*Journal of AppliedMath* 2024, 2(6), 2043.
https://doi.org/10.59400/jam2043

Article

# A water wave scattering problem: Revisited

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**CITATION**

Das G, Banerjea S, Mandal BN. A water wave scattering problem: Revisited. Journal of AppliedMath. 2024; 2(6): 2043. https://doi.org/10.59400/jam2043

**ARTICLE INFO**

Received: 12 October 2024
Accepted: 22 November 2024
Available online: 29 November 2024



**Abstract:** The problem of water wave scattering by a thin vertical wall with a gap submerged in deep water is studied using singular integral equation formulation. The corresponding boundary value problem is reduced to a Cauchy type singular integral equation of first kind in two disjoint intervals where the unknown function satisfying the integral equation has square root zero at the end points of the two intervals. In this case the solution exists if the forcing function satisfies two solvability conditions. The reflection coefficient is determined here using the solvability conditions without solving the integral equation and also the boundary value problem.

**Keywords:** vertical barrier with a gap; disjoint intervals; singular integral equation; solvability conditions; reflection coefficient

## 1. Introduction

The boundary value problem associated with the problem of water wave scattering by a thin vertical barrier submerged in deep water is well known in the literature and studied by many researchers using sophisticated mathematical techniques. Notable among them are [1–4]. Singular integral equation formulation of this class of problems gave rise to many analytical as well as numerical methods of solving Cauchy type, Abel type and hypersingular integral equations ([1,4–10]).

The two-dimensional irrotational motion due to a normally incident time harmonic wave train, with frequency $\sigma$, on a thin vertical wall denoted by $x = 0$; $y \in B \equiv (\alpha, \beta) \cup (\gamma, \infty)$, where $0 < \alpha < \beta < \gamma$; submerged in deep water can be described by a velocity potential denoted by $Re\{\Phi(x,y)e^{-i\sigma t}\}$, where $\Phi(x,y)$ satisfies the boundary value problem described by (1) to (6). Here, the coordinate system is chosen where $y$ axis is taken vertically downwards so that the water occupies the region $y > 0$ with $y = 0$ as mean free surface.

$$\nabla^2\Phi = 0 \; in \; y \geq 0, |x| < \infty \tag{1}$$

$$\frac{\partial \Phi}{\partial y} + K\,\Phi = 0 \; in \; y = 0, |x| < \infty \tag{2}$$

$$\frac{\partial \Phi}{\partial x} = 0 \; on \; x = 0, y \in B \tag{3}$$

$$\nabla\Phi \to 0 \; as \; y \to \infty, 0 < |x| < \infty \tag{4}$$

$$r^{1/2}\nabla\Phi \to 0 \; as \; r \to 0 \tag{5}$$

$$\Phi(x,y)\sim\begin{cases} e^{-Ky+iKx} + R\, e^{-Ky-iKx}\ as\ x \to -\infty \\ T\, e^{-Ky+iKx}\ as\ x \to \infty \end{cases} \tag{6}$$

Here in (2), $K = \frac{\sigma^2}{g}$, with $g$ being the acceleration due to gravity; in (5), $r$ is the distance of any field point from sharp edges of the barrier; in (6), $R$ and $T$ denote the reflection and transmission coefficients (complex) respectively, which are unknown.

In the present paper we shall study the above boundary value problem involving submerged thin vertical wall with a gap. As mentioned apriori, the wall occupies the position $x = 0$, $y \in B$, while the gap in the wall has the position $x = 0$, $\beta < y < \gamma$.

