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Article

Free surface flow over a trapezoidal cavity with surface tension effect

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Abstract: This paper investigates two-dimensional free surface flows over a trapezoidal cavity in a fluid with finite depth. By assuming that the flow is steady, irrotational, and that the fluid is non-viscous and incompressible. We suppose that the surface tension is included in the nonlinear boundary condition derived from Bernoulli's equation. A numerical approach using the series truncation method is employed to solve the problem. Solutions are computed for a symmetric trapezoidal obstacle for each value of the Weber number and the height of obstacle. The influence of the Weber number, the bottom height, and the angle of the obstacle are discussed.

Keywords: free surface flow; potential flow; Weber number; surface tension

1. Introduction

Free-surface two dimensional flows over trapezoidal obstacle are considered. We suppose that the flow is potential, irrotational and the fluid is incompressible and non-viscous. This problem was studied by Hanna et al. [1] with effects of gravity by using a series truncation method. The influence of the shape of the bottom on free surfaces has received much attention in the last century. We can mention, for example, the work of Dias and Vanden-Broeck [2]; Sekhri et al. [3] presented the free surface flow over a triangle obstacle by using a series truncation method, and also Abd-el-Malek [4] used the perturbation technique. Furthermore, the authors King and Bloor computed the solutions of free surface flow past a semi-finite step in the bottom [5], and over a curved topography [6], my work [7] studied the problem of flows over a triangle obstacle by employed the boundary integral equation method. Recently, Mansoor [8] obtained numerical solutions for the free surface flow due to a line sink with surface tension, McLean [9] studied the flow past a plate and others.

In this paper, we study the problem of two-dimensional flow with a trapezoidal obstacle. The angle of the symmetric polygon is β, where $0 < \beta \leq \frac{\pi}{2}$ (see **Figure 1**). We take into account the influence of superficial tension, but the force of gravity is not considered. In this case, the mathematical problem is defined by the number of Weber, and it becomes very difficult to solve analytically because of the nonlinear condition described as a Bernoulli's equation on the free surface, which obliges us to use numerical techniques and methods that depend on conformal transformations to solve it. In the present work, we calculate numerical solutions with surface tension effects by using a series truncation method. This method has been employed by Birkhoff and Zarantonello [10], Asavant and Vanden-Broeck [11], Tuck and Vanden-Broeck [12], my work [13], Vanden-Broeck [14,15], Chedala et al [16] and Doak and Vanden-Broeck [17] to calculate non-linear free surface flow. Far upstream, we suppose that the flow is uniform with constant velocity U and constant depth H.

Because of the presence of surface tension in Bernoulli's equation, we will need the dimensionless parameter called the Weber number α. This number may be written as:

$$\alpha = \frac{\rho U^2 H}{T} \tag{1}$$

where T is the superficial tension, and ρ is the density of the fluid. The formulation of the problem is presented in the next section. The numerical approach is described in section 3. In section 4, we present the numerical results and discussion.

2. Mathematical formulation

We consider a two-dimensional flow of an incompressible, inviscid fluid over a trapezoidal obstacle. In this problem, the flow is steady and irrotational, and one supposes that the flow is of uniform velocity U and of depth H. The fluid domain is bounded below by a horizontal rigid wall EA, DF and the trapezoidal depression forming the angle β, where $0 < \beta \leq \frac{\pi}{2}$, and above by the free surface LM. We introduce the Cartesian coordinates with the origin at the point O (see **Figure 1**), and one supposes that the unit velocity is U and the unit amplitude is H.

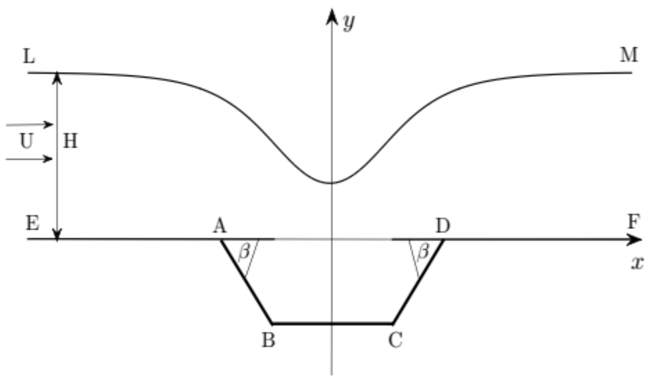

Figure 1. The physical plane $z = x + iy$. The free surface flow LM was calculated for $a = 0.6$, $b = 0.3$, $\beta = \frac{\pi}{3}$ and $\alpha = 70$.

