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Article

Distribution of lattice points in the shifted balls

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Abstract: In this work we study the mean value of the difference between the number of integer points and the volume of a ball as a function of the center of a ball in the unit cube $[0, 1]^3$, applying new method. This mean value is estimated by its possible exact value. Using methods of Fourier analysis, we lead the question to the estimates of double trigonometric integrals. This method allows consider the question on lattice points in domains of arbitrary nature without any symmetry.

Keywords: lattice points; shifted ball; Fourier series; trigonometric integral; eigenvalue

1. Introduction

Through the paper we shall use following notations.

The expression $A << B$ (called the Vinogradov's symbol) for quantities A and positive B means that there exists a constant c that $A \leq cB$. In some cases it is equivalent to the symbol $A = O(B)$.

The symbol Ω for real functions $F(t)$ and $f(t)$ (which is positive) is the negation of the symbol o. So, the notation $F(t) = \Omega(f(t))$ means that there exists a positive constant A such that $F(t) > Af(t)$, as $t \to \infty$.

The expression $F(t) = \Omega_{\pm}(f(t))$ means that there are sequences $x_n \to +\infty$x and $y_m \to +\infty$ such that for positive constant A the relations both $F(x_n) > Af(x_n)$ and $F(y_m) < -Af(y_m)$ are satisfied for all n and m, large enough.

In Number Theory one studies the question on the number of lattice points enclosed into various domains of multidimensional spaces. Let $r(n)$ denote the number of representations of the natural number n in the form

$$n = x^2 + y^2 + z^2,$$

with integral numbers x, y, z. Then the function $T(N)$ defined as

$$\mathrm{T}(N) = \sum_{m=0}^{N} r(n)$$

expresses the number of lattice points in the sphere $x^2 + y^2 + z^2 = N$. From geometric arguments it is clear that the approximate value for the number $T(N)$ will be volume of the ball enclosed in the taken sphere. Denote

$$\mathrm{R} = R(r) = T(r^2) - \frac{4}{3}\pi r^3.$$

The question on estimation of this variance is known as Sphere Problem. From the result of Gauss on lattice points in a circle it follows that $R << r^2$.

In 1935 I. M. Vinogradov using transformation of trigonometric sums found an estimation (see [1])

$$R << r^{1.4+\varepsilon},$$

with arbitrary small constant $\varepsilon > 0$,as r is large. This result was improved by him [2], and independently by Chen [3]. They have proved that

$$R << r^{4/3+\varepsilon}.$$

Further error term was improved as indicated below:
$R << r^{\frac{29}{22}+\varepsilon}$, by Chamizo and Ivaniec [4],
$R << r^{\frac{21}{16}+\varepsilon}$, by Heath-Brown [5],
$R << r^{\frac{17}{14}+\varepsilon}$, by Arkhipova [6].
In 1926 Szego G. proved that $R = \Omega(r\sqrt{log\, r})$. Tsang [7] has shown that

$$\mathrm{R} = \Omega_{\pm}(r\, log^{1/2}\, r).$$

There is a conjecture [2,6] which states that

$$R << r^{1+\varepsilon}. \tag{1}$$

2. Materials and methods

Besides, in series of works [8–10] the problem was studied from other points of view. Authors of those works investigated the fluctuations in the number $N_{\alpha}(r)$ of lattice points reminder term inside a sphere of radius r centered at a point $\alpha \in [0,1)^3$ different from the origin. VinogradovA and SkriganovM [8] got Ω-type results. Despite that in two dimensional case it was established that the relative deviation tends, as radius grows unboundedly, to some mean value which is an absolutely continuous function of the center of a circle, the problem is very difficult in three dimensional case. In the work [9] a similar result was established for shifted balls when a center of a ball satisfies some Diophantine conditions (which is generically fulfilled). In the work [11] it was considered the question on lattice points in the shifted ellipsoids in dimensions $d \geq 8$.

In this work we study the question by applying new estimates for trigonometric integrals. This method is useful for other similar problems to which the methods of indicated works are not applicable.

3. Results and discussion

Denote by $N(r;\theta,\eta,\xi)$ the number of lattice points in the sphere of radius r and the center at the point $(\theta,\eta,\xi) \in [0,1)^3$.

Theorem 1. *Following inequality holds for sufficiently large* $r > 0$:

$$\int_0^1\int_0^1\int_0^1 \left|N(r;\theta,\eta,\xi) - \frac{4}{3}\pi r^3\right|^2 d\theta d\eta d\xi << r^2\, log^4\, r.$$

Corollary 1. *There exists a point* $(\theta,\eta,\xi) \in [0,1)^3$ *for which*

$$\left|N(r;\theta,\eta,\xi) - \frac{4}{3}\pi r^3\right| << r\, log^2\, r.$$

3.1. Some auxiliary lemmas

The statement of the following lemma is known as Sonin's formula [12].

