

Pseudosymmetric normal paracontact metric space forms admitting (α, β) – type almost η –Ricci-Yamabe solitons

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Abstract: In this paper, we have considered normal paracontact metric space forms admitting (α, β) – type almost η –Ricci-Yamabe solitons by means of some curvature tensors. Ricci pseudosymmetry concepts of normal paracontact metric space forms admitting (α, β) – type almost η –Ricci-Yamabe soliton have introduced according to choosing of some special curvature tensors such as Riemann, concircular, projective, W_1 curvature tensor. After that, according to choosing of the curvature tensors, necessary conditions are given for normal paracontact metric space form admitting (α, β) – type almost η –Ricci-Yamabe soliton to be Ricci semisymmetric. Then some characterizations are obtained and some classifications are made under the some conditions.

Keywords: ricci-pseudosymmetric manifold; η –Ricci-Yamabe soliton; normal paracontact metric space form

MSC: 53C15; 53C25; 53D25

1. Introduction

In differential geometry, an interesting problem is whether a compact connected Riemannian manifold is conformally equivalent to a manifold of constant scalar curvature. This problem was formulated by Yamabe in 1960. Yamabe himself gave the affirmative answer, though there were some lacuna in his arguments.

In the past twenty years, the theory of geometric flows has been the most significant geometrical tool to explain the geometric structures in Riemannian geometry. A certain section of solutions on which the metric evolves by dilations and diffeomorphisms plays an important part in the study of singularities of the flows as they appear as possible singularity models. They are often called soliton solutions.

Another important topic of differential geometry is Ricci flow which was developed by Richerd Hamilton in order to solve the century long open problem “Poincare conjecture”. The notion of Yamabe flow also arose parallelly from the work of Hamilton [1].

Hamilton first time introduced the concept of Ricci flow and Yamabe flow simultaneously in 1988. Ricci soliton and Yamabe soliton emerge as the limit of the solutions of the Ricci flow and Yamabe flow, respectively. The notion of Yamabe flow was introduced by Hamilton as a tool for constructing metrics of constant scalar curvature in a given conformal class of Riemannian metrics on Riemannian manifold (Φ, g) , $n \geq 3$. The Yamabe flow is an evolution equation for metrics on a Riemannian manifolds as

follows:

$$\frac{\partial}{\partial t}g(t) = -r(t)g(t), \tag{1}$$

where $r(t)$ denotes the scalar curvature of the metric $g(t)$. Yamabe solitons correspond to self-similar solutions of the Yamabe flow. In dimension $n = 2$ the Yamabe soliton is equivalent to Ricci soliton. However, in dimension $n > 2$, the Yamabe and Ricci solitons do not agree, as the first preserves the conformal class of the metric but the Ricci solitons do not in general.

Over the past twenty years, the theory of geometric flows, such as Ricci flow and Yamabe flow has been the focus of attraction for many geometers. Recently, in 2019, Güler and Crasmareanu introduced the study of a new geometric flow which is a scalar combination of Ricci and Yamabe flow under the name Ricci-Yamabe map [2]. This is also called the Ricci-Yamabe flow of type (α, β) . The Ricci-Yamabe flow is an evolution of the metrics on the Riemannian or semi-Riemannian manifolds, defined as

$$\frac{\partial}{\partial t}g(t) = -2\alpha Ric(t) + \beta r(t)g(t), g_0 = g(0). \tag{2}$$

Due to the sign of involved scalars, the Ricci-Yamabe flow can also be a Riemannian, semi-Riemannian, or singular Riemannian flow. This kind of multiple choice can be useful in some geometrical or physical models, for example, relativistic theories. Therefore, naturally, the Ricci-Yamabe soliton emerges as the limit of the soliton of Ricci-Yamabe flow. This is a strong inspiration for initiating the study of Ricci-Yamabe solitons because although Ricci solitons and Yamabe solitons are the same in two dimensional spaces, there are essentially differences in higher dimensions. An interpolation soliton between Ricci and Yamabe soliton is considered, where the name Ricci-Bourguignon soliton corresponds to Ricci-Bourguignon flow but it depends on a single scalar in [3]. A soliton to the Ricci-Yamabe flow is called a Ricci-Yamabe soliton if it moves only by one parameter group of diffeomorphism and scaling. To be precise, a Ricci-Yamabe soliton on a Riemannian manifold (Φ, g) in [4] is a data set $(g, V, \lambda, \alpha, \beta)$ satisfying

$$\mathcal{L}_V g + 2\alpha S + (2\lambda - \beta r)g = 0, \tag{3}$$

where S is the Ricci tensor, r is the scalar curvature, and \mathcal{L}_V is the Lie-derivative along the vector field. If $\lambda > 0$, $\lambda < 0$, or $\lambda = 0$, then the (Φ, g) is called Ricci-Yamabe expander, Ricci-Yamabe shrinker, or Ricci-Yamabe steady soliton, respectively. Therefore, Equation (2) is called Ricci-Yamabe soliton of (α, β) -type, which is a generalization of Ricci and Yamabe solitons. We note that Ricci-Yamabe soliton of type $(\alpha, 0)$, $(0, \beta)$ -type are α -Ricci soliton and β -Yamabe soliton respectively.

