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Commentary

On the geometry of an almost α-cosymplectic (k, μ, ν)-spaces

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Abstract: The object of the paper is to investigate almost α-cosymplectic (k,μ,ν)-spaces. Some results on almost cosymplectic (k,μ,ν)-spaces with certain conditions are obtained.

Keywords: α-cosymplectic (k,μ,ν)-spaces; W_6-curvaturetensor; W_9-curvature tensor
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1. Introduction

A thorough local description of almost cosymplectic $(-1,\mu,\nu)-$spaces via model spaces is provided in [1] dependent on the function μ. The models are Lie groups with a left-invariant structure that is almost cosymplectic when μ is constant. Although [2] provides model spaces for the almost cosymplectic situation, there aren't enough illustrative instances of almost $\alpha-$cosymplectic manifolds that satisfy [3] with non-constant smooth functions to be discovered in the literature.

The most obvious examples of almost cosymplectic manifolds are the constructions of almost Kaehler manifolds, the real R line, and the circle S1. S. I. Goldberg and K. Yano developed integrability conditions for almost cosymplectic structures on almost contact manifolds. Besides, they discovered that an almost cosymplectic manifold is only cosymplectic when it is locally flat [4].

H. Öztürk studied the notion of almost α-cosymplectic (k,μ,ν)-spaces in terms of a specific curvature condition. The authors in [5] researched the exintence of almost α-cosymplectic (k,μ,ν)-space in 3-dimensional case. The properties of an almost α-cosymplectic manifolds have been studied by several authors [9–11].

In 2022, M. Atçeken studied the invariant submanifolds of an almost α-cosymplectic (k,μ,ν)-space that matched certain geometric requirements so that $Q(\sigma,R)=0$, $Q(S,\sigma)=0$, $Q(S,\widetilde{\nabla}\sigma)=0$, $Q(S,\widetilde{R}\cdot\sigma)=0$, $Q(g,C\cdot R)=0$ and $Q(S,C\cdot\sigma)=0$. He showed that under certain circumstances, these conditions are identical to totally geodesic [6].

Our paper aim is on invariant submanifolds of an almost α-cosymplectic (k,μ,ν)-manifolds, which is inspired by the works mentioned above. In addition, we demonstrate several prerequisites for an α-cosymplectic (k,μ,ν)-manifolds invariant submanifold to be totally geodesic. Then, certain classifications and characterizations have been developed.

2. Preliminaries

An almost contact manifold is an odd-dimensional manifold M^{2n+1} which carries a field ϕ of endomorphisms of the tangent spaces, a vector field ξ, called characteristic or Reeb vector field, and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{1}$$

Here $I : TM^{2n+1} \to TM^{2n+1}$ denotes an identity mapping. Because of (1), it follows that

$$\eta \circ \phi = 0, \ \phi\xi = 0, \ rank(\phi)\text{=}2n. \tag{2}$$

An almost contact manifold $M^{2n+1}(\phi, \xi, \eta)$ called normal if the tensor field $N = [\phi, \phi] + 2d\eta \otimes \xi = 0$, where $[\phi, \phi]$ denote the Nijenhuis tensor field of ϕ. Any almost contact manifold $M^{2n+1}(\phi, \xi, \eta)$ is known to have a Riemannian metric like that

$$g(\phi\alpha_1, \phi\alpha_2) = g(\alpha_1, \alpha_2) - \eta(\alpha_1)\eta(\alpha_2), \tag{3}$$

for all vector fields $\alpha_1, \alpha_2 \in \Gamma(TM)$ [7]. A metric of this type, g, is known as an equipped metric, and the structure (ϕ, η, ξ, g) and manifold $M^{2n+1}(\phi, \eta, \xi, g)$, associated with it, are known as an almost contact metric manifold and are written as $M^{2n+1}(\phi, \eta, \xi, g)$. It is known as the fundamental form of $M^{2n+1}(\phi, \eta, \xi, g)$ when $\Phi(\alpha_1, \alpha_2) = g(\phi\alpha_1, \alpha_2)$. An almost contact metric manifold is said to be a cosymplectic manifold if η and Φ are closed, that is, $d\eta = d\Phi = 0$ [1]. The definition of an almost α-cosymplectic manifold for every real number α is given as follows [8]:

$$d\eta = 0, \ d\Phi = 2\alpha\eta \wedge \Phi. \tag{4}$$

The term $\alpha-$cosymplectic refers to a normal almost $\alpha-$cosymplectic manifold [5]. We refer to references [9–11]. It's commonly known that the following equality holds for the tensor h on the contact metric manifold $M^{2n+1}(\phi, \eta, \xi, g)$, described by $2h = L_\xi\phi$,

$$\widetilde{\nabla}_{\alpha_1}\xi = -\phi\alpha_1 - \phi h\alpha_1, \ h\phi + \phi h = 0, \ trh = tr\phi h = 0, \ h\xi = 0, \tag{5}$$

here,$\widetilde{\nabla}$ is the Levi-Civita connection on M^{2n+1} [12].

The following presented the notation of the $(k, \mu, \nu)-$contact metric manifold, which expands above generalized (k, μ)-spaces:

$$R(\alpha_1, \alpha_2)\xi = \eta(\alpha_2)\left[kI + \mu h + \nu\phi h\right]\alpha_1 + \eta(\alpha_1)\left[kI + \mu h + \nu\phi h\right]\alpha_2, \tag{6}$$

R is the Riemannian curvature tensor of M^{2n+1} and certain smooth functions k, μ and ν on M^{2n+1}, where α_1, α_2 are vector fields [13].

Lemma 1. *Let $M^{2n+1}(\phi, \eta, \xi, g)$ be an almost $\alpha-$cosymplectic $(k, \mu, \nu)-$manifold. Then,*

$$h^2 = (k + \alpha^2)\phi^2, \tag{7}$$

$$\xi(k) = 2(k+\alpha^2)(\nu - 2\alpha), \tag{8}$$

$$\begin{aligned} R(\xi,\alpha_1)\alpha_2 &= k[g(\alpha_1,\alpha_2)\xi - \eta(\alpha_2)\alpha_1] + \mu[g(h\alpha_1,\alpha_2)\xi - \eta(\alpha_2)h\alpha_1] \\ &\quad +\nu[g(\phi h\alpha_1,\alpha_2)\xi - \eta(\alpha_2)\phi h\alpha_1], \end{aligned} \tag{9}$$

$$(\widetilde{\nabla}_{\alpha_1}\phi)\alpha_2 = g(\alpha\phi\alpha_1 + h\alpha_1,\alpha_2)\xi - \eta(\alpha_2)(\alpha\phi\alpha_1 + h\alpha_1), \tag{10}$$

$$\widetilde{\nabla}_{\alpha_1}\xi = -\alpha\phi^2\alpha_1 - \phi h\alpha_1, \tag{11}$$

for any vector fields α_1, α_2 on M^{2n+1} [7].

Suppose that M is an immersed submanifold of $\widetilde{M}^{2n+1}$, which is an almost α—cosymplectic (k,μ,ν)-space. We use $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$ to characterize the tangent and normal subspaces of M in $\widetilde{M}$. The Gauss and Weingarten formulae are given, respectively, by

$$\widetilde{\nabla}_{\alpha_1}\alpha_2 = \nabla_{\alpha_1}\alpha_2 + \sigma(\alpha_1,\alpha_2), \tag{12}$$

and

$$\widetilde{\nabla}_{\alpha_1}\alpha_5 = -A_{\alpha_5}\alpha_1 + \nabla^{\perp}_{\alpha_1}\alpha_5 \tag{13}$$

for all $\alpha_1,\alpha_2 \in \Gamma(TM)$ and $\alpha_5 \in \Gamma(T^{\perp}M)$, σ and A remove the second fundamental form and shape operators of M, respectively, ∇ and $\nabla^{\perp}$ are the induced connections on M and $\Gamma(T^{\perp}M)$. $\Gamma(TM)$ stands for the set of differentiable vector fields on M. They are related by

$$g(A_{\alpha_5}\alpha_1,\alpha_2) = g(\sigma(\alpha_1,\alpha_2),\alpha_5). \tag{14}$$