Lemma 1. *Let a and b bereal numbers* $a < b$. *Then the equality*

$$\sum_{a<n\leq b} f(n) = \int_a^b f(x)dx + \rho(b)f(b) - \rho(a)f(a) - \sigma(b)f'(b) + +\sigma(a)f'(a) + \int_a^b f''(x)\sigma(x)dx,$$

holds, where $f(x)$ *is a function defined in the interval* $(a, b]$ *having continuous derivative of second order,*

$$\rho(x) = \{x\} - 1/2, \sigma(x) = \int_0^x \rho(t)dt.$$

Lemma 2. *The function* $\rho(x)$ *has following Fourier expansion*

$$\rho(x) \sim \sum_{m=-\infty, m\neq 0}^{\infty} g_m e^{2\pi i m x}$$

with $g_m = -1/(2\pi i m)$.

This lemma is evident.

Lemma 3. *Let the function* $f'(x) \geq \delta > 0$ *be monotonically non-increasing in* $[a, b]$, *and* $f'(x) \geq \delta > 0$ *in this segment. Then*

$$\left|\int_a^b e^{2\pi i f(x)} dx\right| \leq 4\delta^{-1}.$$

Lemma 4. *Let*

$$f''(x) \geq A > 0$$

function $f(x)$ *at the interval [a, b]. Then*

$$\left|\int_a^b e^{2\pi i f(x)} dx\right| \leq 12A^{-1/2}.$$

Let's denote by $M(r; \theta, \eta)$ *the number of lattice points in the circle* $(x+\theta)^2 + (y+\eta)^2 \leq r^2$.

Lemma 5. *We have:*

$$\mathrm{M}(r; \theta, \eta) = \pi r^2 + \sum_{-\frac{r}{\sqrt{2}}-\theta<n<\frac{r}{\sqrt{2}}-\theta} \left(\frac{1}{2} - \left\{-\eta + \sqrt{r^2 - (n+\theta)^2}\right\}\right) +$$
$$+\sum_{-\frac{r}{\sqrt{2}}-\theta<n<\frac{r}{\sqrt{2}}-\theta} \left(\frac{1}{2} - \left\{\eta + \sqrt{r^2 - (n+\theta)^2}\right\}\right) +$$
$$\sum_{-\frac{r}{\sqrt{2}}-\eta<n<\frac{r}{\sqrt{2}}-\eta} \left(\frac{1}{2} - \left\{-\theta + \sqrt{r^2 - (n+\eta)^2}\right\}\right) +$$
$$+\sum_{-\frac{r}{\sqrt{2}}-\eta<n<\frac{r}{\sqrt{2}}-\eta} \left(\frac{1}{2} - \left\{\theta + \sqrt{r^2 - (n+\eta)^2}\right\}\right) + \delta_0(r; \theta, \eta); |\delta_0| \leq 3.$$

This Lemma is a consequence of the Sonin's formula [12].

3.2. Lattice points in shifted balls

The Sphere Problem is consisted in finding of asymptotic relation for the number $N(r)$ of lattice points in the sphere

$$x^2 + y^2 + z^2 \leq r^2,$$

as $r \to \infty$, with the possible better error term. The same problem we shall investigate in sifted spheres.

Consider a shifted ball

$$(x+\theta)^2+(y+\eta)^2+(z+\xi)^2 \le r^2$$

with a center $(-\theta,-\eta,-\xi) \in [-1,0]^3$. If we intersect the ball by the plane $z=k \in Z$, we get a disc at the section. The number $N(r;\theta,\eta,\xi)$ of all lattice points in the ball is represented as a sum of number of lattice points laying in all of these discs, as $z=k \in Z$ takes such values for which the sections mentioned above is not empty (**Figure 1**).

Denote by $N_k(r;\theta,\eta,\xi)$ the number of lattice points in the ball lying on the section. Then we have

$$N(r;\theta,\eta,\xi)=\sum_{k=-[r+\xi]}^{[r+\xi]+1} N_k(r;\theta,\eta,\xi).$$

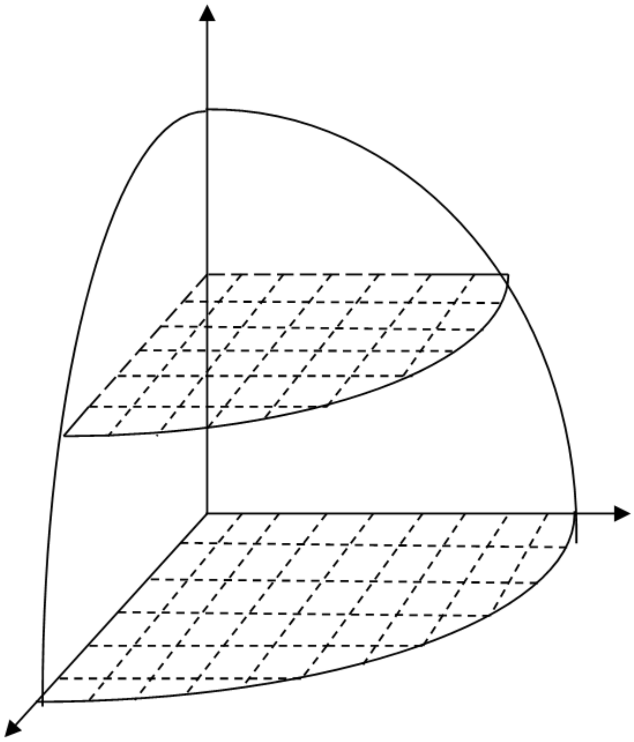

Figure 1. Sections.

