Thus we may assume both faces are with size three. Hence we get two triangles sharing the same edge v_2v. However, this also results in a 4-cycle, which is a contradiction. Hence, each vertex in T adjacent to v_2 is a non-leaf. Since L is a longest path, each vertex adjacent to v_2 is a branching vertex. That means the vertex is adjacent to two pendant vertices. Therefore, H is obtained by identifying the dark-spotted vertex of $\frac{n-7}{3}$ copies of stars with v_2. It can be checked that $e(H) = \frac{5}{3}(n-1)$, and this completes the proof of Claim 1. □

Claim 2. *If $k = 5$, then $3|(n-2)$ and $e(H) = \frac{5}{3}(n-2)+1$.*

Proof. From Observation 1, $S = \{v_0, v'_0, v_1, v_2, u, u_1, u_2, v_3, w, w_1, w_2, v_4, v_5, v'_5\}$. We verify that each vertex

$v \in V(T)\backslash S$ incident to v_2 or v_3 is a branching vertex. Indeed, without loss of generality assume $v \in N(v_2)$. Since L is a longest path in T, the faces incident to v_2v and located on its left side must be either a 3-face or a 4-face. The latter can not happen, as H is a C_4-free graph. Thus we may assume the face is a 3-face and let the leaf forming the 3-face be v', i.e., the 3-face is $(v_2,\ v,\ v',\ v_2)$. For the same reason, we have a leaf v'' adjacent to v_2 such that $(v_2,\ v',\ v'',\ v_2)$ is the 3-face incident to the edge $v'v$. However this results a 4-cycle $(v_2,\ v,\ v',\ v'', v_2)$, which is a contradiction. Therefore, each vertex in $V(T)\backslash S$ adjacent to v_2 or v_3 is a branching vertex. This implies, H is obtained by identifying the black-spotted vertex of $\frac{n-8}{3}$ copies of the star to either v_2 or v_3. It can be checked that $e(H) = \frac{5}{3}(n-2)+1$. This completes the proof of Claim 2. □

Claim 3. *For $k \geq 6$, then H meets either the conditions of Lemma 4 or the conditions of Lemma 5 or the conditions of Lemma 6.*

Proof. Consider the longest path $L = (v_0,\ v_1, v_2,\ v_4, \ldots,\ v_k)$. As L is a longest path, v_1 is a branching vertex and hence it is adjacent to two leaves where v_0 is one of the two vertices. Let the other vertex be v_0'. From the degree condition of Halin graph, $d_T(v_2) \geq 3$. Moreover, every vertex adjacent to v_2 is not a leaf. Since again L is a longest path, each vertex adjacent to v_2 must be a branching vertex. If v_3 is a semi-branching vertex of degree at least 4, then H satisfies the condition of Lemma 4 and we are done. So we may assume that v_3 is not a semi-branching vertex or a semi-branching vertex with $d_T(v_3) = 3$. In the former case, the edge v_2v_3 is an edge with the property that its end vertices are non-leaf and the two faces incident to the edge are with size at least 6, and hence H satisfies the conditions of Lemma 6. In the latter case, since L is a longest path in T, v_2 is not a semi-branching vertex. Hence, the path $(v_2,\ v_3,\ v_4)$ is with a size of at least 6, and hence H meets conditions of Lemma 5. This completes the proof of Claim 3. □

Notice that we finish the proof of Lemma 8 if $3|(n-3)$ or $3|(n-1)$. Indeed, if conditions of Lemma 4 or Lemma 6 happen, then $e(H) \leq e(H') + 2$, where H' is an $(n-1)$-vertex C_4-free Halin graph. If $3|(n-1)$, then by induction we have $e(H) = e(H') + 2 \leq \left(\frac{5}{3}\left[(n-1)-3\right]+3\right)+2 = \frac{5}{3}(n-1)$ and we are done. On the other hand if $3|(n-3)$, then by induction we have, $e(H) = e(H') + 2 \leq \left(\frac{5}{3}\left[(n-1)-2\right]+1\right)+2 = \frac{5}{3}(n-3)+3$ and we are done. On the other hand, suppose conditions of Lemma 5 meet by H. In this case, $e(H) = e(H')+3$, where H' is an $(n-2)$-vertex C_4-free Halin graph. If $3|(n-1)$, then $e(H) = e(H')+3 \leq \left(\frac{5}{3}\left[(n-2)-2\right]+1\right)+3 \leq \frac{5}{3}(n-1)$. If $3|(n-3)$, then $e(H) = e(H')+3 \leq \frac{5}{3}\left[(n-2)-1\right]+3 = \frac{5}{3}(n-3)+3$, and we are again done by induction.