An advanced extension of Ricci soliton is the concept of η -Ricci soliton, defined by Siddiqi and Akyol in [5] and by Cho and Kimura in [6]. Therefore, analogously, we can define the new notion by perturbing the Equation (2) that defines the type of soliton by a multiple of a certain $(0, 2)$ -tensor field $\eta \otimes \eta$. We obtain a slightly more general notion, namely, η -Ricci-Yamabe soliton of type (α, β) defined as:

$$\mathcal{L}_V g + 2\alpha S + (2\lambda - \beta r)g + 2\mu\eta \otimes \eta = 0. \tag{4}$$

Again, let us remark that η -Ricci-Yamabe soliton of type $(\alpha, 0)$ or $(1, 0)$, $(0, \beta)$ or $(0, 1)$ -type are α - η -Ricci soliton (or η -Ricci soliton) and β - η -Yamabe soliton (or η -Yamabe soliton) respectively for more details about these particular cases [7–11].

In particular, if $\mu = 0$ then the notion of η -Ricci-Yamabe soliton $(g, V, \lambda, \mu, \alpha, \beta)$ is reduced to the notion of Ricci-Yamabe soliton $(g, V, \lambda, \alpha, \beta)$. If $\mu \neq 0$, then the η -Ricci-Yamabe soliton is named proper η -Ricci-Yamabe soliton.

According to Pigola et al. if we replace the constant λ in (3) with a smooth function $\lambda \in C^\infty(\Phi)$, called soliton function, then we say (Φ, g) is an almost Ricci soliton.

The study of paracontact geometry was initiated by Kenayuki and Williams [12]. Zamkovoy studied paracontact metric manifolds and their subclasses [13,14]. Recently Welyczko studied curvature and torsion of Frenet Legendre curves in 3-dimensional normal paracontact metric manifolds [15,16]. In recent years, contact metric manifolds and their curvature properties have been studied by many authors in [16–18].

In this paper, we have considered normal paracontact metric space forms admitting (α, β) -type almost η -Ricci-Yamabe solitons by means of some curvature tensors. Ricci pseudosymmetry concepts of normal paracontact metric space forms admitting (α, β) -type almost η -Ricci-Yamabe soliton have introduced according to choice of some special curvature tensors such as Riemann, concircular, projective, W_1 curvature tensor. After that, according to choosing of the curvature tensors, necessary conditions are given for normal paracontact metric space form admitting (α, β) -type almost η -Ricci-Yamabe soliton to be Ricci semisymmetric. Then some characterizations are obtained, and some classifications are made under some conditions.

For simplicity's sake, the normal paracontact metric space form expression will be expressed as *NPMS*-form after this part of the article. Similarly, for brevity, after this part of the article, η -Ricci-Yamabe soliton expressions will be shown as η -*RY**S*, Ricci pseudosymmetric as *Ricci* - *P*, and Ricci semisymmetric as *Ricci* - *S*.

2. Preliminaries

Let's take an n -dimensional differentiable Φ manifold. If it admits a tensor field ϕ of type $(1, 1)$, a contravariant vector field ξ , and a 1-form η satisfying the following conditions:

$$\phi^2 \epsilon_1 = \epsilon_1 - \eta(\epsilon_1) \xi, \quad \phi \xi = 0, \quad \eta(\phi \epsilon_1) = 0, \quad \eta(\xi) = 1, \tag{5}$$

and

$$g(\phi \epsilon_1, \phi \epsilon_2) = g(\epsilon_1, \epsilon_2) - \eta(\epsilon_1) \eta(\epsilon_2), \quad g(\epsilon_1, \xi) = \eta(\epsilon_1), \tag{6}$$

for all $\epsilon_1, \epsilon_2, \xi \in \chi(\Phi)$, (ϕ, ξ, η) is called almost paracontact structure and (Φ, ϕ, ξ, η) is called almost paracontact metric manifold. If the covariant derivative of ϕ satisfies

$$(\nabla_{\epsilon_1} \phi) \epsilon_2 = -g(\epsilon_1, \epsilon_2) \xi - \eta(\epsilon_2) \epsilon_1 + 2\eta(\epsilon_1) \eta(\epsilon_2) \xi, \tag{7}$$

then, Φ is called a normal paracontact metric manifold, where ∇ is Levi-Civita connection. From Equation (7), we can easily see that

$$\phi\epsilon_1 = \nabla_{\epsilon_1}\xi, \tag{8}$$

for any $\epsilon_1 \in \chi(\Phi)$ [12].