In this article, we take into account the effect of superficial tension, but the gravity is negligible. The purpose of this study is to establish the basis of numerical accuracy of the computer program in order to allow satisfactory computation of similar flows with surface tension present. We consider the flow domain in the complex variable $z = x + iy$. Because the flow is potential, we can define the complex function as $f = \phi + i\psi$ in terms of the potential ϕ and the stream function ψ. The complex velocity is given by:

$$\xi = u - iv$$

where u and v are the *x*- and *y*-components of velocity, respectively.

Without loss of generality, we choose $\psi = 0$ on the streamline EABCDF and $\psi = UH$ on free streamline LM. The flow configuration in the complex potential plane $f = \phi + i\psi$ is illustrated in (**Figure 2**).

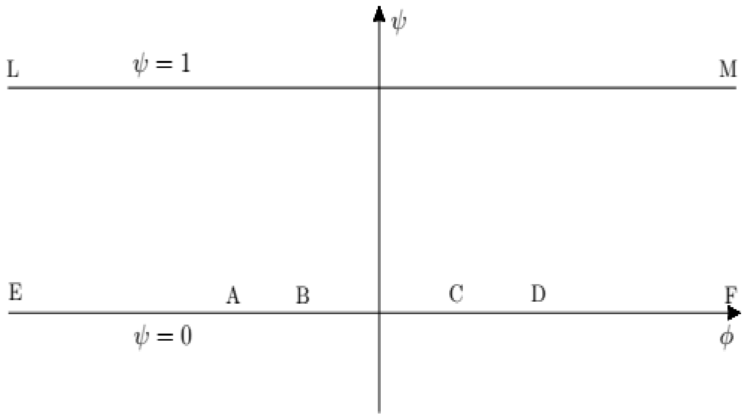

Figure 2. The flow configuration in the f-plane $f = \phi + i\psi$.

On the free streamline LM, in dimensionless variables, Bernoulli's equation is given by:

$$\frac{1}{2}(u^2 + v^2) + \frac{p}{\rho} = \frac{1}{2}U^2 + \frac{p_0}{\rho} = c \tag{2}$$

here p is the pressure of the fluid, p_0 is the atmospheric pressure, ρ is the density of the fluid, and c is a constant.

Laplace's capillary formula is given by:

$$p - p_0 = KT \tag{3}$$

where K is the mean curvature, and T is the superficial tension. Substituting Equation (3) into Equation (2) we obtain:

$$\frac{1}{2}(u^2 + v^2) - \frac{1}{\alpha}K = \frac{1}{2} \tag{4}$$

here α is the Weber number defined in Equation (1). The mathematical problem can be formulated in the potential function ϕ which checks the following conditions:

$$\begin{cases} \Delta\phi = 0 & \text{in the fluid domain} \\ \dfrac{\delta\phi}{\delta y} = 0 & \text{on EA, BC and DF} \\ (\nabla\phi)^2 - \dfrac{2}{\alpha}K = 1 & \text{on free surface} \end{cases} \tag{5}$$

We can write the complex velocity as:

$$\xi = u - iv = e^{\tau - i\theta} \tag{6}$$

where $e^{\tau} = |\xi|$ and θ is the angle between the speed vector and the horizontal.

Hence, Equation (4) becomes:

$$e^{2\tau} - \frac{2}{\alpha}e^{\tau}\left|\frac{\delta\theta}{\delta\phi}\right| = 1 \tag{7}$$

The kinematic conditions can be written as:

$$\begin{cases} \theta = 0, \psi = 0 & \text{on EA, BC and DF} \\ v = u\tan\theta\,, \psi = 0 & \text{on AB and CD} \end{cases} \tag{8}$$

The problem is now to find an analytic function $\tau - i\theta$ satisfying the boundary condition in Equation (7) and the kinematic conditions in Equation (8).