Next, we give our argument on how we finish the proof when $k \geq 6$ and $3|(n-2)$. Since H is a Halin graph and v_{k-3} is a non-leaf vertex, $d_T(v_{k-3}) \geq 3$.

If $d_T(v_{k-3}) = 3$, then there is a vertex, say x such that $x \in N_T(v_{k-3})$. If x is a leaf in T, then it can be seen that the path $(v_{k-4},\ v_{k-3},\ v_{k-2})$ is incident to a face of size at six. Then by Lemma 5 we have an $(n-2)$-vertex C_4-free Halin graph H' such that $e(H) = e(H')+3$. Thus, by induction, $e(H) = e(H')+3 \leq \left(\frac{5}{3}\left[(n-2)-3\right]+3\right)+3 = \frac{5}{3}(n-2)+1$, and we are done by induction. On the other hand, if x is not a leaf in T, then again it can be seen that the two bounded faces incident to the edge $v_{k-3}v_{k-2}$ are with size at least 6 and hence by Lemma 6 we have an $(n-1)$-vertex C_4-free Halin graph H' such that $e(H) = e(H')+1$. This implies by induction $e(H) = e(H')+1 \leq \frac{5}{3}\left[(n-1)-1\right]+1 = \frac{5}{3}(n-2)+1$ and we are done by induction.

Now we may assume that $d_T(v_{k-3}) \geq 4$. From Observation 1, we have a branching vertex $w \in N_T(v_{k-2})$ such that the path $(v_{k-1},\ v_{k-2},\ w)$ is incident to a bounded face in H. Since L is a longest path in T, every vertex $N_T(v_{k-2})\backslash\{v_{k-3}\}$ is a branching vertex. Let F_1 and F_2 be the two bounded faces incident to the edge

$v_{k-3}v_{k-2}$.

Notice that we have a unique pair of leaves incident to each bounded face in H. Denote v_1^u and w_1^u be the leaves such that the edge $v_1^u w_1^u$ is incident to F_1 and w_1^u is a leaf adjacent to the vertex in $N(v_{k-2})\backslash\{v_{k-3}\}$. Similarly denote v_1^l and w_1^l be leaves such that the edge $v_1^l w_1^l$ is incident to F_2 and w_1^l is a leaf adjacent to the vertex in $N(v_{k-2})\backslash\{v_{k-3}\}$. Notice that w_1^u and w_1^l could be v_k or w_1 as discussed in Observation 1. Moreover, notice that both $|F_1|$ and $|F_2|$ are at least 5. If both $|F_1|$ and $|F_2|$ are with size at least 6, then we finish the proof by induction using Lemma 6 considering the edge $v_{k-3}v_{k-2}$. So we may assume one of the two faces is with size 5. Without loss of generality assume $|F_1| = 5$, and hence $u_1^u \in N(v_{k-3})$. Let F_3 be the bounded face in H incident to the path $(v_1,\ v_2,\ u)$ as discussed in Observation 1. We perform the following three operations on H step by step to get a new and equivalent Halin graph H' to H, i.e., $e(H') = e(H)$.

(1) Delete the edges $v_{k-3}v_{k-2}$, $v_1^u w_1^u$ and $v_1^l w_1^l$ from H. The resulting disconnected graph has two components and let C^1 and C^2 be the components containing v_{k-3} and v_{k-2} respectively.

(2) Place the component C^2 in F_3 keeping its shape. Apply rigid motions on C^2 so that the pair of vertices $\{v_2,\ v_{k-2}\}$, $\{v_0',\ w_1^u\}$ and $\{w_1^l, u_1\}$ are joined by an edge after deleting the edge $v_0' u_1$ in C^1.

