Journal of AppliedMath 2025, 3(2), 2176.
https://doi.org/10.59400/jam2176

Article

On the resolutions of the edge ideals of graphs

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CITATION

Barile M. On the resolutions of the edge ideals of graphs. Journal of AppliedMath. 2025; 3(2): 2176. https://doi.org/10.59400/jam2176

ARTICLE INFO

Received: 2 December 2024
Accepted: 7 March 2025
Available online: 26 March 2025

Abstract: The so-called *bridge-friendliness* is a set of divisibility conditions on the minimal generators of monomial ideals. It was introduced by Chau and Kara as a sufficient criterion for the existence of a cellular minimal graded free resolution. It is fulfilled by large classes of monomial ideals, in particular by the edge ideals of acyclic graphs. We present a construction that, given a pair of graphs with bridge-friendly edge ideals, produces a new graph with the same property. An additional assumption is that the starting graphs both have at least one leaf.

Keywords: minimal graded free resolutions; cellular resolutions; Morse theory; edge ideals of graphs; monomial ideals

1. Introduction

Determining a minimal graded free resolutions for a homogeneous ideal in a polynomial ring over a field is a long-standing open question in commutative algebra. The problem is far from being solved even in the special case where the ideal is generated by squarefree quadratic monomials. Such an ideal can be viewed as the ring-theoretic counterpart of a finite simple graph without loops and without isolated vertices, the one obtained by taking the indeterminates as the vertices and the generating monomials as the edges. In this way the ideal becomes the so-called *edge ideal* of the graph. The homological invariants of the ideal that can be derived from the combinatorial properties of the graph have been intensively studied in the course of the last decades. A natural starting point for this investigation is provided by the Taylor-resolution [1], which applies to any monomial ideal, but needs to be refined in order to obtain minimality. Famous subcomplexes are the *Lyubeznik resolution* and the *Scarf complex*. The reader is referred to [2] for a comprehensive survey article on this topic. In recent years, in particular, minimal graded free resolutions have been completely described for the edge ideals of acyclic graphs [3] by means of discrete Morse theory, whereas other authors found classes of edge ideals for which a minimal graded resolution is given by the Lyubeznik resolution [4] and by the Scarf complex [5], respectively. Further classes of monomial ideals derived from graphs were considered in [6]. The method applied in [3] is based on a construction presented in [7], and it was later generalized in [8]. This extension gave rise to the notion of *Barile-Macchia resolution*, which is a refinement of the Taylor resolution and is minimal for monomial ideals that are *bridge-friendly*, i.e., fulfill special divisibility conditions on their minimal monomial generating sets. By now, various classes of bridge-friendly monomial ideals have been determined. In the present paper, we show how new graphs whose edge ideals are bridge-friendly, and thus admit a minimal Barile-Macchia resolution, can be obtained from the combination of others having the same property. These new graphs may contain arbitrarily many cycles.

2. Preliminaries

Given a field K, let $R = K[x_1, \ldots, x_n]$ be a polynomial ring in n indeterminates. Let G be a finite simple graph without loops and isolated vertices on the vertex set $V(G) = \{x_1, \ldots, x_n\}$. The *edge ideal* of G in R is the ideal $I(G)$ generated by all products $x_i x_j$ such that $\{x_i, x_j\}$ is an edge of G. The product $x_i x_j$ will be also called an *edge monomial*. The following construction and the main terminology are taken from [8], Section 2. They are here applied to the special class of edge ideals.

We fix some total order $<$ on the edges of G (equivalently, on the edge monomials in $I(G)$). Given two edge monomials m, m' we will say that m *dominates* m' if $m > m'$. Any non-empty subsequence of the ordered sequence formed by all edge monomials will be called a *symbol*.

Definition 1. *Let σ be symbol and $m \in \sigma$. Then m is called a bridge of σ if* $\mathrm{lcm}(\sigma) = \mathrm{lcm}(\sigma \setminus \{m\})$.