3. Numerical procedure

We map the flow domain in f-plane onto the upper half of the unit disk where the points E, A, B, C, D and F corresponding to the points $t = 1, -a, -b, b, a$ and $t = 1$, respectively, so that the solid boundary goes onto the real diameter and the free surface onto the circumference of the half of the circle (see **Figure 3**).

The relation between the f-plane and the t-plane is given by

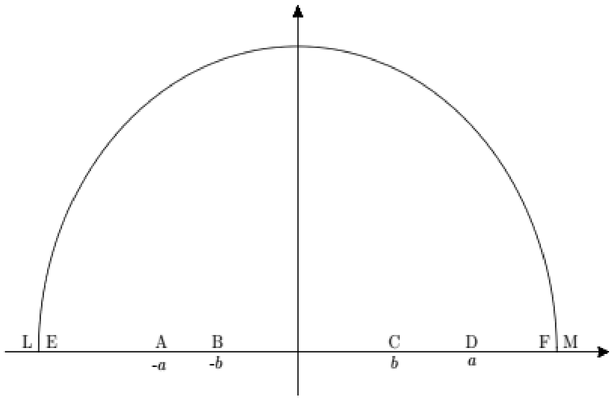

Figure 3. The t-plane.

$$f = \frac{2}{\pi} \ln \frac{1+t}{1-t} \tag{9}$$

Since the complex velocity $\xi(t) = u - iv$ is analytic in all the flow domain except at the points A, B, C and D which correspond to $t = -a, -b, b$ and $t = a$. The local behaviour of the velocity of these points is necessary for following study, we have:

$$\xi(t) \sim O\left(\left(\frac{t^2 - a^2}{1 - a^2 t^2}\right)^{-\beta}\right) \quad \text{as} \quad t \longrightarrow a \tag{10}$$

$$\xi(t) \sim O\left(\left(\frac{t^2 - b^2}{1 - b^2 t^2}\right)^{\beta}\right) \quad \text{as} \quad t \longrightarrow b \tag{11}$$

Now, the complex velocity $\xi(t)$ can be written as:

$$\xi(t) = e^{\tau - i\theta} = \left(\left(\frac{t^2 - a^2}{1 - a^2 t^2}\right)^{-\beta}\right) \times \left(\left(\frac{t^2 - b^2}{1 - bt^2}\right)^{\beta}\right) \times exp\left(\sum_{k=0}^{+\infty} a_k t^{2k}\right) \tag{12}$$

Since Equation (12) satisfies Equations (10) and (11), we expect the series to converge in the half of the disk in the t −plane. We will determine the coefficients a_k by using the conditions of Equation (8).

All points of free surface LM in t-plane are given by the relation.

$$\text{t} = e^{i\sigma}, 0 < \sigma < \pi$$

Using Equation (9) the expression Equation (7) is rewritten as:

$$e^{2\tau(\sigma)} - \frac{2}{\alpha} e^{\tau(\rho)} \left| \frac{\delta\theta(\sigma)}{\delta\phi} \right| = 1 \qquad (13)$$

here $\tau(\sigma)$ and $\theta(\sigma)$ denote the values of τ and θ on free the surface LM. We shall use Equation (13) to determine the unknown coefficients a_k. For solving our problem, we truncate the infinite series in Equation (12) after N terms. We Introduce the N mesh points on the free streamline LM where:

$$\sigma_I = \frac{\pi}{N}\left(I - \frac{1}{2}\right), I = 1,2,\dots,N$$

By using Equation (13) and the last equation, we obtain $\tau(\sigma)$, $\theta(\sigma)$ and $\frac{\delta\theta(\sigma)}{\delta\phi}$ are expressed in terms of coefficients a_k. This leads to the system with N nonlinear equations in N unknowns a_k $(k = 1, \dots., N)$ is solved by the Newton method. The Weber number α, the angle β and the height of the obstacle (the parameters a and b) are four parameters.

The form of the free streamline is obtained by numerically integrating the relations.

$$\begin{cases} \dfrac{\partial x}{\partial \phi} = \dfrac{2}{\pi \sin \sigma} e^{-\sigma} \cos \theta(\sigma) \\ \dfrac{\partial y}{\partial \phi} = \dfrac{2}{\pi \sin \sigma} e^{-\sigma} \sin \theta(\sigma) \end{cases}$$

where $0 < \sigma < \pi$. Most of the calculations were done and presented with $N = 50$.