This type of boundary value problem for various configurations of the vertical barrier is usually solved by integral equation formulation or function theoretic method [1,4,10–13]. The integral equation arising while solving this boundary value problem is usually singular and various analytic and numerical methods are used to solve it. The first kind Cauchy type integral equation formulation is extensively used to solve this kind of boundary value problem. The Cauchy type singular integral equation may be in single interval or in disjoint multiple intervals depending on the configuration of the barrier. The literature on Cauchy type singular integral equation in single interval is quite extensive. However the literature on Cauchy type singular integral equation in disjoint multiple interval is rather limited. This motivated us to reinvestigate this boundary value problem. Banerjea and Mandal [10] studied this boundary value problem using first kind singular integral equation formulation. They used Havelock's expansion of water wave potential, to reduce the boundary value problem to the solution of singular integral equation in double interval whose kernel involves i) Cauchy type singularity; ii) a combination of Cauchy and logarithmic singularity. In their study the unknown function satisfying the integral equation has square root singularity at the end points of the domain of definition. The integral equation was solved by using function theoretic method. It may be mentioned here that the solution of Cauchy integral equation in double interval was solved in [11] using a simple method, and in [12] using function theoretic method. The reflection coefficient was evaluated using the solutions of the two types of integral equations and they found to match with each other. In the present study, following [4,14], we reduced the boundary value problem to a Cauchy singular integral equation in two disjoint intervals. Unlike the analysis in [10], here the unknown function satisfying the integral equation is bounded at all end points of the domain of integral equation. In this case the solution of the integral equation exists if the forcing function satisfies two solvability conditions [12]. Here, the forcing function involved two unknown constants, one of which is the reflection coefficient. These unknowns are obtained from the two solvability conditions without solving the integral equation. The reflection coefficient was found to match with the result in [10]. The first advantage of the method used in this study is that the reflection coefficient is obtained without solving the boundary value problem while in [10], the boundary value problem needed to be solved in order to evaluate the reflection coefficient. The second advantage is that the the expression for $|R|$ in [10] involves evaluation of singular integral whereas, in the present analysis, the expression for $|R|$ doesnot involve singular integral. Thus the result in the present analysis is more amenable to the numerical technique.

This method can be applied to scattering of water waves involving vertical barrier with other configurations. However for barrier with other geometry, the integral equation obtained thus, will have more complicated kernel and the solution method has to be explored as future work.

## 2. Method of solution

To solve this boundary value problem, we introduce a new function $\psi(x,y)$ defined by ([4,14]),

$$\psi(x,y)=\begin{cases} e^{K\,y}\int_{\infty}^{y}\Phi(x,\eta)e^{-K\,\eta}d\eta+\frac{1}{2\,K}e^{-Ky+iKx}+\frac{R}{2\,K}e^{-Ky-iKx}\ (x<0) \\ e^{K\,y}\int_{\infty}^{y}\Phi(x,\eta)e^{-K\,\eta}d\eta+\frac{T}{2\,K}e^{-Ky+iKx}\ (x>0). \end{cases} \tag{7}$$

Then from Equation (1) to (6), we find that $\psi(x,y)$ satisfies:

$$\nabla^2\psi=0\ in\ y\geq 0 \tag{8}$$

$$\psi(x,0)=0(-\infty<x<\infty) \tag{9}$$

$$\psi(+0,y)-\psi(-0,y)=\begin{cases} \frac{2\,R}{K}sinh\,Ky, y\in(0,\alpha), \\ \frac{2\,A}{K}e^{-K(\gamma-y)}-\frac{2\,R}{K}e^{-K\,\gamma}sinh\,K(\gamma-y), y\in(\beta,\gamma) \end{cases} \tag{10}$$

where $A$ is an unknown constant to be determined,

$$\frac{\partial\psi}{\partial x}(+0,y)-\frac{\partial\psi}{\partial x}(-0,y)=0, y\in G\equiv(0,\alpha)\cup(\beta,\gamma) \tag{11}$$

$$\nabla\psi\sim O\left(r^{1/2}\right)\ as\ r\to 0 \tag{12}$$

$$\frac{\partial\psi}{\partial x}(0,y)=\frac{i}{2}(1-R)e^{-K\,y}, y\in B \tag{13}$$

Following [14], a suitable application of Greens integral theorem to the function $\psi(x,y)$ and the Green's function.

$$G_1(x,y;\xi,\eta)=\frac{1}{4\pi}\log\frac{\{(x-\xi)^2+(y-\eta)^2\}\{(x+\xi)^2+(y-\eta)^2\}}{\{(x-\xi)^2+(y+\eta)^2\}\{(x+\xi)^2+(y+\eta)^2\}} \tag{14}$$

Produces a representation of $\psi(x,y)$ as:

$$\psi(x,y)=\int_0^{\infty}\frac{\partial\psi}{\partial\xi}(+0,\eta)\,G_1(x,y;0,\eta)d\eta, x>0 \tag{15}$$

$$\psi(x,y)=\int_0^{\infty}\frac{\partial\psi}{\partial\xi}(-0,\eta)\,G_1(x,y;0,\eta)d\eta, x<0 \tag{16}$$

Using relations Equation (10), (11) and (13), we obtain:

$$\int_0^\infty \frac{\partial\psi}{\partial\xi}(0,\eta)\,G_1(0,y;0,\eta)d\eta = \begin{cases} \frac{R}{K} sinh\,Ky, y \in (0,\alpha), \\ \frac{A}{K}e^{-K(\gamma-y)} - \frac{R}{K}e^{-K\,\gamma} sinh\,K(\gamma-y), y \in (\beta,\gamma) \end{cases}$$