Remark 1. *If $m = xy$, then m is a bridge of σ if and only if in σ there are two edge monomials of the form ax and yb, with which m forms a 3-path or a 3-cycle. In this case, (ax, yb) will be called a pair of sides for m. We will also say that m is a bridge between ax and yb, or that (ax, xy, yb) is a bridge triple (around xy).*

Definition 2. *Let σ be symbol and m an edge monomial. Then m is called a gap of σ if $m \notin \sigma$ and m is a bridge of $\sigma \cup \{m\}$ (or, equivalently,* $\mathrm{lcm}(\sigma \cup \{m\}) = \mathrm{lcm}(\sigma)$). *In this case, m is called a true gap of σ if, in $\sigma \cup \{m\}$, any bridge smaller than m is a bridge in σ.*

Remark 2. *The second part of the preceding definition can be rephrased as follows. If the edge monomial $m = xy$ is a gap of σ, then it is not a true gap if and only if inserting m into σ causes some edge monomial $m' < m$ of σ, which is not a bridge in σ, to become a bridge. This means that xy is a monomial completing a bridge triple around m', i.e., either m' is of the form ax and adding xy produces a bridge triple (ua, ax, xy), or m' is of the form yb and adding xy produces a bridge triple (xy, yb, bv). In any case, m' contains x or y.*

The next definition refers to notions introduced in ([8], Definition 2.16) and is based on the characterization contained in ([8], Theorem 2.24).

Definition 3. *Let σ be a symbol.*

(a) *If σ has a bridge, we define $sb(\sigma)$ as its smallest bridge.*

(b) *We say that σ is type-1 if it has a true gap not dominating any of its bridges.*

(c) *We say that σ is potentially-type-2 it is has a bridge not dominating any of its true gaps. In this case, we will say that σ is type-2 if, whenever σ' is another potentially-type-2 symbol such that $\sigma \setminus \{sb(\sigma)\} = \sigma' \setminus \{sb(\sigma')\}$, we have that $sb(\sigma) < sb(\sigma')$.*

In [8] Chau and Kara introduced a class of cellular resolutions for monomial ideals in R, based on discrete Morse theory. We briefly describe it here in the special case of edge ideals. They first defined the set A of ordered pairs (σ, τ) such that $\tau \subset \sigma$, $|\tau| = |\sigma| - 1$, where τ is type-1 and σ is type-2. Then they showed in ([8], Theorem 2.11) that A is a *homogeneous acyclic matching* on the simplicial complex formed by the sets of minimal monomial generators of $I(G)$. This, according to the theory developed by Batzies and Welker [7], induces a cellular resolution of $I(G)$. Chau and Kara could thus prove the following result. In order to provide the reader with a quick reference, we quote the statement contained in a subsequent paper:

Theorem 1. *Given a total order $<$ on the set of edge monomials in $I(G)$, there is a graded free resolution (called Barile-Macchia resolution) of $I(G)$ over R in which the basis of the ith module is the set of symbols of length i that are neither type-1 nor type-2 (*[4]*, Theorem 2.6).*

The symbols mentioned in the preceding theorem were called *Barile-Macchia-critical* in [8]. They were obtained by means of a recursive procedure ([8], Algorithm 2.9). The Barile-Macchia resolution fails, in general, to be minimal. Moreover, it may vary according to the total order $<$ chosen. In [3] it was proven that, for a suitable order, it is minimal whenever G is an acyclic graph. Then, in [8], it was shown that this result derives from a sufficient criterion which is always fulfilled if G is acyclic. It is based on the following notion.

Definition 4. *Given a total order $<$ on the edge monomials in $I(G)$, we say that $I(G)$ is bridge-friendly with respect to $<$ if every potentially-type-2 symbol is type-2.*

In the sequel, we will say that $I(G)$ is *bridge-friendly* if it so with respect to some total order $<$.