Moreover, if such a manifold has a constant sectional curvature equal to c , then the Riemannian curvature tensor R is given by

$$\begin{aligned} R(\epsilon_1, \epsilon_2)\epsilon_3 &= \frac{c+3}{4} [g(\epsilon_2, \epsilon_3)\epsilon_1 - g(\epsilon_1, \epsilon_3)\epsilon_2] + \frac{c-1}{4} [\eta(\epsilon_1)\eta(\epsilon_3)\epsilon_2 \\ &- \eta(\epsilon_2)\eta(\epsilon_3)\epsilon_1 + g(\epsilon_1, \epsilon_3)\eta(\epsilon_2)\xi - g(\epsilon_2, \epsilon_3)\eta(\epsilon_1)\xi + g(\phi\epsilon_2, \epsilon_3)\phi\epsilon_1 \\ &- g(\phi\epsilon_1, \epsilon_3)\phi\epsilon_2 - 2g(\phi\epsilon_1, \epsilon_2)\phi\epsilon_3], \end{aligned} \tag{9}$$

for any vector fields $\epsilon_1, \epsilon_2, \epsilon_3 \in \chi(\Phi)$ [16].

In a *NPMS*-form by direct calculations, we can easily see that

$$S(\epsilon_1, \epsilon_2) = \frac{c(n-5) + 3n + 1}{4}g(\epsilon_1, \epsilon_2) + \frac{(c-1)(5-n)}{4}\eta(\epsilon_1)\eta(\epsilon_2), \tag{10}$$

from which

$$Q\epsilon_1 = \frac{c(n-5) + 4n + 1}{4}\epsilon_1 + \frac{(c-1)(5-n)}{4}\eta(\epsilon_1)\xi, \tag{11}$$

for any $\epsilon_1, \epsilon_2 \in \chi(\Phi)$, where Q is the Ricci operator and S is the Ricci tensor of Φ .

Lemma 1. *Let Φ be an n -dimensional NPMS-forms. In this case, the following equations are obtained.*

$$R(\xi, \epsilon_1)\epsilon_2 = g(\epsilon_1, \epsilon_2)\xi - \eta(\epsilon_2)\epsilon_1, \tag{12}$$

$$R(\epsilon_1, \xi)\epsilon_2 = -g(\epsilon_1, \epsilon_2)\xi + \eta(\epsilon_2)\epsilon_1, \tag{13}$$

$$R(\epsilon_1, \epsilon_2)\xi = \eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \tag{14}$$

$$\eta(R(\epsilon_1, \epsilon_2)\epsilon_3) = g(\eta(\epsilon_1)\epsilon_2 - \eta(\epsilon_2)\epsilon_1, \epsilon_3) \tag{15}$$

$$S(\epsilon_1, \xi) = (n-1)\eta(\epsilon_1), \tag{16}$$

$$Q\xi = (n-1)\xi, \tag{17}$$

where R, S , and Q are Riemann curvature tensor, Ricci curvature tensor, and Ricci operator, respectively.

Let Φ be a Riemannian manifold, T is $(0, k)$ -type tensor field and A is $(0, 2)$ -type tensor field. In this case, Tachibana tensor field $Q(A, T)$ is defined as

$$\begin{aligned} Q(A, T)(X_1, \dots, X_k; \epsilon_1, \epsilon_2) &= -T((\epsilon_1 \wedge_A \epsilon_2) X_1, \dots, X_k) - \\ &\dots - T(X_1, \dots, X_{k-1}, (\epsilon_1 \wedge_A \epsilon_2) X_k), \end{aligned} \tag{18}$$

where,

$$(\epsilon_1 \wedge_A \epsilon_2) \epsilon_3 = A(\epsilon_2, \epsilon_3) \epsilon_1 - A(\epsilon_1, \epsilon_3) \epsilon_2, \tag{19}$$

$$k \geq 1, X_1, X_2, \dots, X_k, \epsilon_1, \epsilon_2 \in \Gamma(T\Phi).$$

3. (α, β) –type almost η –Ricci-Yamabe solitons on ricci pseudosymmetric and ricci semisymmetric normal paracontact metric space forms

Now let $(g, \xi, \lambda, \mu, \alpha, \beta)$ be (α, β) -type almost η –RYS on NPMS–form. Then we have

$$\begin{aligned} (L_\xi g)(\epsilon_1, \epsilon_2) &= L_\xi g(\epsilon_1, \epsilon_2) - g(L_\xi \epsilon_1, \epsilon_2) - g(\epsilon_1, L_\xi \epsilon_2) \\ &= \xi g(\epsilon_1, \epsilon_2) - g([\xi, \epsilon_1], \epsilon_2) - g(\epsilon_1, [\xi, \epsilon_2]) \\ &= g(\nabla_\xi \epsilon_1, \epsilon_2) + g(\epsilon_1, \nabla_\xi \epsilon_2) - g(\nabla_\xi \epsilon_1, \epsilon_2) \\ &\quad + g(\nabla_{\epsilon_1} \xi, \epsilon_2) - g(\nabla_\xi \epsilon_2, \epsilon_1) + g(\epsilon_1, \nabla_{\epsilon_2} \xi), \end{aligned}$$

for all $\epsilon_1, \epsilon_2 \in \Gamma(T\Phi)$. By using ϕ is symmetric, we have

$$(L_\xi g)(\epsilon_1, \epsilon_2) = 2g(\phi\epsilon_1, \epsilon_2). \tag{20}$$

Thus, in a NPMS–forms, from (4) and (20), we have

$$2\alpha S(\epsilon_1, \epsilon_2) + 2g(\phi\epsilon_1, \epsilon_2) + (2\lambda - \beta r)g(\epsilon_1, \epsilon_2) + 2\mu\eta(\epsilon_1)\eta(\epsilon_2) = 0. \tag{21}$$