Writing $\frac{\partial\psi}{\partial\xi}(0,\eta) = f(\eta)$, the above reduces to a Cauchy type integral equation in two disjoint intervals given by:

$$\frac{1}{\pi}\int_{\eta\in G} f(\eta)\frac{2\eta}{y^2-\eta^2}d\eta = g(y), y \in G \tag{17}$$

where,

$$g(y) = \begin{cases} R\cosh Ky - \frac{i}{2\pi}(1-R)\int_{\eta\in B} e^{-K\,\eta}\frac{2\eta}{y^2-\eta^2}d\eta\,, y \in (0,\alpha), \\ Ae^{-K(\gamma-y)} + Re^{-K\gamma}\cosh K(\gamma-y) \\ -\frac{i}{2\pi}(1-R)\int_{\eta\in B} e^{-K\,\eta}\frac{2\eta}{y^2-\eta^2}d\eta\,, y \in (\beta,\gamma). \end{cases} \tag{18}$$

Thus the forcing function $g(y)$ involves two unknown constants $A$ and $R$. In view of the edge condition Equation (5).

$$\text{f(y)}{\sim}O\left(|y-e_i|^{1/2}\right) as\ y \to e_i \tag{19}$$

where $e_i = 0, \alpha, \beta, \gamma$, for $i$ = 1, 2, 3, 4. The solution of the integral Equation (17) with $f(\eta)$ satisfying the condition Equation (19) is given in [12] and also in the Appendix and it is also noted that the solution of Equation (17) exists if $g(y)$ satisfies two solvability conditions given by:

$$(\int_0^\alpha - \int_\beta^\gamma)\frac{g(u)}{\rho(u)}du = 0\ and\ (\int_0^\alpha - \int_\beta^\gamma)\frac{u^2 g(u)}{\rho(u)}du = 0 \tag{20}$$

where, $\rho(y) = \left|(y^2-\alpha^2)(\beta^2-y^2)(\gamma^2-y^2)\right|^{\frac{1}{2}}$.

It is interesting to note that the function $g(y)$ contains two unknown constants $R$ and $A$ which can be evaluated from the two relations Equation (20). Now substituting the value of $g(y)$ from Equation (18) in relations Equation (20) and solving them, we obtain the value of $R$ and $A$. The expression for $R$ is given by:

$$R = \frac{C(K)\gamma_3''(K) - D(K)\gamma_3(K)}{\gamma_3''(K)[\alpha_1(K) - e^{-K\,c}\alpha_3(K) + C(K)] - \gamma_3(K)\left[\alpha_1''(K) - e^{-K\,c}\alpha_3''(K) + D(K)\right]} \tag{21}$$

where, $\alpha_1(K) = \int_{L_1}\frac{\cosh Ky}{\rho(\text{y})}dy$ , $\alpha_3(K) = \int_{L_3}\frac{\cosh K(c-y)}{\rho(\text{y})}dy$ , $\gamma_3(K) = -\int_{L_3}\frac{e^{-K(c-y)}}{\rho(\text{y})}dy$, denotes double derivative with respect to $K$, $L_1 = (0,\alpha)$, $L_2 = (\alpha,\beta)$, $L_3 = (\beta,\gamma)$, $L_4 = (\gamma,\infty)$, $C(K) = \frac{i}{2\pi}[A_{12}(-K) + A_{14}(-K) - A_{32}(-K) -$

$A_{34}(-K)]$, $D(K) = \frac{i}{2\pi}[B_{12}(-K) + B_{14}(-K) - B_{32}(-K) - B_{34}(-K)]$, $A_{ij}(K) = \int_{L_i} \frac{1}{\rho(\mathrm{y})}(\int_{L_j} \frac{2\eta e^{K\eta}}{\eta^2 - y^2} d\eta) dy$ , $B_{ij}(K) = \int_{L_i} \frac{y^2}{\rho(\mathrm{y})}\left(\int_{L_j} \frac{2\eta e^{K\eta}}{\eta^2 - y^2} d\eta\right) dy$.

## 3. Numerical results

The graphical representation of the reflection coefficient R obtained from the condition Equation (21) in the present study and R obtained in [10] are represented in **Figure 1** for $\frac{\beta}{\alpha} = 1.1, \frac{\gamma}{\alpha} = 1.49$ and in **Figure 2** for $\frac{\beta}{\alpha} = 1.3, \frac{\gamma}{\alpha} = 1.35$. Here we see that $|R|$ obtained from the solvability condition almost matches with the reflection coefficient obtained from [10]. In **Figure 3**, $|R|$ is depicted graphically for various values of $K\alpha$ for different lengths of the gap ($\frac{\gamma - \beta}{\alpha} = 0.2, 1.2, 2.2$) in the vertical barrier. It is observed that for $K\alpha < 1.5$, as the length of the gap increases, the reflection coefficient $|R|$ decreases. This behaviour is expected since larger gap allows the water to pass through it. For $K\alpha > 1.5$, the short waves which are near the free surface of water, are not affected by the barrier.