Remark 3. *1) Note that the property introduced in Definition 4 is purely combinatorial, it is referred to the graph, and is independent of the field K of coefficients; 2)According to Definitions 3 and 4, $I(G)$ is bridge-friendly if and only if, for any pair of potentially-type-2 symbols σ and σ' such that $\sigma \setminus sb(\sigma) = \sigma' \setminus sb(\sigma')$ we have that $sb(\sigma) = sb(\sigma')$, or, equivalently, $\sigma = \sigma'$.*

Now we can state the aforementioned criterion:

Theorem 2. *If $I(G)$ is bridge-friendly, then the corresponding Barile-Macchia resolution is minimal (*[8]*, Theorem 2.29).*

In the next section we will determine a new, large class of graphs G for which the edge ideal $I(G)$ is bridge-friendly.

3. The main result

In the sequel, for the sake of simplicity, we will use the same notation for edges and edge monomials. For our purposes, these two notions will be identified. We first give an important preliminary remark. Suppose that $I(G)$ is bridge-friendly with respect to some total order $<$ on its edge monomials. Let w_1w_2 be a pendant edge of the graph G, i.e., an edge containing a vertex, say w_2, of degree 1 (a so-called *leaf*). Since w_1w_2 is then the only edge monomial divisible by w_2, it cannot be a bridge, nor a gap in any symbol. Hence its position in the arrangement determined by $<$ is irrelevant with respect to the property of being type-1 or (potentially-)type-2, hence, also with respect to bridge-friendliness. Consequently, we may assume, e.g., that w_1w_2 is the greatest or the smallest edge monomial.

We now present the crucial construction. Let G_1 and G_2 be graphs on disjoint vertex sets, and suppose that w_1w_2 is a pendant edge of G_1, where w_2 is a leaf, and w_3w_4 is a pendant edge of G_2, where w_4 is a leaf. We define the *dot product* $G_1 \bullet_{(w_2=w_4)} G_2$

to be the graph obtained by identifying the vertices w_2 and w_4. We will also denote it by $G_1 \bullet G_2$ for short.

The main result is the following. For its statement and its proof we refer to the notation just introduced.

Theorem 3. *If $I(G_1)$ and $I(G_2)$ are bridge-friendly, then so is $I(G_1 \bullet G_2)$.*

Proof. For $i \in \{1, 2\}$, let $<_i$ be a total order on the edge monomials with respect to which $I(G_i)$ is bridge-friendly. According to the preceding remark, we may assume that w_1w_2 is the minimum edge monomial in $I(G_1)$ and w_3w_4 is the maximum edge monomial in $I(G_2)$. We then consider $G_1 \bullet G_2$, in which w_3w_4 is replaced by w_2w_3, and we endow the set of edge monomials in $I(G_1 \bullet G_2)$ with a total order $<$ defined as follows. Given two edge monomials m, m', we set $m > m'$ if either

(i) $m, m' \in I(G_i)$ and $m >_i m'$ for some $i \in \{1, 2\}$, or

(ii) $m \in I(G_1)$ and $m' \in I(G_2)$.

Note that this ordering extends the orderings previously given on the sets of edge monomials of $I(G_1)$ and $I(G_2)$, respectively. Moreover, the edge monomials w_1w_2 and w_2w_3 are consecutive. We will show that $I(G_1 \bullet G_2)$ is bridge-friendly with respect to $<$. Let σ be a symbol of $I(G_1 \bullet G_2)$. Suppose that σ is potentially-type-2. We prove that it is type-2. To this end, we will distinguish between several cases and subcases. We need one preliminary remark, which will be applied in the course of the proof. First note that, by assumption, $sb(\sigma)$ is a bridge of σ not dominating any true gaps in σ. Let $\sigma = \sigma_1 \cup \sigma_2$, where σ_i is the subsymbol formed by all edge monomials of σ belonging to $I(G_i)$. Let σ' be another symbol that is potentially-type-2 and is such that