The section of the ball by a plane $z=k$ is a disc of a radius $\sqrt{r^2-k^2}$. By the lemma 1, one deduces:

$$N_k(r;\theta,\eta,\xi)=\pi(r^2-(k+\xi)^2)+$$

$$+\sum_{-\frac{\sqrt{r^2-(k+\xi)^2}}{\sqrt{2}}-\theta<n<\frac{\sqrt{r^2-(k+\xi)^2}}{\sqrt{2}}-\theta}\left(\frac{1}{2}-\left\{-\eta+\sqrt{r^2-(k+\xi)^2-(n+\theta)^2}\right\}\right)+$$

$$+\sum_{-\frac{\sqrt{r^2-(k+\xi)^2}}{\sqrt{2}}-\theta<n<\frac{\sqrt{r^2-(k+\xi)^2}}{\sqrt{2}}-\theta}\left(\frac{1}{2}-\left\{\eta+\sqrt{r^2-(k+\xi)^2-(n+\theta)^2}\right\}\right)+$$

$$+\sum_{-\frac{\sqrt{r^2-(k+\xi)^2}}{\sqrt{2}}-\eta<m<\frac{\sqrt{r^2-(k+\xi)^2}}{\sqrt{2}}-\eta}\left(\frac{1}{2}-\left\{-\theta+\sqrt{r^2-(k+\xi)^2-(m+\eta)^2}\right\}\right)+$$

$$+\sum_{-\frac{\sqrt{r^2-(k+\xi)^2}}{\sqrt{2}}-\eta<m<\frac{\sqrt{r^2-(k+\xi)^2}}{\sqrt{2}}-\eta}\left(\frac{1}{2}-\left\{\theta+\sqrt{r^2-(k+\xi)^2-(m+\eta)^2}\right\}\right)+$$

$$+\delta_0(r;\theta,\eta,\xi);\ |\delta_0|\le 3.$$

Then we receive (since we shall integrate over ξ, one can assume $-\xi \pm r$ non-integral), using Sonin's formula:

$$\sum_{-r-\xi\le k\le r-\xi}\pi(r^2-(k+\xi)^2)=\int_{-r-\xi}^{r-\xi}\pi(r^2-(z+\xi)^2)\,dz+O(r)=$$

$$=2\pi r^3-\pi\frac{(x+\xi)^3}{3}\Big|_{-r-\xi}^{r-\xi}+O(r)=\frac{4\pi r^3}{3}+O(r).$$

So,

$$\left|N(r;\theta,\eta,\xi)-\frac{4\pi r^3}{3}\right|\le O(r)+$$

$$+\left|\sum_{-r-\xi\le k\le r-\xi}\quad\sum_{-\frac{\sqrt{r^2-(k+\theta)^2}}{\sqrt{2}}-\theta<n<\frac{\sqrt{r^2-(k+\theta)^2}}{\sqrt{2}}-\theta}\left(\frac{1}{2}-\left\{-\eta+\sqrt{r^2-(k+\xi)^2-(n+\theta)^2}\right\}\right)\right|+$$

$$+\left|\sum_{-r-\xi\le k\le r-\xi}\quad\sum_{-\frac{\sqrt{r^2-(k+\theta)^2}}{\sqrt{2}}-\theta<n<\frac{\sqrt{r^2-(k+\theta)^2}}{\sqrt{2}}-\theta}\left(\frac{1}{2}-\left\{\eta+\sqrt{r^2-(k+\xi)^2-(n+\theta)^2}\right\}\right)\right|+$$

$$+\left|\sum_{-r-\xi\le k\le r-\xi}\quad\sum_{-\frac{\sqrt{r^2-(k+\theta)^2}}{\sqrt{2}}-\eta<n<\frac{\sqrt{r^2-(k+\theta)^2}}{\sqrt{2}}-\eta}\left(\frac{1}{2}-\left\{-\theta+\sqrt{r^2-(k+\xi)^2-(n+\eta)^2}\right\}\right)\right|+$$

$$+\left|\sum_{-r-\xi\le k\le r-\xi}\quad\sum_{-\frac{\sqrt{r^2-(k+\theta)^2}}{\sqrt{2}}-\eta<n<\frac{\sqrt{r^2-(k+\theta)^2}}{\sqrt{2}}-\eta}\left(\frac{1}{2}-\left\{\theta+\sqrt{r^2-(k+\xi)^2-(n+\eta)^2}\right\}\right)\right|.$$

3.3. Completion of Theorem's proof

Estimate now the integral

$$\int_0^1\int_0^1\int_0^1\left|N(r;\theta,\eta,\xi)-\frac{4\pi r^3}{3}\right|^2 d\theta d\eta d\xi$$

noting that the all moduli at the right hand side of the previous inequality allows a similar estimation in average. Consequently, we have:

$$\int_0^1\int_0^1\int_0^1\left|N(r;\theta,\eta,\xi)-\frac{4\pi r^3}{3}\right|^2 d\theta d\eta d\xi << r^2+$$

$$+\int_0^1\int_0^1\int_0^1\left|\sum_{-r-\xi\le k\le r-\xi}\sum_{-\frac{\sqrt{r^2-(k+\xi)^2}}{\sqrt{2}}-\theta<n<\frac{\sqrt{r^2-(k+\xi)^2}}{\sqrt{2}}-\theta}\left(\frac{1}{2}-\left\{-\eta+\sqrt{r^2-(k+\xi)^2-(n+\theta)^2}\right\}\right)\right|^2 d\theta d\eta d\xi \qquad (2)$$

