

Some important notes on an almost α -cosymplectic (k, μ, ν) -manifolds

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ABSTRACT: The current work looks at certain geometric requirements that must be satisfied for an invariant submanifold of an almost α -cosymplectic (k, μ, ν) -manifolds to be totally geodesic. Consequently, we obtain some interesting results invariant submanifolds of an almost cosymplectic (k, μ, ν) -manifolds. Additionally, we give an example on 5-dimensional case.

KEYWORDS: α -cosymplectic (k, μ, ν) -manifolds; W_3 -curvature tensor; W_4 -curvature tensor

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1. Introduction

Z. Olszak outlined a few requirements that must be satisfied for a structure to be almost cosymplectic for it to be virtually contact metric. They proved that there are no practically cosymplectic manifolds with non-zero constant sectional curvature in dimensions greater than three. Fortunately, cosymplectic locally flat manifolds with zero sectional curvature do exist. In addition, they investigated a number of restrictions on essentially cosymplectic manifolds with conformally flat surfaces or constant ϕ -sectional curvature^[1].

The most obvious examples of almost cosymplectic manifolds are the constructions of almost Kaehler manifolds, the real R line, and the circle S1. S. I. Goldberg and K. Yano developed integrability conditions for almost cosymplectic structures on almost contact manifolds. Besides, they studied an almost cosymplectic manifold is cosymplectic only in the case it is locally flat^[2].

İ. K peli Erken researched almost α -cosymplectic manifolds. They studied, respectively, projectively flat, conformally flat and concircularly flat almost α -cosymplectic manifolds (with the η -parallel tensor field ϕh). They focused on the almost η -parallel tensor field's characteristics, ϕh ^[3].

In 2022, M. At eken studied at the invariant submanifolds of an almost α -cosymplectic (k, μ, ν) -space that matched certain geometric requirements so that $Q(\sigma, R) = 0$, $Q(S, \sigma) = 0$, $Q(S, \tilde{\nabla}\sigma) = 0$, $Q(S, \tilde{R} \cdot \sigma) = 0$, $Q(g, C \cdot R) = 0$ and $Q(S, C \cdot \sigma) = 0$. They showed that under certain circumstances, these conditions are identical to totally geodesic^[4]. In the following periods, many authors have studied various submanifolds^[6, 7, 10, 13-15, 18].

Our paper aim is on invariant submanifolds of an almost α -cosymplectic (k, μ, ν) -manifolds, which is inspired by the works mentioned above. In addition, we demonstrate several prerequisites for an α -cosymplectic (k, μ, ν) -manifolds invariant submanifold to be totally geodesic. Then, certain classifications and characterizations have been developed.

2. Preliminaries

A field ϕ of endomorphisms of the tangent spaces, an odd-dimensional manifold, a 1-form η fulfilling M^{2n+1} , and a characteristic or Reeb vector field are all components of an almost contact manifold and a vector field ξ

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{1}$$

in which $I : TM^{2n+1} \rightarrow TM^{2n+1}$ indicates a mapping of identities. Because of (1), it follows

$$\eta \circ \phi = 0, \quad \phi\xi = 0, \quad \text{rank}(\phi)=2n. \tag{2}$$

An almost contact manifold $M^{2n+1}(\phi, \xi, \eta)$ is noted to be normal if the tensor field $N = [\phi, \phi] + 2d\eta \otimes \xi = 0$, where $[\phi, \phi]$ denote the Nijenhuis tensor field of ϕ . Any almost contact manifold $M^{2n+1}(\phi, \xi, \eta)$ is known to have a Riemannian metric like that

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2), \tag{3}$$

for all vector fields $X_1, X_2 \in \Gamma(TM)$ ^[8]. A metric of this type, g , is known as an equipped metric, and the structure (ϕ, η, ξ, g) and manifold $M^{2n+1}(\phi, \eta, \xi, g)$, associated with it, are known as an almost contact metric manifolds and are written as $M^{2n+1}(\phi, \eta, \xi, g)$. It is defined for $M^{2n+1}(\phi, \eta, \xi, g)$ to have a 2-form Φ . It is known as the fundamental form of $M^{2n+1}(\phi, \eta, \xi, g)$ when $\Phi(X_1, X_2) = g(\phi X_1, X_2)$. An almost contact metric manifold is referred to as a cosymplectic manifold if η and Φ are closed, that is, $d\eta = d\Phi = 0$ ^[9].