For $\epsilon_2 = \xi$ in (21), this implies that

$$2\alpha S(\xi, \epsilon_1) = (\beta r - 2\lambda - 2\alpha)\eta(\epsilon_1). \tag{22}$$

Taking into account of (16) and (22), we conclude that

$$2n\alpha = \beta r - 2\lambda. \tag{23}$$

Definition 1. Let Φ be an n –dimensional NPMS–form. If there exists a function \mathcal{H}_1 on Φ such that

$$R \cdot S = \mathcal{H}_1 Q(g, S),$$

then the Φ is called Ricci – P.

Also, if $\mathcal{H}_1 = 0$, the Φ is called Ricci – S.

Theorem 1. Let Φ be a NPMS–form and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be (α, β) –type almost η – RYS on Φ . If Φ is a Ricci – P, then we get

$$\mathcal{H}_1 = \frac{2\lambda - \alpha + 1}{1 - \alpha} \text{ or } \mathcal{H}_1 = \frac{\alpha - 2\lambda + 1}{1 + \alpha}.$$

Proof. Let’s assume that NPMS–form Φ be Ricci – P and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be

(α, β) –type almost η – RYS on Φ . That’s mean

$$(R(\epsilon_1, \epsilon_2) \cdot S)(\epsilon_4, \epsilon_5) = \mathcal{H}_1 Q(g, S)(\epsilon_4, \epsilon_5; \epsilon_1, \epsilon_2),$$

for all $\epsilon_1, \epsilon_2, \epsilon_4, \epsilon_5 \in \Gamma(T\Phi)$. From the last equation, we can easily write

$$\begin{aligned} & S(R(\epsilon_1, \epsilon_2)\epsilon_4, \epsilon_5) + S(\epsilon_4, R(\epsilon_1, \epsilon_2)\epsilon_5) \\ &= \mathcal{H}_1 \{S((\epsilon_1 \wedge_g \epsilon_2)\epsilon_4, \epsilon_5) + S(\epsilon_4, (\epsilon_1 \wedge_g \epsilon_2)\epsilon_5)\}. \end{aligned} \tag{24}$$

Setting $\epsilon_5 = \xi$ in (24), we get

$$\begin{aligned} & S(R(\epsilon_1, \epsilon_2)\epsilon_4, \xi) + S(\epsilon_4, R(\epsilon_1, \epsilon_2)\xi) \\ &= \mathcal{H}_1 \{S(g(\epsilon_2, \epsilon_4)\epsilon_1 - g(\epsilon_1, \epsilon_4)\epsilon_2, \xi) \\ &+ S(\epsilon_4, \eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2)\}. \end{aligned} \tag{25}$$

Making use of (14) and (22) in (25), we have

$$\begin{aligned} & S(\epsilon_4, \eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2) \\ &+ \frac{\beta r - 2\lambda - 2\alpha}{2\alpha} \eta(R(\epsilon_1, \epsilon_2)\epsilon_4) \\ &= \mathcal{H}_1 \{(\beta r - 2\lambda - 2\alpha) / 2ag(\eta(\epsilon_1)\epsilon_2 - \eta(\epsilon_2)\epsilon_1, \epsilon_4) \\ &+ S(\epsilon_4, \eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2)\}. \end{aligned} \tag{26}$$

By using (15) in (26), we get

$$\begin{aligned} & \frac{\beta r - 2\lambda - 2\alpha}{2\alpha} g(\eta(\epsilon_1)\epsilon_2 - \eta(\epsilon_2)\epsilon_1, \epsilon_4) \\ &+ S(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \epsilon_4) \\ &= \mathcal{H}_1 \{(\beta r - 2\lambda - 2\alpha) / 2ag(\eta(\epsilon_1)\epsilon_2 - \eta(\epsilon_2)\epsilon_1, \epsilon_4) \\ &+ S(\epsilon_4, \eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2)\}. \end{aligned} \tag{27}$$

If we use (21) in the (27), we can write

$$\left[\left(1 - \frac{2\lambda}{\alpha}\right) - \mathcal{H}_1 \right] g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \epsilon_4) + \frac{1}{\alpha} (\mathcal{H}_1 - 1) g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \phi\epsilon_4) = 0. \tag{28}$$

If we write $\phi\epsilon_4$ instead of ϵ_4 in (28) and make use of (1), we obtain

$$\frac{1}{\alpha} (\mathcal{H}_1 - 1) g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \epsilon_4) + \left[\left(1 - \frac{2\lambda}{\alpha}\right) - \mathcal{H}_1 \right] g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \phi\epsilon_4) = 0. \tag{29}$$

It is clear from (28) and (29) , we get

$$\left\{ \left[\frac{1}{\alpha} (\mathcal{H}_1 - 1) \right]^2 - \left[\left(1 - \frac{2\lambda}{\alpha} \right) - \mathcal{H}_1 \right]^2 \right\} g (\eta (\epsilon_2) \epsilon_1 - \eta (\epsilon_1) \epsilon_2, \epsilon_4) = 0.$$

This completes the proof of Theorem.□

Corollary 1. Let Φ be NPMS–form and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be (α, β) –type almost η – RYS on Φ . If Φ is a Ricci – S, then we observe the following.