Note that under the integral one may suffice with summation over the intervals $r-r^{1/3}-1\le k\le r-1 \vee -r+1\le k\le -r+r^{1/3}+1$, and an error could estimated as

$$<< r + \sum_{|r-k|\le r^{1/3}+2} \sum_{-\frac{\sqrt{r^2-(k+\xi)^2}}{\sqrt{2}}-\theta<n<\frac{\sqrt{r^2-(k+\xi)^2}}{\sqrt{2}}-\theta} 1 << r^{1/3}\sqrt{rr^{1/3}} = r,$$

for sufficiently large r, which is acceptable. When $|k| \le r - r^{1/3} + 1$, by the theorem of Lagrange on finite increments, for some $0 < \mu < \xi$ the inequalities below hold:

$$\left|\frac{\sqrt{r^2-(k+\xi)^2}-\sqrt{r^2-k^2}}{\sqrt{2}}\right| \le \frac{(k+\mu)\xi}{\sqrt{2}\sqrt{r^2-(k+\mu)^2}} \le \frac{r}{\sqrt{r(r-k-\mu)}}.$$

Therefore, from the inequality (2) and said above one deduces

$$\int_0^1\int_0^1\int_0^1 \left|N(r;\theta,\eta,\xi) - \frac{4\pi r^3}{3}\right|^2 d\theta d\eta d\xi <<$$

$$<< r^2 + \int_0^1\int_0^1\int_0^1 |\phi(\theta,\eta,\xi)|^2\, d\theta d\eta d\xi,$$

where,

$$\phi(\theta,\eta,\xi) = \sum_{-r+r^{1/3}-1\le k\le r-r^{1/3}+1} \sum_{-\frac{\sqrt{r^2-k^2}}{\sqrt{2}}<n<\frac{\sqrt{r^2-k^2}}{\sqrt{2}}} \left(\frac{1}{2} - \left\{-\eta + \sqrt{r^2-(k+\xi)^2-(n+\theta)^2}\right\}\right) \quad (3)$$

Now we modify the function represented by double sum on the right hand side of the last equality as follows

$$\phi_0(\theta,\eta,\xi) = \frac{1}{(2\delta)^3}\int_{\theta-\delta}^{\theta+\delta}\int_{\eta-\delta}^{\eta+\delta}\int_{\xi-\delta}^{\xi-\delta} \phi(u,v,w)\, dudvdw. \quad (4)$$

Obviously, that the limits of summations in (3) ensures, at large values of r, the function (4) be defined for all $(\theta,\eta,\xi) \in [0,1]^3$. The function $\rho\left(-\eta + \sqrt{r^2-(k+\xi)^2-(n+\theta)^2}\right)$ is partially-continuous function, and for sufficiently small $\delta > 0$ (its exact value we shall define below) at the points of continuity (θ,η,ξ), for which a cube with the center at this point having an edge 2δ doesn't contain points of discontinuity, we have

$$|\phi_0(\theta,\eta,\xi) - \phi(\theta,\eta,\xi)| =$$

$$= \left|\frac{1}{(2\delta)^3}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}(\phi(\theta+u,\eta+v,\xi+w) - \phi(\theta,\eta,\xi))\, dudvdw\right| \le$$

$$\frac{1}{(2\delta)^3}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}|\phi(\theta+u,\eta+v,\xi+w) - \phi(\theta,\eta,\xi)|\, dudvdw.$$

Let's denote $K = \{(u,v,w)||u-\theta| \le \delta, |v-\eta| \le \delta, |w-\xi| \le \delta\}$, for every (θ,η,ξ). By the theorem on finite increments

$$|\phi(\theta+u,\eta+v,\xi+w) - \phi(\theta,\eta,\xi)| \le \Delta\delta,$$

where,

$$\Delta \le \max_{(u,v,w)\in K}(|\partial\phi/\partial u| + |\partial\phi/\partial v| + |\partial\phi/\partial w|) << r^2.$$