The definition of an almost α -cosymplectic manifold for every real number α is^[11]

$$d\eta = 0, \quad d\Phi = 2\alpha\eta \wedge \Phi. \tag{4}$$

The term α -cosymplectic refers to a normal almost α -cosymplectic manifold^[16]. It's commonly known that the following equality holds for the tensor h on the contact metric manifold $M^{2n+1}(\phi, \eta, \xi, g)$, described by $2h = L_\xi\phi$,

$$\tilde{\nabla}_{X_1}\xi = -\phi X_1 - \phi h X_1, \quad h\phi + \phi h = 0, \quad \text{tr}h = \text{tr}\phi h = 0, \quad h\xi = 0, \tag{5}$$

in this case, $\tilde{\nabla}$ is the Levi-Civita connection on M^{2n+1} ^[5].

The following presented the notation of the (k, μ, ν) -contact metric manifold, which expands above generalized (k, μ) -spaces:

$$R(X_1, X_2)\xi = \eta(X_2) [kI + \mu h + \nu\phi h] X_1 + \eta(X_1) [kI + \mu h + \nu\phi h] X_2, \tag{6}$$

R is the Riemannian curvature tensor of M^{2n+1} and certain smooth functions k, μ and ν on M^{2n+1} , where X_1, X_2 are vector fields^[12].

Lemma 1. Given $M^{2n+1}(\phi, \eta, \xi, g)$ is an almost α -cosymplectic (k, μ, ν) -space, so

$$h^2 = (k + \alpha^2)\phi^2, \tag{7}$$

$$\xi(k) = 2(k + \alpha^2)(\nu - 2\alpha), \tag{8}$$

$$R(\xi, X_1)X_2 = k[g(X_1, X_2)\xi - \eta(X_2)X_1] + \mu[g(hX_1, X_2)\xi - \eta(X_2)hX_1] + \nu[g(\phi hX_1, X_2)\xi - \eta(X_2)\phi hX_1], \tag{9}$$

$$(\tilde{\nabla}_{X_1}\phi)X_2 = g(\alpha\phi x_1 + hx_1, X_2)\xi - \eta(X_2)(\alpha\phi x_1 + hx_1), \tag{10}$$

$$\tilde{\nabla}_{X_1}\xi = -\alpha\phi^2X_1 - \phi hX_1, \tag{11}$$

for any vector fields X_1, X_2 on M^{2n+1} [8].

Let M be an immersed submanifold of \tilde{M}^{2n+1} , which is an almost α -cosymplectic (k, μ, ν) -space. We describe the tangent and normal subspaces of M in \tilde{M} by $\Gamma(TM)$ and $\Gamma(T^\perp M)$. After that, the Gauss and Weingarten formulas are provided, respectively, by

$$\tilde{\nabla}_{X_1}X_2 = \nabla_{X_1}X_2 + \sigma(X_1, X_2), \tag{12}$$

and

$$\tilde{\nabla}_{X_1}X_5 = -A_{X_5}X_1 + \nabla_{X_1}^\perp X_5 \tag{13}$$

for all $X_1, X_2 \in \Gamma(TM)$ and $X_5 \in \Gamma(T^\perp M)$, σ and A are referred to as the second fundamental form and shape operators of M , respectively, ∇ and ∇^\perp are what caused the relationship on M and $\Gamma(T^\perp M)$. $\Gamma(TM)$ stands for the set of differentiable vector fields on M . They are associated by