The first covariant derivative of the second fundamental form σ is defined by

$$(\widetilde{\nabla}_{\alpha_1}\sigma)(\alpha_2,\alpha_3) = \nabla^{\perp}_{\alpha_1}\sigma(\alpha_2,\alpha_3) - \sigma(\nabla_{\alpha_1}\alpha_2,\alpha_3) - \sigma(\alpha_2,\nabla_{\alpha_1}\alpha_3), \tag{15}$$

for all $\alpha_1,\alpha_2,\alpha_3 \in \Gamma(TM)$. If $\widetilde{\nabla}\sigma = 0$, the second fundamental form is parallel, which is considered to be submanifold.

The following Gauss equation results from denoting the Riemannian curvature tensor of the submanifold M by R.

$$\begin{aligned} \widetilde{R}(\alpha_1,\alpha_2)\alpha_3 &= R(\alpha_1,\alpha_2)\alpha_3 + A_{\sigma(\alpha_1,\alpha_3)}\alpha_2 - A_{\sigma(\alpha_2,\alpha_3)}\alpha_1 + (\widetilde{\nabla}_{\alpha_1}\sigma)(\alpha_2,\alpha_3) \\ &\quad -(\widetilde{\nabla}_{\alpha_2}\sigma)(\alpha_1,\alpha_3), \end{aligned} \tag{16}$$

for all $\alpha_1,\alpha_2,\alpha_3 \in \Gamma(TM)$.

$\widetilde{R}\cdot\sigma$ is given by

$$\begin{aligned} (\widetilde{R}(\alpha_1,\alpha_2)\cdot\sigma)(\alpha_4,\alpha_5) &= R^{\perp}(\alpha_1,\alpha_2)\sigma(\alpha_4,\alpha_5) - \sigma(R(\alpha_1,\alpha_2)\alpha_4,\alpha_5) \\ &\quad -\sigma(\alpha_4,R(\alpha_1,\alpha_2)\alpha_5), \end{aligned} \tag{17}$$

where

$$R^{\perp}(\alpha_1,\alpha_2) = [\nabla^{\perp}_{\alpha_1},\nabla^{\perp}_{\alpha_2}] - \nabla^{\perp}_{[\alpha_1,\alpha_2]},$$

indicate the normal bundle's Riemannian curvature tensor.

In fact, for the Riemannian manifold (M^{2n+1}, g), the W_6 curvature tensor is determined by

$$W_6(\alpha_1, \alpha_2)\alpha_3 = R(\alpha_1, \alpha_2)\alpha_3 - \frac{1}{2n}[S(\alpha_2, \alpha_3)\alpha_2 - g(\alpha_1, \alpha_2)Q\alpha_3], \tag{18}$$

for all $\alpha_1, \alpha_2, \alpha_3 \in \Gamma(TM)$ [14].

Similarly, the tensor $W_6 \cdot \sigma$ is defined by

$$\begin{aligned}(W_6(\alpha_1, \alpha_2) \cdot \sigma)(\alpha_4, \alpha_5) &= R^{\perp}(\alpha_1, \alpha_2)\sigma(\alpha_4, \alpha_5) - \sigma(W_6(\alpha_1, \alpha_2)\alpha_4, \alpha_5) \\ &\quad -\sigma(\alpha_4, W_6(\alpha_1, \alpha_2)\alpha_5),\end{aligned} \tag{19}$$

for all $\alpha_1, \alpha_2, \alpha_4, \alpha_5 \in \Gamma(TM)$.