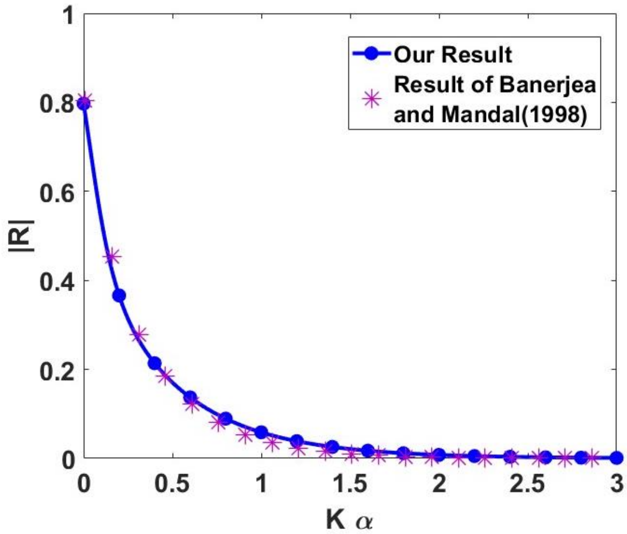


**Figure 1.** Comparison of our results with the results of [10] with $\frac{\beta}{\alpha} = 1.1, \frac{\gamma}{\alpha} = 1.49$.

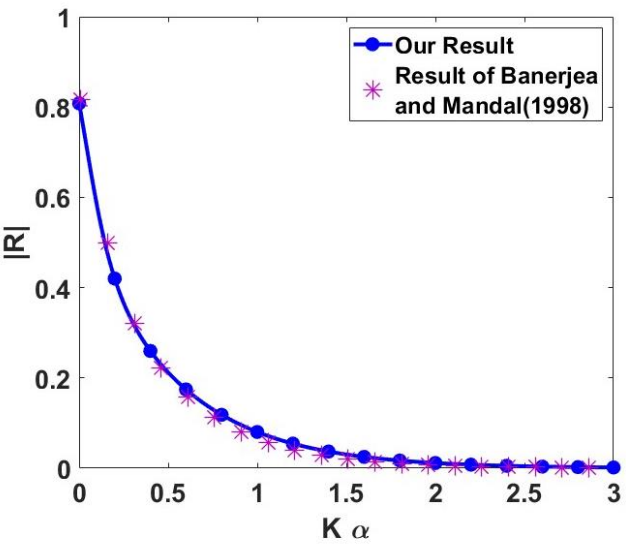


**Figure 2.** Comparison of our results with the results of [10] with $\frac{\beta}{\alpha} = 1.3$, $\frac{\gamma}{\alpha} = 1.35$.