$$g(A_{X_5}X_1, X_2) = g(\sigma(X_1, X_2), X_5). \tag{14}$$

The second fundamental form σ is first covariant derivative is provided by

$$(\tilde{\nabla}_{X_1}\sigma)(X_2, X_3) = \nabla_{X_1}^\perp\sigma(X_2, X_3) - \sigma(\nabla_{X_1}X_2, X_3) - \sigma(X_2, \nabla_{X_1}X_3), \tag{15}$$

for all $X_1, X_2, X_3 \in \Gamma(TM)$. If $\tilde{\nabla}\sigma = 0$, the second fundamental form is parallel, which is considered to be submanifold.

The following Gauss equation results from denoting the Riemannian curvature tensor of the submanifold M by R .

$$\begin{aligned} \tilde{R}(X_1, X_2)X_3 &= R(X_1, X_2)X_3 + A_{\sigma(X_1, X_3)}X_2 - A_{\sigma(X_2, X_3)}X_1 + (\tilde{\nabla}_{X_1}\sigma)(X_2, X_3) \\ &\quad - (\tilde{\nabla}_{X_2}\sigma)(X_1, X_3), \end{aligned} \tag{16}$$

for all $X_1, X_2, X_3 \in \Gamma(TM)$.

$\tilde{R} \cdot \sigma$ is determined by

$$\begin{aligned} (\tilde{R}(X_1, X_2) \cdot \sigma)(X_4, X_5) &= R^\perp(X_1, X_2)\sigma(X_4, X_5) - \sigma(R(X_1, X_2)X_4, X_5) \\ &\quad - \sigma(X_4, R(X_1, X_2)X_5), \end{aligned} \tag{17}$$

where

$$R^\perp(X_1, X_2) = [\nabla_{X_1}^\perp, \nabla_{X_2}^\perp] - \nabla_{[X_1, X_2]}^\perp,$$

indicate the normal bundle's Riemannian curvature tensor.

In fact, for the Riemannian manifold (M^{2n+1}, g) , the W_3 curvature tensor is prescribed by

$$W_3(X_1, X_2)X_3 = R(X_1, X_2)X_3 - \frac{1}{2n}[S(X_1, X_3)X_2 - g(X_2, X_3)QX_1], \tag{18}$$

for all $X_1, X_2, X_3 \in \Gamma(TM)^{[17]}$.

Similarly, the tensor $W_3 \cdot \sigma$ is defined by

$$\begin{aligned} (W_3(X_1, X_2) \cdot \sigma)(X_4, X_5) &= R^\perp(X_1, X_2)\sigma(X_4, X_5) - \sigma(W_3(X_1, X_2)X_4, X_5) \\ &\quad - \sigma(X_4, W_3(X_1, X_2)X_5), \end{aligned} \tag{19}$$

for all $X_1, X_2, X_4, X_5 \in \Gamma(TM)$.

Furthermore, the W_4 -curvature tensor for Riemannian manifold (M^{2n+1}, g) is given by

$$W_4(X_1, X_2)X_3 = R(X_1, X_2)X_3 + \frac{1}{2n}[g(X_1, X_3)QX_2 - g(X_1, X_2)QX_3] \tag{20}$$

for all $X_1, X_2, X_3 \in \Gamma(TM)^{[17]}$.

For a $(0, k)$ -type tensor field $(0, k)$ -type tensor field T and $(0, 2)$ -type tensor field A on a semi-Riemannian manifold (M, g) , a $(0, k + 2)$ -type tensor field Tachibana $Q(A, T)$ is defined as

$$\begin{aligned} Q(A, T)(X_{11}, X_{12}, \dots, X_{1k}; X_1, X_2) &= -T((X_1 \wedge_A X_2)X_{11}, X_{12}, \dots, X_{1k}) \\ &\quad - T(X_{11}, (X_1 \wedge_A X_2)X_{13}, \dots, X_{1k}) \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &\quad - T(X_{11}, X_{12}, \dots, (X_1 \wedge_A X_2)X_{1k}), \end{aligned} \tag{21}$$