- i) Φ is Ricci-Yamabe expander soliton if $\alpha \in (1, \infty)$.
- ii) Φ is Ricci-Yamabe steady soliton if $\alpha = 1$ or $\alpha = -1$.
- iii) Φ is Ricci-Yamabe shrinker soliton if $\alpha \in (-\infty, -1)$.

Specifically, if $\alpha = 1$ and $\beta = 0$, the (α, β) –type almost η – RYS is reduced to a $\alpha - \eta$ –Ricci soliton. In this case, we can state the following theorem.

Theorem 2. Let Φ be NPMS–form and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be $\alpha - \eta$ –Ricci soliton on Φ . If Φ is a Ricci – P, then Φ is either a Ricci-Yamabe steady soliton or $\mathcal{H}_1 = 1 - \lambda$.

Corollary 2. Let Φ be NPMS–form and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be $\alpha - \eta$ –Ricci soliton on Φ . If Φ is a Ricci – S, then Φ is either a Ricci-Yamabe steady soliton or a Ricci-Yamabe expander soliton.

For an n –dimensional semi-Riemann manifold Φ , the concircular curvature tensor is defined as

$$C (\epsilon_1, \epsilon_2) \epsilon_3 = R (\epsilon_1, \epsilon_2) \epsilon_3 - \frac{r}{n(n-1)} [g (\epsilon_2, \epsilon_3) \epsilon_1 - g (\epsilon_1, \epsilon_3) \epsilon_2]. \quad (30)$$

For an n –dimensional NPMS–form, if we choose $\epsilon_3 = \xi$ in (28) , we can write

$$C (\epsilon_1, \epsilon_2) \xi = \left[1 - \frac{r}{n(n-1)} \right] [\eta (\epsilon_2) \epsilon_1 - \eta (\epsilon_1) \epsilon_2], \quad (31)$$

For an n –dimensional NPMS–form, if we choose $\epsilon_3 = \xi$ in (28) , we can write

$$\eta (C (\epsilon_1, \epsilon_2) \epsilon_3) = \left[1 - \frac{r}{n(n-1)} \right] g (\eta (\epsilon_1) \epsilon_2 - \eta (\epsilon_2) \epsilon_1, \epsilon_3). \quad (32)$$

Definition 2. Let Φ be an n –dimensional NPMS–form. If there exists a function \mathcal{H}_2 on Φ such that

$$C \cdot S = \mathcal{H}_2 Q (g, S),$$

then the Φ is called **concircular Ricci – P**.

Also, if $\mathcal{H}_2 = 0$, the Φ is called **concircular Ricci – S**.

Theorem 3. Let Φ be a NPMS–form and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be (α, β) –type almost η – RYS on Φ . If Φ is a concircular Ricci – P, then $\mathcal{H}_2 = \frac{n(n-1) - r}{n(n-1)}$ or $\alpha = \pm 1$.

Proof. Let’s assume that NPMS–form Φ be concircular Ricci–P and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be (α, β) –type almost η – RYS on Φ . That’s mean

$$(C (\epsilon_1, \epsilon_2) \cdot S) (\epsilon_4, \epsilon_5) = \mathcal{H}_2 Q (g, S) (\epsilon_4, \epsilon_5; \epsilon_1, \epsilon_2),$$

for all $\epsilon_1, \epsilon_2, \epsilon_4, \epsilon_5 \in \Gamma(T\Phi)$. From the last equation, we can easily write

$$\begin{aligned} & S(C(\epsilon_1, \epsilon_2)\epsilon_4, \epsilon_5) + S(\epsilon_4, C(\epsilon_1, \epsilon_2)\epsilon_5) \\ &= \mathcal{H}_2 \{S((\epsilon_1 \wedge_g \epsilon_2)\epsilon_4, \epsilon_5) + S(\epsilon_4, (\epsilon_1 \wedge_g \epsilon_2)\epsilon_5)\}. \end{aligned} \tag{33}$$

If we choose $\epsilon_5 = \xi$ in (33), we get

$$\begin{aligned} & S(C(\epsilon_1, \epsilon_2)\epsilon_4, \xi) + S(\epsilon_4, C(\epsilon_1, \epsilon_2)\xi) \\ &= \mathcal{H}_2 \{S(g(\epsilon_2, \epsilon_4)\epsilon_1 - g(\epsilon_1, \epsilon_4)\epsilon_2, \xi) \\ &+ S(\epsilon_4, \eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2)\}. \end{aligned} \tag{34}$$