4. Results and discussion

We calculate approximate solutions for flows over a trapezoidal obstacle by employing the numerical approach described above. We compute these solutions for each value of the Weber number $\alpha \geq \alpha_0$, the height of the obstacle (the parameters a and b) and for several values of the angle β where α_0 is the critical value. For values of α very large, $\alpha \longrightarrow +\infty$, all the coefficients in series (12) are negligibl ($a_k \approx 0$) for all $i \geq 0$. For example, for $\beta = \frac{\pi}{2}$, $a = 0.5$, $b = 0.1$ we have $a_1 = -3.4158 \times 10^{-9}$, $a_{10} = 1.6032 \times 10^{-9}$, $a_{30} = 6.0793 \times 10^{-10}$, $a_{50} = 3.3259 \times 10^{-10}$. This gives the exact solution for $\alpha \longrightarrow +\infty$.

$$\xi(t) = \left(\left(\frac{t^2 - a^2}{1 - a^2 t^2}\right)^{-\beta}\right) \times \left(\left(\frac{t^2 - b^2}{1 - bt^2}\right)^{\beta}\right)$$

For fixed values of α $(0 < \alpha < +\infty)$ the coefficients a_k were found to decrease rapidly. **Table 1** shows some values of coefficients a_k for different values of Weber number α and the angle β.

Table 1. Some values of coefficients a_k of the series Equation (12) and the angle β for some values of Weber number α.

β	α	a_1	a_{30}	a_{50}
	1000	1.9882×10^{-4}	-4.0031×10^{-8}	-7.9161×10^{-9}
$\frac{\pi}{2}$	50	3.1410×10^{-3}	-8.3277×10^{-7}	-1.5954×10^{-7}
	1	1.4981×10^{-1}	-6.6945×10^{-5}	-1.1781×10^{-5}
	1000	1.3119×10^{-4}	-2.6407×10^{-8}	-5.2220×10^{-6}
$\frac{\pi}{3}$	50	2.599×10^{-3}	-5.4932×10^{-7}	-1.0524×10^{-7}
	1	9.9009×10^{-2}	-4.4221×10^{-5}	-7.7831×10^{-6}
	1000	6.3627×10^{-5}	-1.2916×10^{-8}	-3.3792×10^{-9}
$\frac{\pi}{6}$	50	1.2609×10^{-3}	-2.6993×10^{-7}	-5.1813×10^{-8}
	3.5	1.4426×10^{-2}	8.6653×10^{-5}	4.8616×10^{-5}

Figures 4 and **5** show the influence of surface tension on the free-surface profiles for $\beta = \frac{\pi}{3}$ and $\beta = \frac{\pi}{2}$ respectively (the critical value is $\alpha_0 = 0.4$). It should be noted that the free surface elevation decreases when the Weber number α decreases. The effect of the angle β on free surface profiles for the Weber number $\alpha = 100$ and $a = 0.6$, $b = 0.3$ is shown in (**Figure 6**). The influence of the height of obstacle on free-surface profiles is displayed in (**Figure 7**) for some values of b and for fixed values of $\alpha = 200$, $\beta = \frac{\pi}{2}$ and $a = 0.15$. **Figure 8** shows the shape of the free surface for different values of a and for $\alpha = 100$, $\beta = \frac{\pi}{4}$ and $b = 0.8$. We noted that the elevation of the free surface increases when the obstacle height increases. The numerical results show that a solution exists for each value of $\alpha \geq \alpha_0$. Most of the computations were done and presented with $N = 50$.