for all $X_{11}, X_{12}, \dots, X_{1k}, X_1, X_2 \in \chi(M)$, where

$$(X_1 \wedge_A X_2)X_3 = A(X_2, X_3)X_1 - A(X_1, X_3)X_2. \tag{22}$$

3. Invariant Submanifolds of an almost α -Cosymplectic (k, μ, ν) -Space

Let M be an almost α -cosymplectic (k, μ, ν) -space and M be an immersed submanifold of \widetilde{M}^{2n+1} . For any point at $X_1 \in M$, if $\phi(T_{X_1}M) \subseteq T_{X_1}M$, M is said to be an invariant submanifold of $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$ with regard to ϕ . Following, it will be clear that a submanifold that is invariant with regard to ϕ is likewise invariant with respect to h .

Proposition 1. M is an invariant submanifold of an almost α -cosymplectic (k, μ, ν) -space $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$ if is tangent to M . Hence, M is equivalent to the following equality;

$$R(X_1, X_2)\xi = k[\eta(X_2)X_1 - \eta(X_1)X_2] + \mu[\eta(X_2)hX_1 - \eta(X_1)hX_2] + \nu[\eta(X_2)\phi hX_1 - \eta(X_1)\phi hX_2] \tag{23}$$

$$(\nabla_{X_1}\phi)X_2 = g(\alpha\phi X_1 + hX_1, X_2)\xi - \eta(X_2)(\alpha\phi X_1 + hX_1) \tag{24}$$

$$\nabla_{X_1}\xi = -\alpha\phi^2 X_1 - \phi hX_1 \tag{25}$$

$$\phi\sigma(X_1, X_2) = \sigma(\phi X_1, X_2) = \sigma(X_1, \phi X_2), \quad \sigma(X_1, \xi) = 0, \tag{26}$$

where ∇, σ and R stand for M 's shape operator, Riemannian curvature tensor, and the induced Levi-Civita connection on M , respectively.

Proof 1. As the proof is a consequence of straightforward math, we omit it.

We shall assume for the remainder of this work that M is an invariant submanifold of an α -cosymplectic (k, μ, ν) -space $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$. From (5), we have in this instance

$$\phi hX_1 = -h\phi X_1, \tag{27}$$

for all $X_1 \in \Gamma(TM)$, in other words M is also invariant in relation to the tensor field h . \square

Theorem 1. Let M be an invariant submanifold of an almost α -cosymplectic (k, μ, ν) -space $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$.

Then $Q(g, W_3 \cdot \sigma) = 0$ if and only if M is either totally geodesic or $[4k^2 + (\mu^2 + \nu^2)(k + \alpha^2)] = 0$.

Proof 2. We suppose that $Q(g, W_3 \cdot \sigma) = 0$. This means that

$$(W_3(X_1, X_2) \cdot \sigma)((X_3 \wedge_g X_6)X_4, X_5) + (W_3(X_1, X_2) \cdot \sigma)(X_4, (X_3 \wedge_g X_6)X_5) = 0,$$

for all $X_1, X_2, X_4, X_5, X_3, X_6 \in \Gamma(TM)$, which implies that

$$(W_3(X_1, X_2) \cdot \sigma) + (g(X_4, X_6)X_3 - g(X_3, X_4)X_6, X_5) + (W_3(X_1, X_2) \cdot \sigma) + (X_4, g(X_5, X_6)X_3 - g(X_3, X_5)X_6) = 0. \tag{28}$$