Furthermore, the W_9-curvature tensor for Riemannian manifold (M^{2n+1}, g) is given by

$$W_9(\alpha_1, \alpha_2)\alpha_3 = R(\alpha_1, \alpha_2)\alpha_3 + \frac{1}{2n}[S(\alpha_1, \alpha_2)\alpha_3 - g(\alpha_2, \alpha_3)Q\alpha_1] \tag{20}$$

for all $\alpha_1, \alpha_2, \alpha_3 \in \Gamma(TM)$ [14].

For a $(0, k)-$type tensor field T, $k \geq 1$ and a $(0, 2)$-type tensor field A on a Riemannian manifold (M, g), $Q(A, T)$ is defined by

$$\begin{aligned}Q(A, T)(\alpha_{11}, \alpha_{12}, ..., \alpha_{1k}; \alpha_1, \alpha_2) &= -T((\alpha_1 \wedge_A \alpha_2)\alpha_{11}, \alpha_{12}, ..., \alpha_{1k}) \\ &- T(\alpha_{11}, (\alpha_1 \wedge_A \alpha_2)\alpha_{13}, ..., \alpha_{1k}) \\ &\quad . \\ &\quad . \\ &\quad . \\ &- T(\alpha_{11}, \alpha_{12}, ..., (\alpha_1 \wedge_A \alpha_2)\alpha_{1k}),\end{aligned} \tag{21}$$

for all $\alpha_{11}, \alpha_{12}, ..., \alpha_{1k}, \alpha_1, \alpha_2 \in \Gamma(TM)$, where

$$(\alpha_1 \wedge_A \alpha_2)\alpha_3 = A(\alpha_2, \alpha_3)\alpha_1 - A(\alpha_1, \alpha_3)\alpha_2. \tag{22}$$

3. On the geometry of an almost α-cosymplectic (k, μ, ν)-spaces

Now, assume that M is an immersed submanifold of $\widetilde{M}^{2n+1}$ and that M is an almost $\alpha-$cosymplectic (k, μ, ν)-space. For any point at $\alpha_1 \in M$, $\phi(T_{\alpha_1}M) \subseteq T_{\alpha_1}M$, then M is said to be an invariant submanifold of $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$ with regard to ϕ. A submanifold that is invariant with respect to ϕ will thereafter be seen to be invariant with respect to h.

Proposition 1. *If ξ is tangent to M, then M is an invariant submanifold of an almost $\alpha-$cosymplectic (k, μ, ν)-space $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$. Hence, we have following* *equalities;*

$$R(\alpha_1,\alpha_2)\xi = k[\eta(\alpha_2)\alpha_1-\eta(\alpha_1)\alpha_2]+\mu[\eta(\alpha_2)h\alpha_1-\eta(\alpha_1)h\alpha_2] +\nu[\eta(\alpha_2)\phi h\alpha_1-\eta(\alpha_1)\phi h\alpha_2] \tag{23}$$

$$(\nabla_{\alpha_1}\phi)\alpha_2 = g(\alpha\phi\alpha_1+h\alpha_1,\alpha_2)\xi-\eta(\alpha_2)(\alpha\phi\alpha_1+h\alpha_1) \tag{24}$$

$$\nabla_{\alpha_1}\xi = -\alpha\phi^2\alpha_1-\phi h\alpha_1 \tag{25}$$

$$\phi\sigma(\alpha_1,\alpha_2)=\sigma(\phi\alpha_1,\alpha_2)=\sigma(\alpha_1,\phi\alpha_2),\quad \sigma(\alpha_1,\xi)=0, \tag{26}$$

where ∇,σ and R stand for M's shape operator, Riemannian curvature tensor, and the induced Levi-Civita connection on M, respectively.