$$\sigma \setminus \{sb(\sigma)\} = \sigma' \setminus \{sb(\sigma')\}. \tag{1}$$

Let $\sigma' = \sigma'_1 \cup \sigma'_2$, where the meaning of the notation is obvious. Now, if $sb(\sigma) = sb(\sigma')$, there is nothing to prove. So suppose that $sb(\sigma) \neq sb(\sigma')$. Then, in view of Equation (1), $sb(\sigma') \notin \sigma$. Moreover, we may assume that $sb(\sigma) > sb(\sigma')$. Note that, by assumption,

$$\text{lcm}(sb(\sigma') \cup \sigma \setminus \{sb(\sigma)\}) = \text{lcm}(sb(\sigma') \cup \sigma' \setminus \{sb(\sigma')\}) =$$

$$\text{lcm}(\sigma' \setminus \{sb(\sigma')\}) = \text{lcm}(\sigma \setminus \{sb(\sigma)\}).$$

It follows that $\text{lcm}(sb(\sigma') \cup \sigma) = \text{lcm}(\sigma)$.

We have thus established our

Preliminary remark: The bridge $sb(\sigma')$ is a gap of σ. More precisely, if $sb(\sigma')$ is a bridge between m_1 and m_2 in σ', it is a gap between m_1 and m_2 in σ. Since, by assumption, $sb(\sigma) > sb(\sigma')$, and σ is potentially-type-2, this gap cannot be a true gap.

Case 1 $w_1w_2 \notin \sigma$. We preliminarily show that in this case any true gap in σ_1 is a true gap in σ. Now, let m be a gap in σ_1. Then $m \neq x_1x_2$, because x_1x_2 is a pendant edge of G_1. Moreover, m is, a fortiori, a gap in σ. So assume that m is not a true gap in σ. Then, according to Remark 2, m completes a 2-walk in σ to a 3-walk. This walk must be contained in σ_1, since $w_1w_2 \notin \sigma$ implies that no edge of σ_1 is adjacent to an edge of σ_2 (see **Figure 1**). It follows that m is not a true gap in σ_1, which proves our claim.

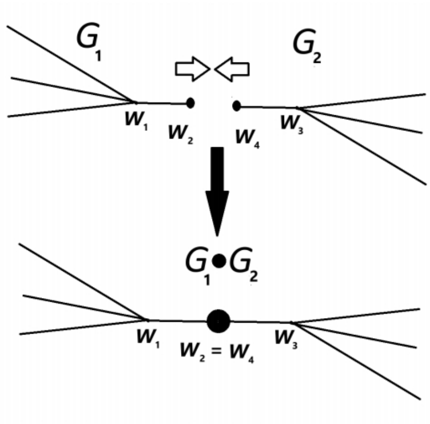

Figure 1. The construction of the dot product $G_1 \bullet G_2$.

Case 1.1 $sb(\sigma) \in I(G_1)$. Since $sb(\sigma) \neq w_1w_2$, we then have that $sb(\sigma) = sb(\sigma_1)$. But $sb(\sigma_1)$ cannot dominate any true gap in σ_1, because, in view of the remark in Case 1, the same true gap would be dominated by $sb(\sigma)$ in σ, against the assumption that σ is potentially-type-2. Hence σ_1 is potentially-type-2 with respect to $<_1$.

Case 1.1.1 $sb(\sigma') \in I(G_1)$.