In (28), putting $X_2 = X_4 = X_3 = X_5 = \xi$ and using (18), (19),(23), we observe

$$\begin{aligned} & (W_3(X_1, \xi) \cdot \sigma)(\eta(X_6)\xi - X_6, \xi) = (W_3(X_1, \xi) \cdot \sigma)(\eta(X_6)\xi, \xi) \\ & - (W_3(X_1, \xi) \cdot \sigma)(X_6, \xi) \\ = & R^\perp(X_1, \xi)\sigma(\eta(X_6)\xi, \xi) - \sigma(\eta(X_6)W_3(X_1, \xi)\xi, \xi) \\ & - \sigma(\eta(X_6)\xi, W_3(X_1, \xi)\xi) - R^\perp(X_1, \xi)\sigma(X_6, \xi) \\ & + \sigma(W_3(X_1, \xi)X_6, \xi) + \sigma(X_6, W_3(X_1, \xi)\xi) = 0. \end{aligned} \tag{29}$$

Non-zero components of the (29) vectors give us (6) and (16),

$$\sigma(W_3(X_1, \xi)\xi, X_6) = \sigma(X_6, 2kX_1 + \mu hX_1 + \nu \phi hX_1) = 0. \tag{30}$$

Also taking ϕX_1 instead of X_1 in (30) and by virtue of lemma 2.1 and proposition 1, we have

$$-2k\sigma(\phi hX_1, X_6) - \mu(k + \alpha^2)\sigma(\phi X_1, X_6) + \nu(k + \alpha^2)\sigma(X_1, X_6) = 0. \quad (31)$$

(30) and (31) implies that

$$[4k^2 + (\mu^2 + \nu^2)(k + \alpha^2)] = 0 \text{ or } \sigma = 0.$$

The proof is finished as a result. \square

Theorem 2. Let M be an invariant submanifold of an almost α -cosymplectic (k, μ, ν) -space $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$.

Then $Q(S, W_3 \cdot \sigma) = 0$ if and only if M is either totally geodesic or $2nk [4k^2 + (\mu^2 + \nu^2)(k + \alpha^2)] = 0$.

Proof 3. We believe that $Q(S, W_3 \cdot \sigma) = 0$, resulting in that

$$Q(S, W_3(X_1, X_2) \cdot \sigma)(X_4, X_5; X_3, X_6) = 0,$$

for all $X_1, X_2, X_4, X_5, X_3, X_6 \in \Gamma(TM)$, by virtue of (19) and (21), we obtain

$$\begin{aligned} &S(X_3, X_4)(W_3(X_1, X_2) \cdot \sigma)(X_6, X_5) - S(X_6, X_4)(W_3(X_1, X_2) \cdot \sigma)(X_3, X_5) \\ &+ S(X_3, X_5)(W_3(X_1, X_2) \cdot \sigma)(X_4, X_6) - S(X_6, X_5)(W_3(X_1, X_2) \cdot \sigma)(X_4, X_3) = 0. \end{aligned} \quad (32)$$

Expanding (32) and putting $X_2 = X_4 = X_3 = X_5 = \xi$, non-zero components is

$$2nk\sigma(X_6, W_3(X_1, \xi)\xi). \quad (33)$$

As a result, by combining the previous equation and applying (20), we determine that

$$2nk\sigma(X_6, 2kX_1) + 2nk\sigma(X_6, \mu hX_1) + 2nk\nu\sigma(X_6, \phi hX_1) = 0. \quad (34)$$

On the other hand, substituting ϕX_1 for X_1 and taking into account (7) and (26), we conclude that $2nk [4k^2 + (\mu^2 + \nu^2)(k + \alpha^2)] \sigma(hX_1, X_6) = 0$, which follows that, $2nk [4k^2 + (\mu^2 + \nu^2)(k + \alpha^2)] = 0$ or $\sigma = 0$. \square

Theorem 3. Let M be an invariant submanifold of an almost α -cosymplectic (k, μ, ν) -space $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$.

Then $Q(g, W_4 \cdot \sigma) = 0$ if and only if M is either totally geodesic or $[k^2 + (\mu^2 + \nu^2)(k + \alpha^2)] = 0$.