So, for the set of points of continuity, doesn't containing points from neighborhoods of a type K with the center at points of discontinuity, the relation below is satisfied:

$$\phi_0(\theta,\eta,\xi) = \phi(\theta,\eta,\xi) + O(r^2\delta). \tag{5}$$

Estimate the measure of a union of all neighborhoods of a view K for points of discontinuity (θ,η,ξ) in the unite cube $[0,1]^3$. The points of discontinuity of the function $\rho\left(-\eta+\sqrt{r^2-(k+\xi)^2-(n+\theta)^2}\right)$ are the points where the number $-\eta+\sqrt{r^2-(k+\xi)^2-(n+\theta)^2}$ is an integral number:

$$-\eta+\sqrt{r^2-(k+\xi)^2-(n+\theta)^2} = m$$

Since $(\theta,\eta,\xi) \in [0,1]^3$, then for the given lattice point (n,m,k) near the sphere points of discontinuity are placed at the intersection of the sphere $x^2+y^2+z^2=r^2$ with the cube $(n,m,k)+[0,1]^3$. The measure of a union of all K -neighborhoods of the points of discontinuity for every such cube is $O(\delta)$. So, desired measure of a union of all K -neighborhoods for the points of discontinuity in the unite cube $[0,1]^3$ is of $O(r^2\delta)$. For points of this set we have

$$|\phi_0(\theta,\eta,\xi) - \phi(\theta,\eta,\xi)| << r^2.$$

All of reasoning above yields:

$$\int_0^1\int_0^1\int_0^1 \left|N(r;\theta,\eta,\xi)-\frac{4\pi r^3}{3}\right|^2 d\theta d\eta d\xi << r^2 + \int_0^1\int_0^1\int_0^1 |\phi(\theta,\eta,\xi)|^2\, d\theta d\eta d\xi <<<< r^2 +$$
$$\int_0^1\int_0^1\int_0^1 (|\phi_0(\theta,\eta,\xi)|^2 + |\phi_0(\theta,\eta,\xi)-\phi(\theta,\eta,\xi)|^2)\, d\theta d\eta d\xi <<<< r^2 + r^4\delta^2 + r^6\delta + \tag{6}$$
$$\int_0^1\int_0^1\int_0^1 |\phi_0(\theta,\eta,\xi)|^2\, d\theta d\eta d\xi.$$

Applying Parseval's identity we get:

$$\int_0^1\int_0^1\int_0^1 |\phi_0(\theta,\eta,\xi)|^2\, d\theta d\eta d\xi = \sum_{q=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} \left|c_{pqs}\right|^2 \tag{7}$$

here

$$c_{pqs} = \int_0^1\int_0^1\int_0^1 \phi_0(\theta,\eta,\xi) e^{-2\pi i(p\theta+q\eta+s\xi)}\, d\theta d\eta d\xi.$$

We have to estimate the multiple series on the right hand side of the equality (7).

Let's consider first an expansion of the function $\phi(\theta,\eta,\xi)$ into trigonometric series. We have

$$\phi(\theta,\eta,\xi) \sim \sum_{q=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} b_{pqs}\, e^{2\pi i(p\theta+q\eta+s\xi)};$$

here

$$b_{pqs} = \int_0^1\int_0^1\int_0^1 \phi(\theta,\eta,\xi) e^{-2\pi i(p\theta+q\eta+s\xi)}\, d\theta d\eta d\xi.$$

Dissect inner double sum under the multiple integral in the expression (3), splitting the interval of summation over k into the union of sub-intervals (in which $|r-k|$ varies) of a view

$$I_j = \left(2^{j-1}r^{1/3}, 2^j r^{1/3}\right], j = 1,\ldots,J,$$

where J is a largest natural number for which $2^J r^{1/3} \leq r/\sqrt{2}$. For $j = 0$ we put $J_0 = [r/\sqrt{2}, r]$. So, the number of such sub-intervals is $O(\log r)$. Then we get

$$b_{pqs} = b_{pqs}^0 + b_{pqs}^1 + \cdots + b_{pqs}^J, \tag{8}$$

And

$$b_{pqs}^j = \int_0^1 \int_0^1 \int_0^1 \left(\sum_{|r-k| \in I_j} \sum_{-\frac{\sqrt{r^2-k^2}}{\sqrt{2}} < n < \frac{\sqrt{r^2-k^2}}{\sqrt{2}}} \rho\left(-\eta + \sqrt{r^2 - (k+\xi)^2 - (n+\theta)^2}\right) \right) e^{-2\pi i (p(n+\theta) + q\eta + s(k+\xi))}\, d\theta d\eta d\xi = \int_0^1 d\eta \iint_{\Pi_j} \left(\frac{1}{2} - \left\{-\eta + \sqrt{r^2 - z^2 - x^2}\right\}\right) e^{-2\pi i (pz + q\eta + sx)} dxdy \tag{9}$$

and Π_j is a union of all unite quadrates, every of which has left lower vertex at the point(k, n) with $|r-k| \in I_j, -\frac{\sqrt{r^2-k^2}}{\sqrt{2}} < n < \frac{\sqrt{r^2-k^2}}{\sqrt{2}}$. By the lemma 2,

$$b_{pqs}^j = \sum_{m=-\infty, m\neq 0}^{\infty} g_m \int_0^1 d\eta \iint_{\Pi_j} e^{2\pi i m\left(-\eta + \sqrt{r^2-z^2-x^2}\right)} e^{-2\pi i (p\xi + q\eta + s\theta)}\, d\theta d\eta d\xi = \\ = g_q \iint_{\Pi_j} e^{2\pi i (q\sqrt{r^2-z^2-y^2} - pz - sx)}\, dxdy. \tag{10}$$

Recall now definition (4). It is evident that the coefficients c_{pqs} of the expansion of the function.