(3) Join the pair of vertices $\{v_1^u,\ v_1^l\}$ with an edge and denote the resulting graph by H'.

Since $d_T(v_{k-3}) \geq 4$ we have, $d_{H'}(v_{k-3}) \geq 3$. Moreover $v(H') = v(H)$ and all the leaves of T form the outer face of H'. Thus, H' is a Halin graph equivalent to H. However, it may happen that H' may contain a C_4. If a 4-cycle exists in H', then it must contain an edges in $\{v_0' w_1^u,\ w_1^l u_1,\ v_2 v_{k-2},\ v_1^u v_1^l\}$. Since the two faces incident to the edge $v_2 v_{k-2}$ in H' are of size at least 6, then using Lemma 3 for any cycle $\mathcal{C}$ containing an edge in $\{v_0' w_1^u,\ w_1^l u_1,\ v_2 v_{k-2}\}$ we have, $|\mathcal{C}| \geq 6$. This implies, if the Halin graph H' contains a 4-cycle, then it must contain the edge $v_1^u v_1^l$.

If H' is C_4-free graph, then we can finish the proof by induction using Lemma 6. Indeed, the two bounded faces incident to the edge $v_2 v_{k-2}$ in H' are of size at least 6. From Lemma 6 we have $e(H') = e(H'')+1$, where H'' is an $(n-1)$-vertex C_4-free Halin graph associated to H' in the lemma. Therefore, $e(H) = e(H') + 1 \leq \frac{5}{3}\left[(n-1) - 1\right] + 1 = \frac{5}{3}(n-2) + 1$.

Now we assume H' contains a 4-cycle. As explained earlier, the cycle contains $v_1^u v_1^l$. Such a 4-cycle happens when $v_1^l \in N(v)$, where v is in $N_H(v_{k-3})$, or $v_1^l \in N_H(v_{k-4})$ or $v_1^l \in N_H(v_{k-3})$ and at least one of the edges in $\{v_{k-3}v_1^u,\ v_{k-3}v_1^l\}$ is incident to a 3-face in H. Let the face, other than the F_1, and incident to $v_{k-3}v_1^u$ be F_4. Denote the associated leaf by v_2^u such that $v_2^u v_1^u$ is incident to F_4. We distinguish the three situations separately to complete the proof.

2.1. Case 1: When $v_1^l \in N_H(v)$, where $v \in N_H(v_{k-3})$ and v is a branching vertex

Notice that F_4 may or may not be a 3-face. We finish the proof by induction. Indeed, if F_4 is not a 3-face, then from Lemma 4 using the semi-branching vertex v_{k-3} and then applying Lemma 6 using the edge $v_{k-3}v_{k-2}$, we get an $(n-2)$-vertex C_4-free Halin graph, H^* such that $e(H) = e(H^*) + 3 \leq \left(\frac{5}{3}\left[(n-2) - 3\right] + 3\right) + 3 = \frac{5}{3}(n-2) + 1$. On the other hand, if F_4 is a 3-face, then applying Lemma 4 on the the semi-branching vertex v_{k-3} twice and then using Lemma 6 on the edge $v_{k-3}v_{k-2}$ we get an $(n-3)$-vertex C_4-free Halin graph H^*. Moreover we have $e(H) = e(H^*) + 5 \leq \left(\frac{5}{3}\left[(n-3) - 2\right] + 1\right) + 5 = \frac{5}{3}(n-2) + 1$, and we are done by induction.

2.2. Case 2: When $v_1^l \in N_H(v_{k-4})$

Actually, this may happen when $k \geq 7$. If F_4 is not a 3-face, then we can still finish the proof by induction using Lemma 4 considering v_{k-3} as a semi-branching vertex and then applying Lemma 6 using the edge $v_{k-3}v_{k-2}$ as the two faces incident to the edge are with size at least 6. Observe that the resulting graph is an $(n-2)$-vertex C_4-free Halin graph which only miss 3 edges. The same argument holds to finish the proof by induction if F_4 is a 3-face and $d_T(v_{k-3}) \geq 5$. In this case the resulting graph is an $(n-3)$-vertex C_4-free Halin graph which only miss 5 edges. On the other hand, if $d_T(v_{k-3}) = 4$ and F_4 is a 3-face, then we apply Lemma 4 on H using the v_2^u and then Lemma 5 using the leaf v_1^u, so that we get an $(n-3)$-vertex C_4-free Halin graph which only miss 5 edges. This we can finish the proof by induction as shown in Case 1 above.