By using of (22) and (31) in (34), we have

$$\begin{aligned} & AS(\epsilon_4, \eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2) \\ &+ \frac{\beta r - 2\lambda - 2\alpha}{2\alpha} \eta(C(\epsilon_1, \epsilon_2)\epsilon_4) \\ &= \mathcal{H}_2 \{(\beta r - 2\lambda - 2\alpha) / 2ag(\eta(\epsilon_1)\epsilon_2 - \eta(\epsilon_2)\epsilon_1, \epsilon_4) \\ &+ S(\epsilon_4, \eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2)\}, \end{aligned} \tag{35}$$

where $A = 1 - \frac{r}{n(n-1)}$. Substituting (32) into (35), we have

$$\begin{aligned} & \frac{A(\beta r - 2\lambda - 2\alpha)}{2\alpha} g(\eta(\epsilon_1)\epsilon_2 - \eta(\epsilon_2)\epsilon_1, \epsilon_4) \\ &+ AS(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \epsilon_4) \\ &= \mathcal{H}_2 \{(\beta r - 2\lambda - 2\alpha) / 2ag(\eta(\epsilon_1)\epsilon_2 - \eta(\epsilon_2)\epsilon_1, \epsilon_4) \\ &+ S(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \epsilon_4)\}. \end{aligned} \tag{36}$$

If we use (21) in the (36), we can write

$$(A - \mathcal{H}_2)g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \epsilon_4) + \left(\frac{\mathcal{H}_2 - A}{\alpha}\right)g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \phi\epsilon_4) = 0. \tag{37}$$

If we write $\phi\epsilon_4$ instead of ϵ_4 in (37) and make use of (1), we obtain

$$\left(\frac{\mathcal{H}_2 - A}{\alpha}\right)g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \epsilon_4) + (A - \mathcal{H}_2)g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \phi\epsilon_4) = 0. \tag{38}$$

By means of (37) and (38), we conclude

$$\left\{ \left[\frac{\mathcal{H}_2 - A}{\alpha} \right]^2 - (A - \mathcal{H}_2)^2 \right\} g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \epsilon_4) = 0,$$

and so we get

$$(\mathcal{H}_2 - A)^2 (1 - \alpha^2) g (\eta (\epsilon_2) \epsilon_1 - \eta (\epsilon_1) \epsilon_2, \epsilon_4) = 0.$$

This completes the proof of Theorem.□

Corollary 3. Let Φ be a NPMS-forms and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be (α, β) -type almost η -RYS on Φ . If Φ is a concircular Ricci - S, then Φ is either constant scalar curvature $r = n(n - 1)$ or $\alpha = \pm 1$.

Corollary 4. Let Φ be a NPMS-forms and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be α - η -Ricci-soliton on Φ . If Φ is a concircular Ricci - P, then $\mathcal{H}_2 = \frac{n(n - 1) - r}{n(n - 1)}$.

For an n -dimensional semi-Riemann manifold Φ , the projective curvature tensor is defined as

$$P (\epsilon_1, \epsilon_2) \epsilon_3 = R (\epsilon_1, \epsilon_2) \epsilon_3 - \frac{1}{n - 1} [S (\epsilon_2, \epsilon_3) \epsilon_1 - S (\epsilon_1, \epsilon_3) \epsilon_2]. \quad (39)$$

For an n -dimensional NPMS-form, if we choose $\epsilon_3 = \xi$ in (37), we can write

$$P (\epsilon_1, \epsilon_2) \xi = 0, \quad (40)$$

and similarly, if we take the inner product of both sides of (37) by ξ , we get

$$\eta (P (\epsilon_1, \epsilon_2) \epsilon_3) = 0. \quad (41)$$

Definition 3. Let Φ be an n -dimensional NPMS-form. If there exists a function \mathcal{H}_3 on Φ such that

$$P \cdot S = \mathcal{H}_3 Q (g, S),$$

then the Φ is said to be **projective Ricci - P**.

Also, if $\mathcal{H}_3 = 0$, the Φ is called **projective Ricci - S**.

Theorem 4. Let Φ be a NPMS-form and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be (α, β) -type almost η -RYS on Φ . If Φ is a projective Ricci - P, then Φ is a projective Ricci - S or $\alpha = \pm 1$.