Case 1.1.1.1 $sb(\sigma') \neq w_1w_2$. Then $sb(\sigma') = sb(\sigma_1')$. Moreover, $w_1w_2 \notin \sigma'$, because otherwise, in view of Equation (1), we would have $w_1w_2 \in \sigma$, against our present assumption. Thus, as in the preceding Case 1.1, we deduce that σ_1' is potentially-type-2 with respect to $<_1$. Now the equality (1) between

$$\sigma \setminus sb(\sigma) = (\sigma_1 \setminus sb(\sigma_1)) \cup \sigma_2$$

and

$$\sigma' \setminus sb(\sigma') = (\sigma_1' \setminus sb(\sigma_1')) \cup \sigma_2'$$

implies that

$$\sigma_2 = \sigma_2'$$

and

$$\sigma_1 \setminus sb(\sigma_1) = \sigma_1' \setminus sb(\sigma_1'). \tag{2}$$

Finally, since $I(G_1)$ is bridge-friendly with respect to $<_1$, we conclude that $\sigma_1 = \sigma_1'$, whence $\sigma = \sigma'$, as desired. This settles Case 1.1.1.1.

Case 1.1.1.2 $sb(\sigma') = w_1w_2$. Then, according to the preliminary remark, w_1w_2 is a gap in σ. Since, by assumption, $sb(\sigma) > w_1w_2$, and σ is potentially-type-2, it, however, cannot be a true gap. This means that the insertion of w_1w_2 into σ produces some smaller new bridge, which, by definition of $<$, must belong to $I(G_2)$, and will therefore contain w_2 (see the end of Remark 2). Hence this new bridge is w_2w_3. The new bridge triple will thus contain an edge monomial $w_3y \in \sigma$ for some vertex y. Now, since w_2w_3 and w_3y belong to $I(G_2)$, they are both distinct from $sb(\sigma)$, so that $w_2w_3, w_3y \in \sigma'$. But then w_2w_3 is a bridge in σ'. Since $w_2w_3 < w_1w_2 = sb(\sigma')$, this provides a contradiction, and shows that the present subcase is impossible. This settles

Case 1.1.1.

Case 1.1.2 $sb(\sigma') \in I(G_2)$. Let $sb(\sigma') = xy$, a bridge between the edge monomials ax and yb. Note that, since $w_1w_2 \notin \sigma$ and $w_1w_2 \neq sb(\sigma')$, by Equation (1) we have that $w_1w_2 \notin \sigma'$. Hence in σ and σ' no element of $I(G_1)$ is adjacent to an element of $I(G_2)$. Thus ax and yb both belong to $I(G_2)$. Moreover, by our preliminary remark, xy is a gap in σ, but not a true one. Hence, in view of Remark 2, in $\sigma \cup \{xy\}$ we either have a bridge triple (ua, ax, xy) or a bridge triple (xy, yb, bv), where, in both cases, the middle edge monomial is a bridge smaller than xy. Note that such a bridge triple must be contained in $I(G_2)$. Since $sb(\sigma_1) \in I(G_1)$, it follows that it is contained in σ', as well. But this contradicts the minimality of $sb(\sigma')$. Hence this subcase is impossible. Case 1.1 is thus settled.

Case 1.2 $sb(\sigma) \in I(G_2)$. In this case $sb(\sigma') \in I(G_2)$, as well. The assumption $w_1w_2 \notin \sigma$ implies that w_2w_3 cannot be a bridge in σ (see **Figure 1**), whence $sb(\sigma) = sb(\sigma_2)$. Moreover, since we are assuming that $sb(\sigma') < sb(\sigma)$, and w_2w_3 is the maximum edge monomial in $I(G_2)$, we also have that $sb(\sigma') \neq w_2w_3$, whence $sb(\sigma') = sb(\sigma_2')$. Now, σ_2, σ_2' are potentially-type-2, because any true gap following $sb(\sigma_2)$ or $sb(\sigma_2')$ in σ_2 or σ_2' would be a true gap in σ or in σ'. Since $I(G_2)$ is bridge-friendly with respect to $<_2$, it follows that $\sigma_2' = \sigma_2$, whence $\sigma = \sigma'$. This settles Case 1.

Case 2 $w_1w_2 \in \sigma$.