2.3. Case 3: When $v_1^l \in N_H(v_{k-3})$ and at least one of the edges in $\{v_{k-3}v_1^u,\ v_{k-3}v_1^l\}$ is incident to a 3-face in H.

Let the face other than the F_2 and incident to $v_{k-3}v_1^l$ be F_5. Let v_2^l be a leaf in H such that the edge $v_1^l v_2^l$ is incident to F_5. F_5 may or may not be a 3-face. If both F_4 and F_5 are 3-faces, then $d_{H'}(v_{k-3}) \geq 5$. In this case, delete the vertices v_1^u and v_1^l from H' and then add the edge $v_2^u v_2^l$. This leaves an $(n-2)$-vertex C_4-free Halin graph, say H^*, with 4 edges reduced. Next apply Lemma 6 on the edge v_2v_{k-2} on H^*, the resulting graph becomes an $(n-3)$-vertex C_4-free Halin graph missing only 5 edges from the original graph H. With this, we can complete the proof by induction as given in Case 1 above. Finally, assume F_4 is a 3-face but not F_5. In this case, $d_{H'}(v_{k-3}) \geq 4$. It can be checked that deleting v_1^u and adding the edge $v_2^u v_1^l$ in H' leaves an $(n-1)$-vertex C_4-free Halin graph which misses only two edges. Applying Lemma 6 on the edge v_2v_{k-2} results an $(n-2)$-vertex C_4-free Halin graph which loses only 3 edges from the original graph H. Again in this case we can finish the induction as stated in Case 1. This completes the proof of Lemma 8.

3. Basis of the induction steps

To finish the proof by induction, we verify the bound when $n = 16,\ 17$, and 18. The following lemmas give the details.

Lemma 9. *For a* 16*-vertex* C_4*-free Halin graph* H*,* $e(H) \leq 25$*, i.e.,* $e(H) \leq \frac{5}{3}(n-1)$ *where* $n = 16$.

Proof. We prove the statement addressing different situations for which the length of the longest path a characteristic tree may possibly contain. Let T be the characteristic tree of H. Let L be a longest path in T and k be its length. It is very trivial to check that there is no H when $k \leq 3$. □

Claim 4. *k is at most 6.*

Proof. Let $k = 7$. From Observation 1 we have, $L = (v_0,\ v_1,\ v_2,\ \ldots,\ v_7)$ and $S = \{v_0, v_0', v_1, v_2, u, u_1, u_2, v_7, v_7', v_6, v_5, w,$ However, such vertex assignment leaves the non-leaf vertices v_3 and v_4 in the characteristic tree with degree 2, which is a contradiction to the definition of a Halin graph. Hence the maximum possible choice of k is 6. This completes the proof of Claim 4. □

Claim 5. *If* $k = 6$*, then* $e(H) = 24$.

Proof. Denote $L = (v_0,\ v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ v_6)$. Based on the notations Observation 1 we have, $S = \{v_0, v_0', v_1, v_2, u, u_1, u_2, v_6, v_6', v_5, v_4, w, w_1, w_2\}$. Since v_3 is a non-leaf in T, it must be adjacent with a vertex, say v_3'. Since $|S \cup \{v_3, v_3'\}| = 16$, then every vertex of T is now labeled. Clearly $P = \{v_0, v_0', u_1, u_2, v_3', w_1, w_2, v_6, v_6'\}$ is the set of all pendant vertices of T. Therefore, $e(H) = e(T) + |P| = 15 + 9 = 24$. This completes the proof of Claim 5. It is easy to see that **Figure 4** is the only characteristic tree T meeting the case. □

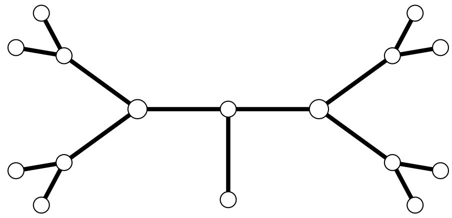

Figure 4. The characteristic tree of a Halin graph on 16 vertices.