Proof 4. We suppose that $Q(g, W_4 \cdot \sigma) = 0$. This means that

$$(W_4(X_1, X_2) \cdot \sigma)((X_3 \wedge_g X_6)X_4, X_5) + (W_4(X_1, X_2) \cdot \sigma)(X_4, (X_3 \wedge_g X_6)X_5) = 0,$$

for all $X_1, X_2, X_4, X_5, X_3, X_6 \in \Gamma(TM)$, it suggests that

$$\begin{aligned} &(W_4(X_1, X_2) \cdot \sigma) + (g(X_4, X_6)X_3 - g(X_3, X_4)X_6, X_5) + (W_4(X_1, X_2) \cdot \sigma) \\ &+ (X_4, g(X_5, X_6)X_3 - g(X_3, X_5)X_6) = 0. \end{aligned} \quad (35)$$

In (35), putting $X_2 = X_4 = X_3 = X_5 = \xi$ and using (6), (20), we observe

$$\begin{aligned} & (W_4(X_1, \xi) \cdot \sigma)(\eta(X_6)\xi - X_6, \xi) = (W_4(X_1, \xi) \cdot \sigma)(\eta(X_6)\xi, \xi) \\ & - (W_4(X_1, \xi) \cdot \sigma)(X_6, \xi) \\ = & R^\perp(X_1, \xi)\sigma(\eta(X_6)\xi, \xi) - \sigma(\eta(X_6)W_4(X_1, \xi)\xi, \xi) \\ & - \sigma(\eta(X_6)\xi, W_4(X_1, \xi)\xi) - R^\perp(X_1, \xi)\sigma(X_6, \xi) \\ & + \sigma(W_4(X_1, \xi)X_6, \xi) + \sigma(X_6, W_4(X_1, \xi)\xi) = 0. \end{aligned} \tag{36}$$

Non-zero components of the vectors (36) provide us (17) and (20) as a reference.

$$\sigma(W_{45}(X_1, \xi)\xi, X_6) = \sigma(X_6, kX_1 + \mu hX_1 + \nu \phi hX_1) = 0. \tag{37}$$

Substituting ϕX_1 for X_1 in (37) and considering the equations (1) and (7), then we get

$$k\sigma(X_6, \phi X_1) - \mu\sigma(X_6, \phi hX_1) + \nu\sigma(X_6, hX_1) = 0. \tag{38}$$

From (37) and (38), we conclude that

$$[k^2 + (\mu^2 + \nu^2)(k + \alpha^2)] \sigma(X_6, hX_1) = 0$$

So, the proof is finished. \square

Theorem 4. Let M be an invariant submanifold of an almost α -cosymplectic (k, μ, ν) -space $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$. Then $Q(S, W_4 \cdot \sigma) = 0$ if and only if M is either totally geodesic or $2nk [k^2 + (\mu^2 + \nu^2)(k + \alpha^2)] = 0$.

Proof 5. Let us assume that $Q(S, W_4 \cdot \sigma) = 0$. Thus, it follows

$$Q(S, W_4(X_1, X_2) \cdot \sigma)(X_4, X_5; X_3, X_6) = 0,$$

for all $X_1, X_2, X_4, X_5, X_3, X_6 \in \Gamma(TM)$, by virtue of (17) and (20), we deduce that

$$\begin{aligned} & S(X_3, X_4)(W_4(X_1, X_2) \cdot \sigma)(X_6, X_5) - S(X_6, X_4)(W_4(X_1, X_2) \cdot \sigma)(X_3, X_5) \\ & + S(X_3, X_5)(W_4(X_1, X_2) \cdot \sigma)(X_4, X_6) - S(X_6, X_5)(W_4(X_1, X_2) \cdot \sigma)(X_4, X_3) = 0. \end{aligned} \tag{39}$$

By setting $X_2 = X_4 = X_3 = X_5 = \xi$ in the last equation and non-zero components is

$$2nk\sigma(X_6, W_4(X_1, \xi)\xi). \tag{40}$$

On the other hand (40) can be written as follows:

$$2nk\sigma(X_6, kX_1 + \mu hX_1 + \nu \phi hX_1) = 0. \tag{41}$$

In the same way, by using (37) and (38), we get $2nk [k^2 + (\mu^2 + \nu^2)(k + \alpha^2)] \sigma(hx_1, X_6) = 0$, this means that, $2nk [k^2 + (\mu^2 + \nu^2)(k + \alpha^2)] = 0$ or $\sigma = 0$.