Case 2.1 $sb(\sigma) = w_1w_2$. Since $sb(\sigma') < w_1w_2$, we have that $sb(\sigma') \in I(G_2)$. Let $sb(\sigma') = xy$, a bridge between the edge monomials $ax, yb \in \sigma'$. From Equation (1) we also have that $ax, yb \in \sigma$ and, according to our preliminary remark, in σ the edge monomial xy is a gap, but not a true gap. Hence the insertion of xy into σ creates a new bridge $m' < xy$. This bridge, however, cannot be such in σ'. Now, by virtue of Equation (1), the only edge monomial of σ that is missing in σ' is w_1w_2. This implies that w_1w_2 is either this new bridge m' or part of a pair of sides for m'. The former case is impossible, because $w_1w_2 > xy$. In the latter case, the new bridge m', which belongs to $I(G_2)$, must be w_2w_3. But this, once again, yields a contradiction, since, in the total order we have fixed, w_2w_3 is the greatest edge monomial of $I(G_2)$, so that $w_2w_3 \geq xy$. Hence this subcase is impossible.

Case 2.2 $sb(\sigma) > w_1w_2$. In this case $sb(\sigma) = sb(\sigma_1)$.

Case 2.2.1 $sb(\sigma') = w_1w_2$. It follows that in σ' there is an edge monomial xw_1 that, together with w_2w_3, forms a pair of sides for the bridge w_1w_2. Once again, by the preliminary remark, w_1w_2 is a gap in σ, between xw_1 and w_2w_3, but cannot be a true gap. Hence the insertion of w_1w_2 into σ produces a new bridge m' smaller than w_1w_2, so that m' must belong to $I(G_2)$. On the other hand, by Remark 2, this new bridge m' contains w_1 or w_2. Hence $m' = w_2w_3$, and, therefore, each pair of sides for m' is formed by w_1w_2 and some edge monomial $w_3x \in I(G_2)$. Now, since $w_2w_3, w_3x \neq sb(\sigma)$, we have that $w_2w_3, w_3x \in \sigma'$. But then in σ' there is the bridge triple (w_1w_2, w_2w_3, w_3x). Thus w_2w_3 is a bridge of σ' smaller than w_1w_2, a contradiction.

Case 2.2.2 $sb(\sigma') = w_2w_3$. Then in σ' there is a bridge triple (w_1w_2, w_2w_3, w_3y), and, by the preliminary remark and Remark 2, inserting w_2w_3 in σ produces a new smaller bridge. This new bridge can only be of the form w_3y, with some edge monomial yz as its side, and, in view of Equation (1), w_3y and yz must belong to σ'. But then the minimality of the bridge w_2w_3 in σ' is contradicted. The present subcase is thus impossible.

Case 2.2.3 $sb(\sigma') < w_2w_3$. Let $xy = sb(\sigma')$, a bridge between ax and yb. Note that $w_2 \notin \{x, y\}$, so that a, b are vertices of G_2. By the preliminary remark and Remark 2, adding xy to σ produces a new bridge $m' < xy$, where m' is the middle term of a bridge triple (ua, ax, xy) or (xy, yb, bv). On the other hand, m' lies in $I(G_2)$, and $m' \neq w_2w_3$. Hence all edges adjacent to m' belong to $I(G_2)$, as well, which implies that ua and bv are both different from $sb(\sigma)$ and therefore $ua \in \sigma'$ or $bv \in \sigma'$. Moreover, the fact that $m' \neq sb(\sigma)$ implies that $m' \in \sigma'$. We thus conclude that σ' contains one of the above bridge triples around m', so that m' is a bridge of σ'. But this contradicts the minimality of $sb(\sigma')$, which shows that this subcase is impossible.