Claim 6. *There is no H when $k = 5$.*

Proof. Denote $L = (v_0,\ v_1,\ v_2,\ v_3,\ v_4, v_5)$. From the discussion in Observation 1, $S = \{v_0, v_0', v_1, v_2, u, u_1, u_2, v_5, v_5', v_4, v_3$ There are two remaining vertices which are not assigned yet. Let this vertex be labeled as x_1 and x_2. For a clear reason, the vertices are adjacent to either v_2 or v_3. Without loss of generality suppose v_2 is such a vertex. In this case, there is a vertex in $\{x_1, x_2\}$, say x_1, is adjacent to v_2 such that xv_2 and uu_2 are incident to the same 4-face in H. But this is a contradiction to the fact that H is a C_4-free Halin graph. This completes the proof of Claim 6. □

Claim 7. *If $k = 4$, then $e(H) = 25$.*

Proof. Let $L = (v_0,\ v_1,\ v_2,\ v_3.\ v_4)$. By Observation 1, $S = \{v_0, v_0', v_1, v_2, u, u_1, u_2, v_4, v_4', v_3\}$. There are 6 vertices remaining, and label the vertices as $x_1, x_2, \ldots, x_6$. Let x_1 be adjacent to v_2, and suppose for contradiction x_1 be a leaf in T. It can be checked that the two faces incident to the edge v_2x_1 in H are of size 4 or 3. Moreover, non of the faces are of size 4. On the other hand, if both faces are with size 3, then again a 4-cycle will be obtained as the two 3-cycles sharing an edge forms a 4-cycle. This is again a contradiction. Thus, each vertex in $R = \{x_1, x_2, \ldots, x_6\}$ adjacent to v_2 is not pendant, and a vertex in R adjacent to v_2 is again adjacent to two pendant vertices in R. Therefore, T is obtained by identifying the dark-spotted vertex of three stars shown in **Figure 2** with the vertex v_2. The resulting graph is T_{16} which is shown in **Figure 1**. It can be calculated that $e(H) = e(T) + 10 = 25$. This completes the proof of Claim 7 and Lemma 9. □

Lemma 10. *For a 17-vertex C_4-free Halin graph H, $e(H) \leq 26$, i.e., $e(H) \leq \frac{5}{3}(n-2)+1$ where $n = 17$.*

Proof. We give similar proof to the one given in Lemma 1. Let H be a C_4-free Halin graph on 17 vertices, and T denote its corresponding characteristic tree. Let L be a longest path in T with length k. It can be checked that $k \geq 4$. □

Claim 8. *k is at most 6.*

Proof. By Observation 1 we have, $S = \{v_0, v_0', v_1, v_2, u, u_1, u_2, v_k, v_k', v_{k-1}, v_{k-2}, w, w_1, w_2\}$. Since T has 17 vertices and $|S| = 14$, three vertices are still not used. It can be seen that if $k \geq 7$, there exists a non-leaf vertex in L with degree 2, and this violates the definition of Halin graphs. Therefore $k \leq 6$. This completes the proof of Claim 8. □

Claim 9. *If $k = 6$, then $e(H) = 26$.*

Proof. In this case $S = \{v_0, v_0', v_1, v_2, u, u_1, u_2, v_6, v_6', v_5, v_4, w, w_1, w_2\}$. Notice that the vertex v_3 in L is a non-leaf in T, and there are two vertices in T, say x_1 and x_2, that are not in L. It is easy to check that non of the vertices is incident to v_2 or v_4, and both vertices are pendant and incident to v_3. Clearly H is C_4-free and $e(H) = e(T) + 10 = 26 = \frac{5}{3}(n-2)+1$ where $n = 17$. There are two possible non-isomorphic characteristic trees, see **Figure 5**. This completes the proof of Claim 10. □