$\phi_0(\theta, \eta, \xi)$ into Fourier series can be expressed as below:

$$c_{pqs} = \frac{\sin p\,\delta}{p\delta} \frac{\sin q\,\delta}{q\delta} \frac{\sin s\,\delta}{s\delta} b_{pqs} \tag{11}$$

Let's denote

$$\mathrm{F}(z, x) = q\sqrt{r^2 - z^2 - x^2} - pz - sx.$$

We have

$$\partial \mathrm{F}/\partial z = -\frac{qz}{\sqrt{r^2-z^2-x^2}} - p, \partial F/\partial x = -\frac{qx}{\sqrt{r^2-z^2-x^2}} - s.$$

Since for $|r-k| \in I_j$ with $2^{j-1} r^{1/3} = K_j$ we have $K_j \leq |r-z| \leq 2K_j$ and $|x| \leq \frac{\sqrt{r^2-z^2}}{\sqrt{2}}$, then one has

$$\frac{qz}{\sqrt{r^2-z^2-x^2}} \leq \frac{qr}{\sqrt{(r^2-z^2)/2}} \leq \frac{\sqrt{2}qr}{\sqrt{rK_j}} \leq 2qr^{1/2} K_j^{-1/2},$$

$$\frac{qx}{\sqrt{r^2-z^2-x^2}} \leq \frac{q\sqrt{(r^2-z^2)/2}}{\sqrt{(r^2-z^2)/2}} \leq q.$$

When $|r-k| \in I_j$, and $|p| \geq 3qr^{1/2} K_j^{-1/2}$ or $|s| \geq 3qr^{1/2} K_j^{-1/2}$, we can estimate the integral (10) by taking of repeated integration, and applying the lemma 3, as follows:

$$\iint_{\Pi_j} e^{2\pi i F(z,x)} dzdx << \min(r|p|^{-1}, r|s|^{-1}) \tag{12}$$

Estimate now the integral over Π_j when both conditions $|p| \leq 3qr^{1/2}K_j^{-1/2}$ and $|s| \leq 3qr^{1/2}K_j^{-1/2}$ are satisfied. For this purpose we first transform the domain Π_j by such way that its boundary stands smooth. Namely, for each $z, |r-z| \in I_j$ we let x to vary in the interval

$$-\frac{\sqrt{r^2-z^2}}{\sqrt{2}} < x < \frac{\sqrt{r^2-z^2}}{\sqrt{2}}.$$

After of such transformation there arise an error of order $W_j = O(K_j)$. So,

$$\iint_{\Pi_j} e^{2\pi i F(z,x)}\, dzdx = \iint_{P_j} e^{2\pi i F(z,x)}\, dzdx + W_j,$$

where $P_j = \left\{(z,x) | K_j < |r-z| \leq 2K_j, -\frac{\sqrt{r^2-z^2}}{\sqrt{2}} < x < \frac{\sqrt{r^2-z^2}}{\sqrt{2}}\right\}$ and

$$\begin{aligned} W_j &= \iint_{\Pi_j} e^{2\pi i F(z,x)} dzdx - \iint_{P_j} e^{2\pi i F(z,x)} dzdx = \\ &= \iint_{\Pi_j \setminus P_j} e^{2\pi i F(z,x)} dzdx - \iint_{P_j \setminus \Pi_j} e^{2\pi i F(z,x)} dzdx \end{aligned} \tag{13}$$

Consider first the case of large gradients, i.e., the case when $\sqrt{(\partial F/\partial z)^2 + (\partial F/\partial x)^2} \geq H$, for some positive H. Applying corollary to the lemma 1 of the work [13] one can write

$$\iint_{P_j} e^{2\pi i F(z,x)}\, dzdx = \int_m^M e^{2\pi i u} du \int_{F(z,x)=u} \frac{ds}{\sqrt{G}}; \tag{14}$$

here m and M mean a minimal and maximal values of the function $F(z,x)$ correspondingly, and $G = (\partial F/\partial x)^2 + (\partial F/\partial y)^2$. Consider equation $F(z,x) = u$ which is explicitly written as below

$$q\sqrt{r^2 - z^2 - x^2} - pz - sx = u,$$

or

$$q^2 r^2 = u^2 + (p^2 + q^2) z^2 + (s^2 + q^2) x^2 + 2puz + 2sux + 2pszx$$

Applying the lemma 3 ([14] (p. 93)), we find such a dissection of the domain P_j into finite number of sub-domains $P_{j1}, P_{j2}, \ldots, P_{jt}$ (t doesn't depend of r) for every of which the surface integral

$$\int_{P_{ji}, F(z,x)=u} \frac{ds}{\sqrt{G}} \tag{15}$$

is a monotone function in u. Then applying second mean value theorem for the integral (14), one gets

$$\iint_{P_j} e^{2\pi i F(z,x)}\, dzdx << \max_{u,i,j} \int_{P_{ji}, F(z,x)=u} \frac{ds}{\sqrt{G}}.$$

Make exchange of variables under the integral (15) by formulae

$$\xi = \partial F/\partial z, \omega = \partial F/\partial \omega.$$

Then lemma 2 of the work [14] (p. 92) gives

$$\int_{P_{ji},F(z,x)=u} \frac{ds}{\sqrt{G}} = \int_{\Pi} \frac{|det\, B| d\sigma}{\sqrt{G'}};$$