Proof. Let's assume that NPMS-form Φ be projective Ricci-P and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be (α, β) -type almost η -RYS on Φ . That's mean

$$(P (\epsilon_1, \epsilon_2) \cdot S) (\epsilon_4, \epsilon_5) = \mathcal{H}_3 Q (g, S) (\epsilon_4, \epsilon_5; \epsilon_1, \epsilon_2),$$

for all $\epsilon_1, \epsilon_2, \epsilon_4, \epsilon_5 \in \Gamma (T\Phi)$. From the last equation, we can easily write

$$\begin{aligned} & S (P (\epsilon_1, \epsilon_2) \epsilon_4, \epsilon_5) + S (\epsilon_4, P (\epsilon_1, \epsilon_2) \epsilon_5) \\ &= \mathcal{H}_3 \{S ((\epsilon_1 \wedge_g \epsilon_2) \epsilon_4, \epsilon_5) + S (\epsilon_4, (\epsilon_1 \wedge_g \epsilon_2) \epsilon_5)\}. \end{aligned} \quad (42)$$

If we choose $\epsilon_5 = \xi$ in (42), we get

$$\begin{aligned} & S(P(\epsilon_1, \epsilon_2)\epsilon_4, \xi) + S(\epsilon_4, P(\epsilon_1, \epsilon_2)\xi) \\ &= \mathcal{H}_3 \{S(g(\epsilon_2, \epsilon_4)\epsilon_1 - g(\epsilon_1, \epsilon_4)\epsilon_2, \xi) \\ &+ S(\epsilon_4, \eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2)\}. \end{aligned} \tag{43}$$

If we make use of (22) and (40) in (43), we have

$$\begin{aligned} & \frac{\beta r - 2\lambda - 2\alpha}{2\alpha} \eta(P(\epsilon_1, \epsilon_2)\epsilon_4) \\ &= \mathcal{H}_3 \{(\beta r - 2\lambda - 2\alpha) / 2ag(\eta(\epsilon_1)\epsilon_2 - \eta(\epsilon_2)\epsilon_1, \epsilon_4) \\ &+ S(\epsilon_4, \eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2)\}. \end{aligned} \tag{44}$$

If we use (41) in the (44), we get

$$\begin{aligned} & \mathcal{H}_3 \{(\beta r - 2\lambda - 2\alpha) / 2ag(\eta(\epsilon_1)\epsilon_2 - \eta(\epsilon_2)\epsilon_1, \epsilon_4) \\ &+ S(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \epsilon_4)\} = 0. \end{aligned} \tag{45}$$

If we use (21) in the (45), we can write

$$\mathcal{H}_3 \left[g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \epsilon_4) - \frac{1}{\alpha} g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \phi\epsilon_4) \right] = 0. \tag{46}$$

If we write $\phi\epsilon_4$ instead of ϵ_4 in (46) and make use of (1), we obtain

$$\mathcal{H}_3 \left[\frac{-1}{\alpha} g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \epsilon_4) + g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \phi\epsilon_4) \right] = 0. \tag{47}$$

It is clear from (46) and (47), we obtain

$$\mathcal{H}_3 \left(1 - \frac{1}{\alpha^2} \right) g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \epsilon_4) = 0.$$

This completes the proof of Theorem. \square

Corollary 5. Let Φ be a NPMS-form and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be (α, β) -type almost η -RYS on Φ . If Φ is a projective Ricci - P, then Φ is a projective Ricci - S or the following results are observed depending on the state of α .

- i) Let $\alpha = 1$.
 - a) Φ is Ricci-Yamabe expander soliton if $\beta r > 2n$.
 - b) Φ is Ricci-Yamabe steady soliton if $\beta r = 2n$.
 - c) Φ is Ricci-Yamabe shrinker soliton if $\beta r < 2n$.
- ii) Let $\alpha = -1$.
 - a) Φ is Ricci-Yamabe expander soliton if $\beta r > -2n$.
 - b) Φ is Ricci-Yamabe steady soliton if $\beta r = -2n$.
 - c) Φ is Ricci-Yamabe shrinker soliton if $\beta r < -2n$.

For an n -dimensional semi-Riemann manifold Φ , the W_1 -curvature tensor is defined as

$$W_1(\epsilon_1, \epsilon_2) \epsilon_3 = R(\epsilon_1, \epsilon_2) \epsilon_3 + \frac{1}{n-1} [S(\epsilon_2, \epsilon_3) \epsilon_1 - S(\epsilon_1, \epsilon_3) \epsilon_2]. \quad (48)$$

For an n -dimensional $NPM S$ -form, if we choose $\epsilon_3 = \xi$ in (48), we can write

$$W_1(\epsilon_1, \epsilon_2) \xi = 2[\eta(\epsilon_2) \epsilon_1 - \eta(\epsilon_1) \epsilon_2], \quad (49)$$

and similarly, if we take the inner product of both of sides of (48) by ξ , we get

$$\eta(W_1(\epsilon_1, \epsilon_2) \epsilon_3) = 2g(\eta(\epsilon_1) \epsilon_2 - \eta(\epsilon_2) \epsilon_1, \epsilon_3). \quad (50)$$

Definition 4. Let Φ be an n -dimensional $NPM S$ -form. If there exists a function \mathcal{H}_4 on Φ such that

$$W_1 \cdot S = \mathcal{H}_4 Q(g, S),$$

then the Φ is called W_1 -Ricci - P .

Also, if $\mathcal{H}_4 = 0$, the Φ is called W_1 -Ricci - S .