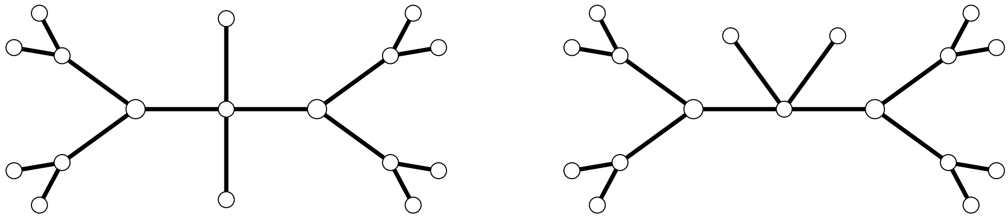

Figure 5. Characteristic tree of a 17-vertex Halin graphs.

Claim 10. *If* $k = 5$*, then* $e(H) = 26$.

Proof. By Observation 1, $S = \{v_0, v_0', v_1, v_2, u, u_1, u_2, v_5, v_5', v_4, v_3, w, w_1, w_2\}$. There are three vertices in T which are not labeled yet. Denote the vertices as x_1, x_2 and x_3, If one of the three vertices is adjacent to v_2 (similarly v_3) and is a pendant vertex in T, then all the remaining two vertices are pendant vertices and adjacent to v_2 or v_3. One of the three edges forms a 4-face containing the edge in $\{v_1v_0, v_2u, v_3w, v_4v_5\}$. But this results in a contradiction, as H is C_4-free. Therefore, the only possible situation that T exists is when the three vertices are connected to the L by identifying the black-spotted vertex of the star, see **Figure 2**, with either v_2 or v_3. In this case, we get T isomorphic to T_{17}, which is shown in **Figure 1**. It can be seen that T is C_4-free and $e(H) = 26$. □

Claim 11. *There is no* H *when* $k = 4$.

Proof. In this case $S = \{v_0, v_0', v_1, v_2, u, u_1, u_2, v_4, v_4', v_3\}$. $|S| = 10$. There are 7 vertices, say $x_1, x_2 \ldots, x_7$ not not labeled in T. None of these vertices is adjacent to v_1 or v_3. In other words, if any of the seven vertices is adjacent to a vertex in L, then it is with v_2. It can be seen that there is a vertex in $\{x_1, x_2, \ldots, x_7\}$, which is adjacent to v_2 and is a pendant in T. Suppose x_1 is such a vertex. By the choice of the path L, x_1v_2 can not be incident to a face of size at least 5. In other words, the two faces incident to the edge are either a 3-face or a 4-face. But in any possibility, H contains a C_4, which is a contradiction. This completes the proof of Claim 11 and Lemma 10. □

Lemma 11. *For an* 18*-vertex* C_4*-free Halin graph* H*,* $e(H) \leq 28$*, i.e.,* $e(H) \leq \frac{5}{3}(n-3)+3$ *where* $n = 18$.

Proof. Let T be the characteristic tree of H, and $L = (v_0,\ v_1,\ v_2,\ \ldots,\ v_k)$ be a longest path in T. It can be checked that $k \geq 4$. □

Claim 12. k *is at most 7.*

Proof. By Observation 1, $|S| = 14$. We remain four vertices that are not labeled yet. If three of the vertices are already in L, then at least two vertices, which are non-leaf, become degree-2. This is a contradiction and therefore $k \leq 7$. This completes the proof of Claim 12. □

Claim 13. *If* $k = 7$*,* $e(H) = 27$.

Proof. Here $S = \{v_0, v_0', v_1, v_2, u, u_1, u_2, v_7, v_7', v_6, v_5, w, w_1, w_2\}$. Observe that v_3 and v_4 are vertices in L but not addressed yet, as the vertices are degree-2 and should be with the degree at least 3. There are two remaining vertices, say x_1 and x_2, which are not on L but in T. From the degree condition of a Halin graph, v_3 and v_4 are adjacent to only one vertex in $\{x_1, x_2\}$. Let $x_1 \in N_T(v_3)$, and $x_2 \in N_T(v_4)$. Notice that the two edges can not be incident to the same face. Otherwise, the $x_1x_2 \in E(H)$ and we get 4-cycle $(v_4,\ v_3,\ x_1,\ x_2,\ v_4)$, which is a contradiction. Therefore the two edges must be on opposite sides of L in the planar embedding of H. H

which is shown in **Figure 6** is the only Halin graph with such property. It can be seen that H is C_4-free and $e(H) = e(T) + 10 = 27$. This completes the proof of Claim 13. □