Here Π is a pre-image of the domain P_{ij} at taken transformation, $d\sigma$ means a surface element at the surface (line) which is a pre-image of the considered surface (line). Jacobi matrix of the tran-sformation is

$$B^{-1} = A = q(r^2 - z^2 - x^2)^{-3/2} \begin{pmatrix} -r^2 + z^2 & zx \\ zx & -r^2 + x^2 \end{pmatrix},$$

and

$$G' = (B^T \cdot B\nabla, \nabla); \nabla = (\partial F/\partial z, \partial F/\partial x)$$

is a quadratic form. Denoting by λ a minimal eigenvalue of the matrix $B^T \cdot B$. we have

$$G' = (B^T \cdot B\nabla, \nabla) \geq \lambda(\nabla, \nabla).$$

Let $L = \max_{(z,x)\in P_{ij}} \|\nabla\|$. Dissect the interval (H, L) into no more than $O(log(L/H))$ subintervals of a view $2^h H \leq \|\nabla\| \leq 2^{h+1} H$. Since $\lambda^{-1} \leq \|A\|$, then one may observe that

$$\int_{\substack{P_{ji},F(z,x)=u, \\ 2^h H \leq \|\nabla\| \leq 2^{h+1} H}} \frac{ds}{\sqrt{G}} \leq \int_{P_{ji}, 2^h H \leq \sqrt{\xi^2+\omega^2} \leq 2^{h+1} H} \frac{|det\, B| d\sigma}{2^h H \sqrt{\lambda}} <<$$

$$<< 2\pi \left(\max_{P_{ij}} \|A\| (det\, A)^{-1} \right) \int_{F(z,x)=0, 2^h H \leq \sqrt{\xi^2+\omega^2} \leq 2^{h+1} H} \frac{d\sigma}{2^h H},$$

where $\|A\|$means Euclidean norm of the matrix A. Since

$$det \begin{pmatrix} -r^2 + z^2 & zx \\ zx & -r^2 + x^2 \end{pmatrix} = r^2(r^2 - z^2 - x^2),$$

then we have following estimation:

$$\|A\| (det\, A)^{-1} << \frac{(r^2 - z^2 - x^2)^2}{q^2 r^2} q(r^2 - z^2 - x^2)^{-3/2} r^2 << \sqrt{rK_j} q^{-1}.$$

Do not destroying the generality, we assume that the mapping $\xi = \partial F/\partial z, \omega = \partial F/\partial x$ is bijective in Π_j. Let $z = z(\xi, \omega), y = y(\xi, \omega)$ is an inverse mapping. The equation $F(z, x) = u$equivalently defines a line

$$F(z(\xi, \omega), x(\xi, \omega)) = f(\xi, \omega) = u$$

along which is taken the surface integral:

$$\int_{F(z,x)=0, 2^h H \leq \sqrt{\xi^2+\omega^2} \leq 2^{h+1} H} \frac{d\sigma}{2^h H}$$

We can dissect the line defined by the equation above into no more that finite number of parts in each of which one of partial derivatives of the function, say $\left(\frac{\partial f}{\partial \xi}\right)$

takes values such that $\left(\frac{\partial f}{\partial \xi}\right) \geq \left(\frac{\partial f}{\partial \omega}\right)$, we can estimate the surface integral applying the result proven in [14] (p. 91):

$$\mathrm{d}\sigma = \left(\frac{\partial f}{\partial \xi}\right)^{-1} \sqrt{\left(\frac{\partial f}{\partial \xi}\right)^2 + \left(\frac{\partial f}{\partial \omega}\right)^2}\, d\xi$$

as below

$$\int_{F(z,x)=u, 2^h H \leq \sqrt{\xi^2+\omega^2} \leq 2^{h+1}H} \frac{d\sigma}{2^h H} << \frac{1}{2^h H} \int_0^{2^{k+1}H} d\xi << 1.$$

So, we get the bound applying the mentioned above estimation

$$\iint_{\Pi_j} e^{2\pi i F(x,y)}\, dxdy << \sqrt{rK_j} q^{-1}\, log(\, L/H). \tag{16}$$

Consider now the case of small gradient, i.e., the part of the integral taken over the sub-domain Π of Π_j where $\sqrt{(\partial F/\partial z)^2 + (\partial F/\partial x)^2} \leq H$, for some positive H. Making exchange of variables $\xi = \partial F/\partial z, \omega = \partial F/\partial x$ we estimate the integral trivially

$$\iint_{\Pi} e^{2\pi i F(z,x)}\, dzdx = \iint_{\Pi, \xi^2+\omega^2 \leq H^2} (det\, A)^{-1} d\xi d\omega \leq \pi H^2 G_0^{-1},$$

where G_0 means a minimal value of the determinant $det\, A$. As it was shown above