Case 2.2.4 $sb(\sigma') > w_1w_2$. In this case $sb(\sigma') = sb(\sigma_1')$. Recall from Case 2.2 that $sb(\sigma) = sb(\sigma_1)$. Hence, as in Case 1.1.1.1, equality (2) holds. If σ_1 and σ_1' are both potentially-type-2 in $I(G_1)$, then $\sigma_1 = \sigma_1'$, so that $\sigma = \sigma'$, as desired. If σ_1 is not potentially-type-2 in $I(G_1)$, then σ_1 has a true gap smaller than $sb(\sigma_1)$ and not present in σ. Note that this gap is certainly a gap in σ. Hence the insertion of its bridge must produce, in σ, a smaller bridge that is not formed in σ_1. This is possible only if this new bridge has one side in $I(G_2)$, which means that it is w_1w_2. This implies that $w_2w_3 \in \sigma$. Since $w_2w_3 \neq sb(\sigma)$, we thus have that $w_2w_3 \in \sigma'$. On the other hand, $w_1w_2 \in \sigma'$, because $w_1w_2 \in \sigma$ and $w_1w_2 \neq sb(\sigma)$. Hence in σ and in σ' there are w_1w_2 and w_2w_3, but w_1w_2 is not a bridge, nor in σ, nor in σ'. Thus in σ and σ' there are no edge monomials of the form cw_1 with $c \neq w_2$. This implies that w_1w_2 is not part of a pair of sides of any bridge of σ_1 or σ_1'. Therefore, the removal of w_1w_2 from σ_1 or σ_1' preserves all the existing bridges. In particular, if we set $\bar{\sigma}_1 = \sigma \setminus \{w_1w_2\}$ and $\bar{\sigma}_1' = \sigma_1' \setminus \{w_1w_2\}$, we have that $sb(\bar{\sigma}_1) = sb(\sigma_1)$ and $sb(\bar{\sigma}_1') = sb(\sigma_1')$. On the other hand, the removal of w_1w_2 from σ or σ' does not produce any new gap in σ_1 or σ_1'. It follows that $\bar{\sigma}_1$ and $\bar{\sigma}_1'$ are potentially-type-2, as well. Now, from Equation (2) we have $\bar{\sigma}_1 \setminus \{sb(\bar{\sigma}_1)\} = \bar{\sigma}_1' \setminus \{sb(\bar{\sigma}_1')\}$. By virtue of the bridge-friendliness of $I(G_1)$, this implies that $\bar{\sigma}_1 = \bar{\sigma}_1'$, whence $\sigma_1 = \sigma_1'$, and finally, $\sigma = \sigma'$. This settles Case 2.2, Case 2 and completes the proof. □

From [3] we know that $I(G)$ is bridge-friendly if G is acyclic. The unicyclic graphs G for which $I(G)$ is bridge-friendly have been completely characterized:

Theorem 4. *If G is a unicyclic graph (*[4]*, Theorem 4.9), then $I(G)$ is bridge-friendly if and only if one of the following two cases occurs:*

(i) *G contains a 3-cycle or a 5-cycle having a vertex of degree 2;*

(ii) *G contains a 6-cycle having two opposite vertices of degree 2.*

Example 1. *In order to illustrate the notions and results presented above and provide the reader with a concrete example we examine the edge ideal of the 4-cycle on the vertices x_1, x_2, x_3, x_4, i.e., the ideal of the polynomial ring $R = K[x_1, x_2, x_3, x_4]$ defined as $I(C_4) = (x_1x_2, x_2x_3, x_3x_4, x_4x_1)$. We also fix the following total order on its edge monomials: $x_1x_4 > x_1x_2 > x_2x_3 > x_3x_4$. We then consider the symbols $\sigma = (x_1x_2, x_2x_3, x_3x_4)$ and $\sigma' = (x_1x_4, x_1x_2, x_3x_4)$. Note that the edge monomial*

x_2x_3 is the only bridge of σ, so that $sb(\sigma) = x_2x_3$ and, similarly, $sb(\sigma') = x_1x_4$. Moreover, σ is potentially-type-2, because the only edge monomial smaller that x_2x_3 is x_3x_4, which belongs to σ, and thus is not a gap. Also σ' is potentially-type-2, but for a different reason. The edge monomial missing in σ', namely x_2x_3, is a gap between x_1x_2 and x_3x_4, and it is smaller that x_1x_4. It, however, is not a true gap: this can be easily seen, since adding x_2x_3 to σ' causes x_3x_4 to become a new bridge. Now we have that

$$\sigma \setminus \{sb(\sigma)\} = \sigma' \setminus \{sb(\sigma')\} = (x_1x_2, x_3x_4).$$