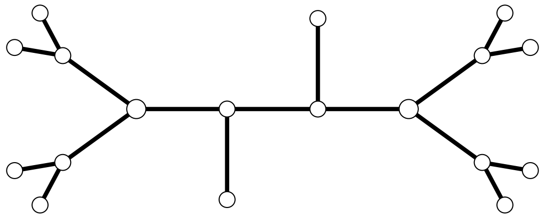

Figure 6. Characteristic tree of an 18-vertex Halin graph.

Claim 14. *If $k = 6$, $e(H) \leq 28$.*

Proof. $S = \{v_0, v_0', v_1, v_2, u, u_1, u_2, v_6, v_6', v_5, v_4, w, w_1, w_2\}$ and v_3 is a non-leaf in L. Moreover, there are three vertices, say x_1, x_2, and x_3 which are not in L. Thus, v_3 must be adjacent to one of the three vertices. If any of the remaining three vertices is adjacent to a vertex in $\{v_2, v_4\}$, it is easy to get a 4-cycle, which is a contradiction. We have two possible graphs.

The first graph is when all the three vertices, x_1, x_2, and x_3 are adjacent to v_3. It is easy to see that, not all edges, x_1v_3, x_2v_3, and x_3v_3, are on the same side of L. Otherwise, H contains a C_4. In this case, the characteristic tree is T_{18} and is shown in **Figure 1**. Clearly $e(h) = e(T) + 11 = 28$.

The second graph is obtained by identifying the dark-spotted vertex of the star with the v_3. The graph is shown in **Figure 7**. Here $e(H) = e(T) + 10 = 27$. This completes the proof of Claim 14. □

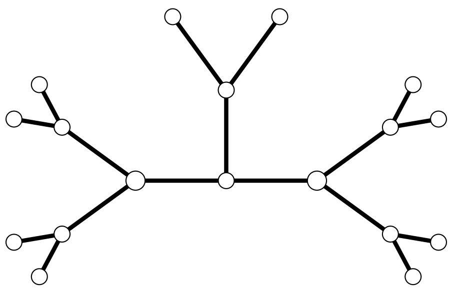

Figure 7. Characteristic tree of an 18-vertex Halin graph.

Claim 15. *There is no H when $k = 4$ or 5.*

Proof. Let $k = 5$. In this case we have, $S = \{v_0, v_0', v_1, v_2, u, u_1, u_2, v_5, v_5', v_4, v_3, w, w_1, w_2\}$. and there are four vertices, say $x_1,\ x_2,\ x_3$ and x_4, which are not in L. If any of the vertices is incident to L, then it must be adjacent to either v_2 or v_3. Without loss of generality let x_1 be adjacent to v_2. Then the two faces which are incident to the edge x_1v_2 are either both 3-face or both 4-face or a mix of the two. But in all three cases, H contains a 4-cycle, which is a contradiction. Therefore, x_1 must be adjacent to two pendant vertices, say x_2 and x_3. However, this results in a vertex x_4 which is not incident to any of the vertex in L. Which is a contradiction as T is a tree.

A similar argument can be given to show that we do not have an 18-vertex C_4-free Halin graph such that the characteristic tree has the longest path of length 4. This completes the proof of Claim 15 and Lemma 11. □

4. Conjectures and concluding remarks

As mentioned earlier in the beginning, Bondy and Lovász proved that a Halin graph is pancyclic if every non-leaf in its characteristic tree is of degree at least 4. It is also remarked that, if the characteristic tree contains a vertex of degree three, cycles of all lengths will still be in the graph with a possible exception of an even-length cycle. The following is our conjecture concerning the sharp upper bound of the Halin Turán number of the 6-cycle.

Conjecture 1. *For* $n \geq 21$,

$$\mathrm{ex}_{\mathcal{H}}(n, C_6) \leq \frac{8}{5}(n-1).$$

Conflict of interest

The author declares no conflict of interest.

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