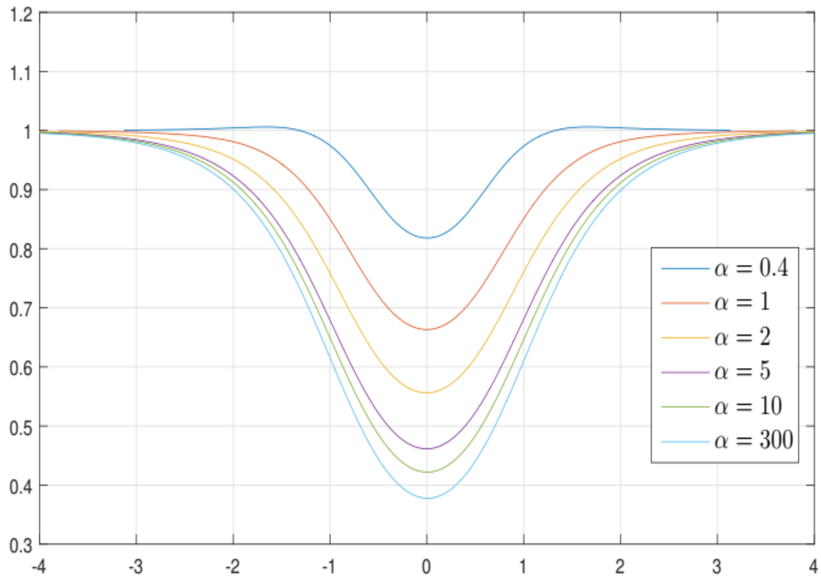

Figure 4. The shape of free surface for $a = 0.8$, $b = 0.4$, $\beta = \frac{\pi}{3}$ and some values of Weber number α.

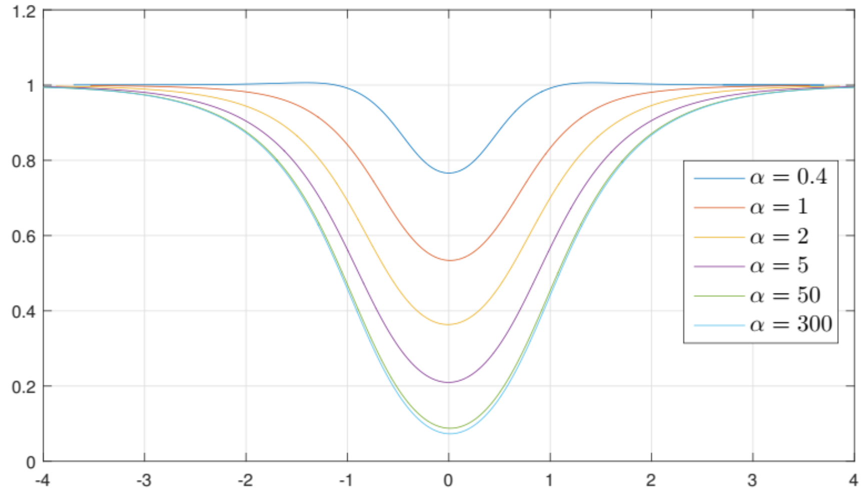

Figure 5. The shape of free surface for $a = 0.8$, $b = 0.4$, $\beta = \frac{\pi}{2}$ and some values of Weber number α.

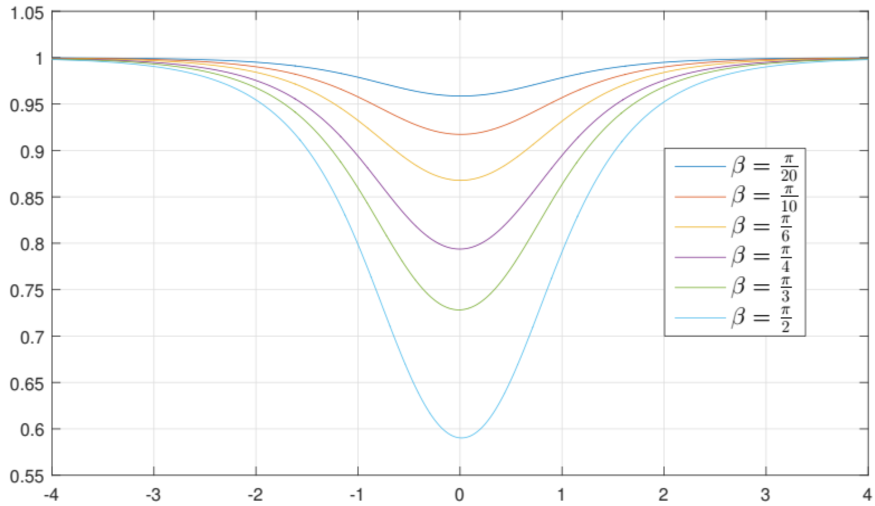

Figure 6. The effect of the angle of obstacle β on the shape of free surface for $a = 0.6$, $b = 0.3$ and $\alpha = 100$.