Since $sb(\sigma) < sb(\sigma')$, it follows that σ' is not type-2. This implies that, with respect to the given total order, the ideal $I(C_4)$ is not bridge-friendly. Actually, from Theorem 4 we know that $I(C_4)$ is not bridge-friendly with respect to any total order.

Further computations yield the following list of type-1 and type-2 symbols:

- $(x_1x_4, x_1x_2, x_2x_3, \underline{x_3x_4})$;
- $(\boldsymbol{x_1x_4}, x_1x_2, \boldsymbol{x_2x_3})_{\boxed{x_3x_4}}$, $(x_1x_4, x_2x_3, \underline{x_3x_4})$, $(x_1x_2, \underline{x_2x_3}, x_3x_4)$;
- $(\boldsymbol{x_1x_4}, \boldsymbol{x_2x_3})_{\boxed{x_3x_4}}$, $(\boldsymbol{x_1x_2}, \boldsymbol{x_3x_4})_{\boxed{x_2x_3}}$.

The sides of the true gaps (in the type-1 symbols) are in boldface, whereas the smallest bridges (in the type-2 symbols) are underlined. The smallest true gaps of the type-1 symbols are given in the boxed subscripts. The remaining symbols are the Barile-Macchia-critical ones, and, according to Theorem 1, those of length i are the generators of the ith module of a graded free resolution of $I(C_4)$ over R:

- (x_1x_4, x_1x_2, x_3x_4);
- (x_1x_4, x_1x_2), (x_1x_4, x_3x_4), (x_1x_2, x_2x_3), (x_2x_3, x_3x_4);
- (x_1x_4), (x_2x_3), (x_2x_3), (x_3x_4).

Note that the first symbol is σ', which has no true gaps, and therefore is not type-1, whereas the remaining symbols have no bridges and no gaps.

The resulting resolution turns out to be minimal:

$$0 \longrightarrow R(-4) \longrightarrow R(-3)^4 \longrightarrow R(-2)^4 \longrightarrow I(C_4) \longrightarrow 0.$$

This example shows that the converse of Theorem 2 is not true: bridge-friendliness is only a sufficient, not a necessary condition for the existence of a minimal Barile-Macchia resolution.

4. Final remarks

Theorem 4, together with our Theorem 3, allows us to construct, for any positive integer n, a graph G with n cycles for which $I(G)$ is bridge-friendly. The definition of dot product is simple, and can be easily applied in a recursive procedure. It requires that the vertices of G_1 and G_2 to be identified are both leaves, but this assumption is not to be viewed as a limit, since it cannot be removed without losing bridge-friendliness. Thanks to the classification presented in ([4], Proposition 4.3), various counterexamples are at hand. One can, e.g., consider the case where G_1 is the graph  (a 3-cycle with two spikes) and G_2 has one single edge. Then $I(G_1)$ is bridge-friendly by Theorem 4, but the same result tells us that the edge ideal of the graph  (a 3-cycle with three spikes) is not.

One reasonable conjecture, which is not contradicted by any of the non-bridge-friendly graphs considered in [4], is the following: the dot product of two bridge-friendly graphs is still bridge-friendly, if the two vertices that are identified do not belong to any cycle. This result, which would arise as a natural generalization of our Theorem 3, would uncover a huge class of bridge-friendly graphs, all accessible through elementary constructions. Moreover, it would immediately provide a new, easy inductive proof of the bridge-friendliness of acyclic graphs. It could also open the way to a combinatorial characterization of bridge-friendliness.

Conflict of interest: The author declares no conflict of interest.

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