Proof. As the proof is a consequence of straightforward math, we omit it. □

We shall assume for the remainder of this work that M is an invariant submanifold of an $\alpha-$cosymplectic (k,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. From Equation (5), we have in this instance

$$\phi h\alpha_1 = -h\phi\alpha_1, \tag{27}$$

for all $\alpha_1\in\Gamma(TM)$, which means that M is also invariant in relation to the tensor field h.

Theorem 1. *Let M be an invariant submanifold of an almost $\alpha-$cosymplectic (k,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. Then $Q(g,W_6\cdot\sigma)=0$ if and only if M is either totally geodesic or $\left[k^2+(\mu^2+\nu^2)(k+\alpha^2)\right]=0$.*

Proof. We suppose that $Q(g,W_6\cdot\sigma)=0$. This means that

$$(W_6(\alpha_1,\alpha_2)\cdot\sigma)((\alpha_3\wedge_g\alpha_6)\alpha_4,\alpha_5)+(W_6(\alpha_1,\alpha_2)\cdot\sigma)(\alpha_4,(\alpha_3\wedge_g\alpha_6)\alpha_5)=0,$$

for all $\alpha_1,\alpha_2,\alpha_4,\alpha_5,\alpha_3,\alpha_6\in\Gamma(TM)$, which implies that

$$(W_6(\alpha_1,\alpha_2)\cdot\sigma)+(g(\alpha_4,\alpha_6)\alpha_3-g(\alpha_3,\alpha_4)\alpha_6,\alpha_5)+(W_6(\alpha_1,\alpha_2)\cdot\sigma) +(\alpha_4,g(\alpha_5,\alpha_6)\alpha_3-g(\alpha_3,\alpha_5)\alpha_6)=0. \tag{28}$$

In (28), putting $\alpha_2=\alpha_4=\alpha_3=\alpha_5=\xi$ and using (18), (19), (23), we observe

$$\begin{aligned} & (W_6(\alpha_1,\xi)\cdot\sigma)(\eta(\alpha_6)\xi-\alpha_6,\xi)=(W_6(\alpha_1,\xi)\cdot\sigma)(\eta(\alpha_6)\xi,\xi) \\ & -(W_6(\alpha_1,\xi)\cdot\sigma)(\alpha_6,\xi) \\ = \; & R^{\perp}(\alpha_1,\xi)\sigma(\eta(\alpha_6)\xi,\xi)-\sigma(\eta(\alpha_6)W_6(\alpha_1,\xi)\xi,\xi) \\ & -\sigma(\eta(\alpha_6)\xi,W_6(\alpha_1,\xi)\xi)-R^{\perp}(\alpha_1,\xi)\sigma(\alpha_6,\xi) \\ & +\sigma(W_6(\alpha_1,\xi)\alpha_6,\xi)+\sigma(\alpha_6,W_6(\alpha_1,\xi)\xi)=0. \end{aligned} \tag{29}$$

In view of (6) and (16), non-zero components of (29) vectors give us

$$\sigma(W_6(\alpha_1,\xi)\xi,\alpha_6)=\sigma(\alpha_6,k\alpha_1+\mu h\alpha_1+\nu\phi h\alpha_1)=0. \tag{30}$$

Also taking $\phi\alpha_1$ instead of α_1 in (30) and by virtue of Lemma 1 and Proposition

1, we have

$$-k\sigma(\phi h\alpha_1, \alpha_6) - \mu(k+\alpha^2)\sigma(\phi\alpha_1, \alpha_6) + \nu(k+\alpha^2)\sigma(\alpha_1, \alpha_6) = 0. \tag{31}$$

(30) and (31) implies that

$$\left[k^2 + (\mu^2+\nu^2)(k+\alpha^2)\right] = 0 \text{ or } \sigma = 0,$$

which completes the proof. □

Theorem 2. *Let M be an invariant submanifold of an almost $\alpha-$cosymplectic (k, μ, ν)-space $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$. Then $Q(S, W_6 \cdot \sigma) = 0$ if and only if M is either totally geodesic or $2nk\left[k^2 + (\mu^2+\nu^2)(k+\alpha^2)\right] = 0$.*