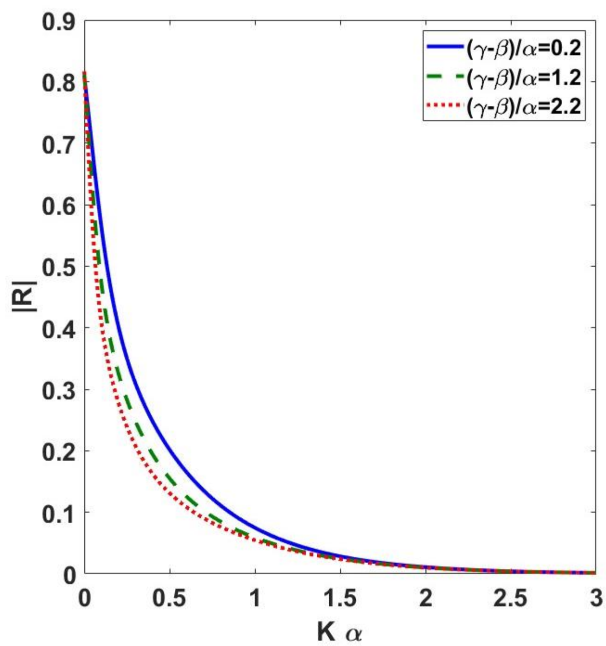


**Figure 3.** $|R|$ against $K\,\alpha$ for different values of the gap of the barrier.

## 4. Conclusion

In the present study the boundary value problem corresponding to the problem of water wave scattering by a thin vertical wall with a gap submerged in deep water is studied by reducing it to a Cauchy type singular integral equation in two disjoint intervals. The unknown function satisfying the integral equation is bounded at all four end points of the domain of definition. The forcing function in this integral equation contains two unknown constants *A* and *R*, which are obtained from the two solvabilty conditions without solving the integral equation. The reflection coefficient |R| thus obtained is compared with the results in [10] and matching of two results are observed. An advantage of the method used in present analysis is that the expression for |R| in [10] involves evaluation of singular integral whereas, in the present analysis, the expression for |R| doesnot involve singular integral. Thus the result in the present analysis is more amenable to the numerical technique.

**Author contributions:** Conceptualization, BNM; methodology, SB and BNM; software, GD; validation, GD and SB; formal analysis, GD and SB; writing—original draft preparation, GD; Writing—review and editing, SB and BNM.

**Conflict of interest:** The authors declare no conflict of interest.

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## Appendix

Using a simple transformation, the integral Equation (17) can be reduced to the following form:

$$\int_L \frac{\varphi(t)}{t-x}dt = \chi(x), x \in L \equiv (a,b) \cup (c,d)$$
$$\varphi(t) \sim O\left(|t-p|^{1/2}\right), t \to p, p = a,b,c,d \tag{A1}$$

To solve Equation (A1), let

$$F(z) = \frac{1}{2\pi i}\int_a^b \frac{\varphi(t)}{t-z}dt + \frac{1}{2\pi i}\int_c^d \frac{\varphi(t)}{t-z}dt \tag{A2}$$

so that $F(z)$ is a sectionally analytic function in entire complex $z$ plane cut along $(a,b) \cup (c,d)$ and $F(z) \sim O(|z|^{-1})\ as\ |z| \to \infty$.

Application of Plemeli Sokhotskii formula [12] to Equation (A2) produces the following two relations for $x \in (a,b) \cup (c,d)$.

$$F^+(x) + F^-(x) = \frac{\chi(x)}{i\,\pi} \tag{A3}$$

$$F^+(x) - F^-(x) = \varphi(x) \tag{A4}$$

where $F^{\pm}(x) = \lim_{y \to 0\pm} F(z)$.

The relation Equation (A3) is a Riemann Hilbert problem [12] for the sectionally analytic function $F(z)$ in the complex $z$-plane, cut along $(a,b) \cup (c,d)$ on the real axis. Its solution is given by

$$F_0(z)\left[\frac{1}{2\pi i}\int_a^b \frac{\chi(t)}{\pi i\, F_0^+(t)}\frac{dt}{t-z} + \frac{1}{2\pi i}\int_c^d \frac{\chi(t)}{\pi i\, F_0^+(t)}\frac{dt}{t-z}\right] \tag{A5}$$

where $F_0(z) = \{(z-a)(z-b)(z-c)(z-d)\}^{\frac{1}{2}}$ and $F_0^{\pm}(x) = \lim_{y \to 0\pm} F(z)$.

Again applying Plemeli Sokhotskii formula [12] to Equation (A5) we obtain $F^{\pm}(x)$ in (a, b) ∪ (c, d) and then using Equation (A4) we obtain the solution of Equation (A1) as:

$$\varphi(x) = m(x)\frac{R(x)}{\pi^2}\left[\int_a^b \frac{\chi(t)}{R(t)}\frac{dt}{t-x} - \int_c^d \frac{\chi(t)}{R(t)}\frac{dt}{t-x}\right], x \in L \equiv (a,b) \cup (c,d) \tag{A6}$$

where, $R(x) = \{|(x-a)(x-b)(x-c)(x-d)|\}^{1/2}$ and $m(x) = \begin{cases} -1, a < x < b \\ 1, c < x < d \end{cases}$.

Noting the behaviour of $F(z)$ at infinity in Equation (A2), we obtain from Equation (A5),

$$F(z) = \frac{F_0(z)}{2\pi i}\left(\int_a^b - \int_c^d\right)\frac{\chi(t)}{\pi F_0^+(t)}\frac{1}{z}\left(1 + \frac{t}{z} + \frac{t^2}{z^2} + \cdots\right)dt$$

which implies $\left(\int_a^b - \int_c^d\right)\frac{\chi(t)}{R(t)}dt = 0$ and $\left(\int_a^b - \int_c^d\right)\frac{t\,\chi(t)}{R(t)}dt = 0$.

These are two solvability conditions which are to be satisfied by $\chi(t)$ in order that the solution Equation (A6) of the integral Equation (A1) exists.