Theorem 5. Let Φ be a $NPM S$ -form and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be (α, β) -type almost η -RYS on Φ . If Φ is a W_1 -Ricci - P , then $\mathcal{H}_4 = 2, \alpha = \pm 1$.

Proof. Let's assume that $NPM S$ -form Φ be W_1 -Ricci - P and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be (α, β) -type almost η -RYS on Φ . That's mean

$$(W_1(\epsilon_1, \epsilon_2) \cdot S)(\epsilon_4, \epsilon_5) = \mathcal{H}_4 Q(g, S)(\epsilon_4, \epsilon_5; \epsilon_1, \epsilon_2),$$

for all $\epsilon_1, \epsilon_2, \epsilon_4, \epsilon_5 \in \Gamma(T\Phi)$. From the last equation, we can easily write

$$\begin{aligned} & S(W_1(\epsilon_1, \epsilon_2) \epsilon_4, \epsilon_5) + S(\epsilon_4, W_1(\epsilon_1, \epsilon_2) \epsilon_5) \\ &= \mathcal{H}_4 \{S((\epsilon_1 \wedge_g \epsilon_2) \epsilon_4, \epsilon_5) + S(\epsilon_4, (\epsilon_1 \wedge_g \epsilon_2) \epsilon_5)\}. \end{aligned} \quad (51)$$

If we choose $\epsilon_5 = \xi$ in (51), we get

$$\begin{aligned} & S(W_1(\epsilon_1, \epsilon_2) \epsilon_4, \xi) + S(\epsilon_4, W_1(\epsilon_1, \epsilon_2) \xi) \\ &= \mathcal{H}_4 \{S(g(\epsilon_2, \epsilon_4) \epsilon_1 - g(\epsilon_1, \epsilon_4) \epsilon_2, \xi) \\ &+ S(\epsilon_4, \eta(\epsilon_1) \epsilon_2 - \eta(\epsilon_2) \epsilon_1)\}. \end{aligned} \quad (52)$$

By means of (22) and (49) in (52), we have

$$\begin{aligned}
 & 2S(\epsilon_4, \eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2) \\
 & + \frac{\beta r - 2\lambda - 2\alpha}{2\alpha} \eta(W_1(\epsilon_1, \epsilon_2)\epsilon_4) \\
 & = \mathcal{H}_4\{(\beta r - 2\lambda - 2\alpha)/2ag(\eta(\epsilon_1)\epsilon_2 - \eta(\epsilon_2)\epsilon_1, \epsilon_4) \\
 & + S(\epsilon_4, \eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2)\}.
 \end{aligned} \tag{53}$$

If we use (50) in the (53), we get

$$\begin{aligned}
 & \frac{\beta r - 2\lambda - 2\alpha}{\alpha} g(\eta(\epsilon_1)\epsilon_2 - \eta(\epsilon_2)\epsilon_1, \epsilon_4) \\
 & + 2S(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \epsilon_4) \\
 & = \mathcal{H}_4\{(\beta r - 2\lambda - 2\alpha)/2ag(\eta(\epsilon_1)\epsilon_2 - \eta(\epsilon_2)\epsilon_1, \epsilon_4) \\
 & + S(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \epsilon_4)\}.
 \end{aligned} \tag{54}$$

If we use (21) in the (54), we can write

$$(2 - \mathcal{H}_4)g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \epsilon_4) + \frac{1}{\alpha}(\mathcal{H}_4 - 2)g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \phi\epsilon_4) = 0. \tag{55}$$

If we write $\phi\epsilon_4$ instead of ϵ_4 in (55) and make use of (1), we obtain

$$\frac{1}{\alpha}(\mathcal{H}_4 - 2)g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \epsilon_4) + (2 - \mathcal{H}_4)g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \phi\epsilon_4) = 0. \tag{56}$$

It is clear from (55) and (56), we get

$$\left\{ \left[\frac{1}{\alpha}(\mathcal{H}_4 - 2) \right]^2 - (2 - \mathcal{H}_4)^2 \right\} g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \epsilon_4) = 0,$$

and so we have

$$(\alpha^2 - 1)(\mathcal{H}_4 - 2)^2 g(\eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \epsilon_4) = 0.$$

This completes the proof of Theorem. \square

Corollary 6. Let Φ be a NPMS-forms and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be (α, β) -type almost η -RYS on Φ . If Φ is a W_1 -Ricci-S, then $\alpha = \pm 1$, that is, if Φ is W_1 -Ricci-S, then the following results are observed depending on the state of α .

- i) Let $\alpha = 1$.
 - a) Φ is Ricci-Yamabe expander soliton if $\beta r > 2n$.
 - b) Φ is Ricci-Yamabe steady soliton if $\beta r = 2n$.
 - c) Φ is Ricci-Yamabe shrinker soliton if $\beta r < 2n$.
- ii) Let $\alpha = -1$.
 - a) Φ is Ricci-Yamabe expander soliton if $\beta r > -2n$.

- b) Φ is Ricci-Yamabe steady soliton if $\beta r = -2n$.
- c) Φ is Ricci-Yamabe shrinker soliton if $\beta r < -2n$.

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