$$det\, A = q^2(r^2 - z^2 - x^2)^{-2} r^2 >> q^2 K_j^{-2}.$$

So, in the case of small gradients we get

$$\iint_{\Pi} e^{2\pi i F(z,x)}\, dzdx << H^2 K_j^2 q^{-2}.$$

Taking $H = r^{1/4} K_j^{-3/4} q^{1/2}$ we get a final estimation

$$\iint_{\Pi_j} e^{2\pi i F(x,y)}\, dxdy << \sqrt{rK_j} q^{-1}\, log\, r.$$

Hence for b_{pqs}^j we have got estimation

$$b_{pqs}^j << \sqrt{rK_j} q^{-2}\, log\, r.$$

Therefore, summing over all $|p|, |s| \leq 3qr^{1/2} K_j^{-1/2}$ we receive

$$\sum_{|p|,|s| \leq 3q\sqrt{rK_j^{-1}}} \left|b_{pqs}^j\right|^2 << q^2 r K_j^{-1} r K_j q^{-4} \log^2 r << r^2 q^{-2} \log^2 r \tag{17}$$

From the bound (12) it follows that for pairs doesn't satisfying the conditions above we may apply the estimation

$$\iint_{\Pi_j} e^{2\pi i F(x,y)}\, dxdy << r|p|^{-1/2}|s|^{-1/2}. \tag{18}$$

Returning back to the relation (7), we can write in accordance with the inequality (2):

$$\int_0^1 \int_0^1 \int_0^1 |\phi_0(\theta, \eta, \xi)|^2\, d\theta d\eta d\xi = W + \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \left|c_{pqs}\right|^2,$$

where W is a contribution of error term W_j into the sum (7).

To estimate the contribution of W_j we must estimate the integrals on the right hand side of the last chain of relations (13). Both integrals have the same estimation, for which we suffice with the estimation of the first of them. Fixing z consider integral

$$W_j(z) = \int_a^b e^{2\pi i F(z,x)} dx,$$

where $[a, b]$ denotes a segment being a closure of the set $\{x|(z, x) \in \Pi_j \backslash P_j\}$. It is clear that its length doesn't exceed 1. Applying theorem of Lagrange on finite increments we find that the segment $[a - 1, b + 1]$ contains $O(1 + q(rK_j)^{-1/2})$ integral numbers s for which at some $x\, \frac{qx}{\sqrt{r^2-z^2-x^2}} = s$. For such s, we estimate the trigonometric integral above trivially, or by using of the lemma 4:

$$\int_a^b e^{2\pi i F(z,x)} dx << min(1, q^{-1/2}(rK_j)^{1/4}),$$

since

$$\frac{\partial^2 F}{\partial x^2} \sqrt{(r^2 - z^2 - x^2)^3} >> q/\sqrt{rK_j}.$$

For other values of s we can estimate the integral by using of the lemma 3 as a value

$$W_j(z) = \int_a^b e^{2\pi i F(z,x)} dx << |s - s_0|^{-1},$$

and here s_0 denotes the value of the function $F(z, x)$ at the points a or b at which $|s - s_0|$ takes minimal value. It is obvious that $|s - s_0| \geq 1$. Summing now over all $|s| \leq 3q\sqrt{rK_j^{-1}}$ we get:

$$\sum_{|s| \leq 3q\sqrt{rK_j^{-1}}} \left|W_j(z)\right|^2 << 1 + q(rK_j)^{-1//2} q^{-1}(rK_j)^{1//2} << 1$$

So, summarizing over all k,$|r - k| \in I_j$the getting inequality and counting the factor$g_q$before the integral in (15), we get contribution of error term W_j into the multiple sum on the right hand side of the relation (12):

$$\sum_{|p|,|s| \leq 3q\sqrt{rK_j^{-1}}} \left|W_j(k)\right|^2 << K_j^2 q^{-2} q\sqrt{rK_j^{-1}} << q^{-1}\sqrt{r}K_j^{3/2} << q^{-1}r^2.$$

Taking $\delta = r^{-4}$ we get for the multiple series following estimation in accordance with (8-18):

$$\sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \left|c_{pqs}\right|^2 << r^2 \log r + \sum_{q=1}^{\infty} r^2 q^{-2} \log^2 r \left(\log^2 r + \sum_{p=1}^{\infty} \frac{sin^2(p\delta)}{p^3\delta^2} \sum_{s=1}^{\infty} \frac{sin^2(s\delta)}{s^3\delta^2}\right).$$

Let's estimate inner sums at the right hand side of the last inequality. We have:

$$\sum_{s=1}^{\infty} \frac{sin^2(s\delta)}{s^3\delta^2} << \sum_{s \leq r^4} \frac{1}{s} + r^8 \sum_{s > r^4} \frac{1}{s^3} << \log r.$$

So, finally we get the following inequality:

$$\int_0^1 \int_0^1 \int_0^1 \left| N(r;\theta,\eta,\xi) - \frac{4\pi r^3}{3} \right|^2 d\theta d\eta d\xi << r^2\, log^4\, r,$$

which completes the proof of the theorem.

4. Discussion

Main idea of the article is a use of Fourier analysis to reduce double trigonometric sums to double trigonometric integral using (9–10). The important result of the article is an estimation (16) of double trigonometric integral. This extends the possibilities of the method for more general domains. There is a possibility of application of the results to initial problem for not shifted domains.

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