Proof. We believe that $Q(S, W_6 \cdot \sigma) = 0$, which follows that

$$Q(S, W_6(\alpha_1, \alpha_2) \cdot \sigma)(\alpha_4, \alpha_5; \alpha_3, \alpha_6) = 0,$$

for all $\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_3, \alpha_6 \in \Gamma(TM)$. By virtue of (19) and (21), we obtain

$$\begin{aligned}&S(\alpha_3, \alpha_4)(W_6(\alpha_1, \alpha_2)\cdot\sigma)(\alpha_6, \alpha_5) - S(\alpha_6, \alpha_4)(W_6(\alpha_1, \alpha_2)\cdot\sigma)(\alpha_3, \alpha_5)\\ &+S(\alpha_3, \alpha_5)(W_6(\alpha_1, \alpha_2)\cdot\sigma)(\alpha_4, \alpha_6)\\ &-S(\alpha_6, \alpha_5)(W_6(\alpha_1, \alpha_2)\cdot\sigma)(\alpha_4, \alpha_3) = 0.\end{aligned} \tag{32}$$

Expanding (32) and putting $\alpha_2 = \alpha_4 = \alpha_3 = \alpha_5 = \xi$, non-zero components are

$$2nk\sigma(\alpha_6, W_6(\alpha_1, \xi)\xi). \tag{33}$$

As a result, by combining the previous equation and applying (20), we determine that

$$2nk\sigma(\alpha_6, k\alpha_1) + 2nk\sigma(\alpha_6, \mu h\alpha_1) + 2nk\nu\sigma(\alpha_6, \phi h\alpha_1) = 0. \tag{34}$$

On the other hand, substituting $\phi\alpha_1$ for α_1 and taking into account (7) and (26), we conclude that $2nk\left[k^2 + (\mu^2+\nu^2)(k+\alpha^2)\right]\sigma(hx_1, \alpha_6) = 0$, which yields $2nk\left[k^2 + (\mu^2\ +\nu^2)(k+\alpha^2)\right] = 0$ or $\sigma = 0$. □

Theorem 3. *Let M be an invariant submanifold of an almost $\alpha-$cosymplectic (k, μ, ν)-space $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$. Then $Q(g, W_9 \cdot \sigma) = 0$ if and only if M is either totally geodesic or $\left[k^2 + (\mu^2+\nu^2)(k+\alpha^2)\right] = 0$.*

Proof. We assume that $Q(g, W_9 \cdot \sigma) = 0$. This means that

$$(W_9(\alpha_1, \alpha_2)\cdot\sigma)((\alpha_3 \wedge_g \alpha_6)\alpha_4, \alpha_5) + (W_9(\alpha_1, \alpha_2)\cdot\sigma)(\alpha_4, (\alpha_3 \wedge_g \alpha_6)\alpha_5) = 0,$$

for all $\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_3, \alpha_6 \in \Gamma(TM)$. Then, we have

$$\begin{aligned}&(W_9(\alpha_1, \alpha_2)\cdot\sigma) + (g(\alpha_4, \alpha_6)\alpha_3 - g(\alpha_3, \alpha_4)\alpha_6, \alpha_5) + (W_9(\alpha_1, \alpha_2)\cdot\sigma)\\ &+(\alpha_4, g(\alpha_5, \alpha_6)\alpha_3 - g(\alpha_3, \alpha_5)\alpha_6) = 0.\end{aligned} \tag{35}$$