This supports our claim. \square

Example 1. Let $M = \{(X_1, X_2, X_3, X_4, X_5) \in \mathbb{R}^5, X_5 \neq \pm 1, 0\}$ and we take

$$\begin{aligned} e_1 &= (X_5 + 1) \frac{\partial}{\partial X_1}, & e_2 &= \frac{1}{X_5 - 1} \frac{\partial}{\partial X_2}, & e_3 &= \frac{1}{2} (X_5 + 1)^2 \frac{\partial}{\partial X_3}, \\ e_4 &= \frac{5}{X_5 - 1} \frac{\partial}{\partial X_4}, & e_5 &= (X_5 - 1) \frac{\partial}{\partial X_5} \end{aligned}$$

are vector fields on M that are linearly independent. Additionally, we define the $(1, 1)$ -type tensor field ϕ by $\phi e_1 = e_2, \phi e_2 = -e_1, \phi e_3 = e_4, \phi e_4 = -e_3$ and $\phi e_5 = 0$.

The Riemannian metric tensor g is further provided by

$$g(e_i, e_j) = \{1, i = j; \quad 0, i \neq j\}.$$

Direct calculations make it simple for us to realize that

$$\phi^2 X_1 = -X_1 + \eta(X_1)\xi, \quad \eta(X_1) = g(X_1, \xi)$$

and

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2).$$

Thus $M^5(\phi, \xi, \eta, g)$ is an almost contact metric manifold in five dimensions. We have the components that are not zero from the Lie-operator.

$$\begin{aligned} [e_1, e_5] &= -(X_5 - 1)e_1, & [e_2, e_5] &= (X_5 + 1)e_2, & [e_3, e_5] &= -(X_5 - 1)e_3, \\ [e_4, e_5] &= (X_5 + 1)e_4. \end{aligned}$$

Additionally, by ∇ denoting the Levi-Civita connection on M with, we may achieve the non-zero components using Koszul's formula.

$$\begin{aligned} \nabla_{e_1} e_5 &= -(X_5 - 1)e_1, & \nabla_{e_2} e_5 &= (X_5 + 1)e_2, & \nabla_{e_3} e_5 &= -(X_5 - 1)e_3, \\ \nabla_{e_4} e_5 &= (X_5 + 1)e_4. \end{aligned}$$

Putting the relationships from above to

$$\nabla_{X_1} e_5 = X_1 - \eta(X_1)e_5 - \phi h X_1,$$

what we can see

$$h e_1 = -X_5 e_2, \quad h e_2 = -X_5 e_1, \quad h e_3 = -X_5 e_4, \quad h e_4 = -X_5 e_3 \text{ and } h e_5 = 0.$$

By direct calculations, we get

$$\begin{aligned} R(e_1, e_5)e_5 &= ke_1 + \mu he_1 + \nu \phi he_1 = 2(X_5 - 1)e_1, \\ R(e_2, e_5)e_5 &= ke_2 + \mu he_2 + \nu \phi he_2 = -2X_5(X_5 + 1)e_2, \\ R(e_3, e_5)e_5 &= ke_3 + \mu he_3 + \nu \phi he_3 = 2(X_5 + 1)e_3, \end{aligned}$$

and

$$R(e_4, e_5)e_5 = ke_4 + \mu he_4 + \nu \phi he_4 = -2X_5(X_5 + 1)e_4,$$

which imply that $k = -(X_5 + 1)$, $\mu = 0$ and $\nu = 2 - \frac{1}{X_5} + X_5$.

Conflict of interest

The author declares no conflict of interest.

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