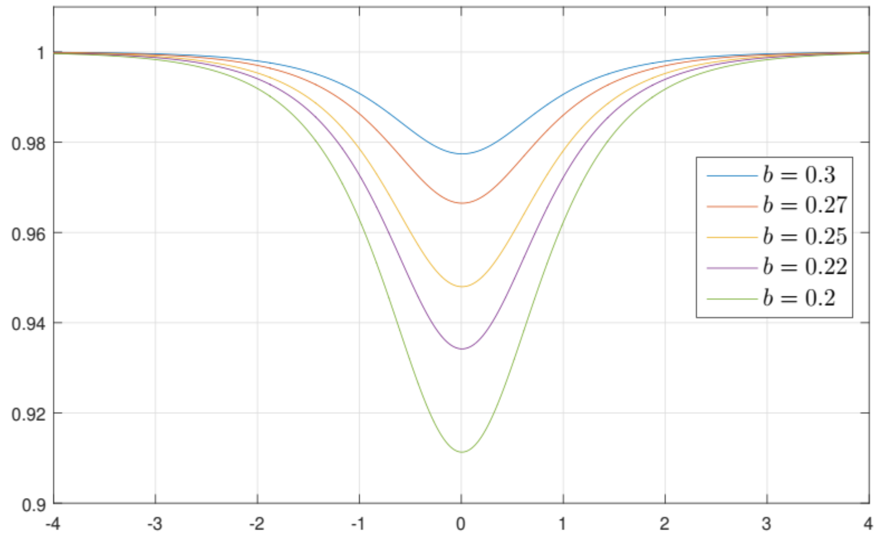

Figure 7. The shape of free surface for $a = 0.15$, $\alpha = 200$, $\beta = \frac{\pi}{2}$ and some values of b.

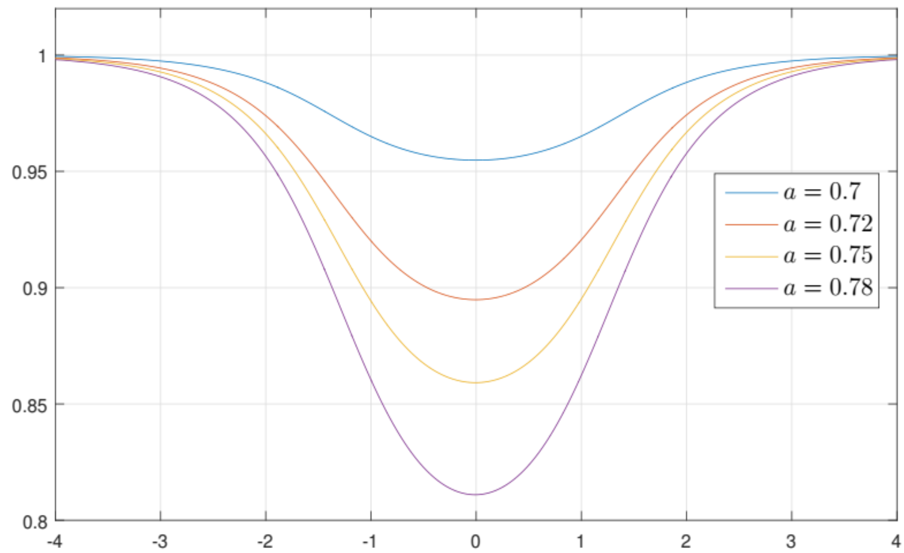

Figure 8. The shape of free surface for $b = 0.8$, $\alpha = 100$, $\beta = \frac{\pi}{4}$ and some values of a.

5. Conclusion

In this work, we presented the problem of free surface fluid flow over a trapezoidal obstacle with the influence of surface tension. We studied this problem in the case when the force of gravity is negligible. The non-linear integral equations have been solved using a series truncation method. We computed solutions for different values of Weber number α, the angles of trapezoidal obstacles, and the height of obstacles. We noted that for the Weber number decreases, the free surface elevation decreases, and the numerical computations show that a solution exists for $\alpha \geq \alpha_0$ where α_0 is a critical value.

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