In (35), taking $\alpha_2 = \alpha_4 = \alpha_3 = \alpha_5 = \xi$ and making use of (6), (20), we obtain

$$\begin{aligned} & (W_9(\alpha_1, \xi) \cdot \sigma)(\eta(\alpha_6)\xi - \alpha_6, \xi) = (W_9(\alpha_1, \xi) \cdot \sigma)(\eta(\alpha_6)\xi, \xi) \\ & -(W_9(\alpha_1, \xi) \cdot \sigma)(\alpha_6, \xi) \\ = \; & R^{\perp}(\alpha_1, \xi)\sigma(\eta(\alpha_6)\xi, \xi) - \sigma(\eta(\alpha_6)W_9(\alpha_1, \xi)\xi, \xi) \\ & -\sigma(\eta(\alpha_6)\xi, W_9(\alpha_1, \xi)\xi) - R^{\perp}(\alpha_1, \xi)\sigma(\alpha_6, \xi) \\ & +\sigma(W_9(\alpha_1, \xi)\alpha_6, \xi) + \sigma(\alpha_6, W_9(\alpha_1, \xi)\xi) = 0. \end{aligned} \tag{36}$$

In view of (17) and (20), non-zero components of (36) vectors give us

$$\sigma(W_9(\alpha_1, \xi)\xi, \alpha_6) = \sigma(\alpha_6, k\alpha_1 + \mu h\alpha_1 + \nu\phi h\alpha_1) = 0. \tag{37}$$

Substituting $\phi\alpha_1$ for α_1 in (37) and considering the Equations (1) and (7), then we get

$$k\sigma(\alpha_6, \phi\alpha_1) - \mu\sigma(\alpha_6, \phi h\alpha_1) + \nu\sigma(\alpha_6, h\alpha_1) = 0. \tag{38}$$

From (37) and (38), we conclude that

$$\left[k^2 + (\mu^2 + \nu^2)(k + \alpha^2)\right] \sigma(\alpha_6, h\alpha_1) = 0$$

Therefore, we get the requested result. □

Theorem 4. *Let M be an invariant submanifold of an almost $\alpha-$cosymplectic (k, μ, ν)-space $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$. Then $Q(S, W_9 \cdot \sigma) = 0$ if and only if M is either totally geodesic or $2nk\left[k^2 + (\mu^2 + \nu^2)(k + \alpha^2)\right] = 0$.*

Proof. Let us assume that $Q(S, W_9 \cdot \sigma) = 0$. It follows that

$$Q(S, W_9(\alpha_1, \alpha_2) \cdot \sigma)(\alpha_4, \alpha_5; \alpha_3, \alpha_6) = 0,$$

for all $\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_3, \alpha_6 \in \Gamma(TM)$. Due to (17) and (20), we deduce that

$$\begin{aligned} & S(\alpha_3, \alpha_4)(W_9(\alpha_1, \alpha_2) \cdot \sigma)(\alpha_6, \alpha_5) - S(\alpha_6, \alpha_4)(W_9(\alpha_1, \alpha_2) \cdot \sigma)(\alpha_3, \alpha_5) \\ & +S(\alpha_3, \alpha_5)(W_9(\alpha_1, \alpha_2) \cdot \sigma)(\alpha_4, \alpha_6) \\ & -S(\alpha_6, \alpha_5)(W_9(\alpha_1, \alpha_2) \cdot \sigma)(\alpha_4, \alpha_3) = 0. \end{aligned} \tag{39}$$

By setting $\alpha_2 = \alpha_4 = \alpha_3 = \alpha_5 = \xi$ in the last equation and non-zero components is

$$2nk\sigma(\alpha_6, W_9(\alpha_1, \xi)\xi), \tag{40}$$

and hence

$$2nk\sigma(\alpha_6, k\alpha_1 + \mu h\alpha_1 + \nu\phi h\alpha_1) = 0. \tag{41}$$

In the same way, by using (37) and (38), we get $2nk\left[k^2 + (\mu^2 + \nu^2)(k + \alpha^2)\right]$ $\sigma(hx_1, \alpha_6) = 0$, this means that, $2nk\left[k^2 + (\mu^2 + \nu^2)(k + \alpha^2)\right] = 0$ or $\sigma = 0$. This proves our assertion. □

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