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Original Research Article

On the existence of a positive solution to a boundary value problem for one nonlinear functional-differential equation of the second order

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ABSTRACT: This article considers a boundary value problem for one non-linear second-order functional differential equation on the segment [0, 1] with an integral boundary condition at one of the ends of the segment. Using the well-known Go-Krasnoselsky theorem, sufficient conditions for the existence of at least one positive solution to the problem under consideration are established. A non-trivial example is given, illustrating the fulfillment of the conditions for the unique solvability of the problem posed.

KEYWORDS: boundary problem; functionally-differential equation; positive solution; cone; green's function

1. Introduction

A fairly large number of works are devoted to the solvability of nonlinear differential equations and systems, in which, in particular, the existence of positive solutions, their behavior, asymptotic behavior, etc., are considered, as are the methods of functional analysis based on the use of the technique of nonlinear analysis, the theory of which is associated with the names of F. Riess, M. G. Kreina, L. V. Kantorovich, G. Freudenthal, G. Birkhoff, and others. Subsequently, they were developed by M. A. Krasnoselsky and his students, L. A. Ladyzhensky, I. A. Bakhtin, V. Y. Stetsenko, Y. V. Pokorny, and others.

Boundary value problems with boundary conditions in integral form constitute a very interesting and important class of boundary value problems and arise in various fields of applied mathematics and physics, in particular heat conduction, groundwater flows, thermoelasticity, and plasma physics. Such problems were considered; for example, see the study of Cabada and Iglesias[1], Benchohra et al.[2], Ahmad and Nieto[3], Belarbi et al.[4], Abdelkader and Benchohra[5]. However, there are relatively few works devoted to directly positive solutions of integral boundary value problems for nonlinear functional-differential equations.

In this article, an attempt is made to fill this gap, sufficient conditions for the existence of at least one positive solution for a second-order nonlinear functional-differential equation with an integral boundary condition at one end of the research segment are obtained. In a similar formulation, the problems were previously considered by Abduragimov[6,7]. The results obtained continue the author's research on this topic.

2. Main results

Denote by C the space $C\,[0,1]$, L_p $(1 < p < \infty)$ the space $L_p(0,1)$ and W^2 the space of real functions on $[0,1]$ with an absolutely continuous derivative.

Consider the boundary value problem

$$x''(t) + f(t,(Tx)(t)) = 0, \qquad 0 < t < 1 \tag{1}$$

$$x(0) = 0 \tag{2}$$

$$x'(1) = \int_0^1 g(s)x(s)ds \tag{3}$$

where $T: C \to L_p$ is a linear positive continuous operator, $g(t)$ is a non-negative summable function on $[0,1]$ such that $\int_0^1 sg(s)s < 1$, the function $f(t,u)$ is non-negative on $[0,1] \times [0,\infty)$, satisfies the Carathéodory condition, and $f(\cdot,\ 0) \equiv 0$.

Definition 1. *By a positive solution to problem (1)–(3) we mean a function $x \in W^2$, positive in the interval $(0,1)$, satisfying Equation (1) and boundary conditions (2), (3) almost everywhere on the specified interval.*

Consider the equivalent of problems (1)–(3) integral equation

$$x(t) = \int_0^1 G(t,s)f\big(s,(Tx)(s)\big)ds + \frac{t}{1-\mu}\int_0^1 g(\tau)\left[\int_0^1 G(\tau,s)f\big(s,(Tx)(s)\big)ds\right]d\tau, 0 \le t \le 1 \tag{4}$$

where,

$$G(t,s) = \begin{cases} s, & 0 \le s \le t, \\ t, & t \le s \le 1, \end{cases}$$

$$\mu = \int_0^1 sg(s)ds.$$

It is easy to see that the function $G(t,s)$ has the properties

1) $G(t,s) > 0,\ t,s \in (0,1)$;

2) $ts \le G(t,s) \le s,\ t,s \in [0,1]$.

Let us rewrite Equation (4) in the form

$$x(t) = \int_0^1 \tilde{G}(t,s)f\big(s,(Tx)(s)\big)ds,\ 0 \le t \le 1 \tag{5}$$

where $\tilde{G}(t,s)$ is the Green's function of the operator $-\frac{d^2}{dt^2}$ with boundary conditions (2), (3)

$$\tilde{G}(t,s) = G(t,s) + \frac{t}{1-\mu}\int_0^1 G(\tau,s)g(\tau)d\tau.$$

Suppose that $f(t,u)$ in the domain $[0,1] \times [0,\ \infty)$ satisfies condition

$$f(t,u) \le bu^{\frac{p}{q}}, \qquad p,q \in (1,\infty) \tag{6}$$

where $b > 0$.

Condition (6) ensures the action of the Nemytsky operator $N: L_p \to L_q$, defined by the relation $(Ny)(t) = f(t,y(t))$ for each $y \in L_p$.

In operator form, Equation (5) can be represented as follows

$$x = \tilde{G}NTx$$

where $\tilde{G}: L_q \to C$, $\big(\tilde{G}u\big)(t) = \int_0^1 \tilde{G}(t,s)u(s)ds$ is the Green operator.

Let's put

$$A = \tilde{G}NT,$$

where the A is defined by the equality

$$(Ax)(t) = \int_0^1 \tilde{G}(t,s)f\big(s,(Tx)(s)\big)ds,\ 0 \le t \le 1.$$

Denote by $\widetilde{K}$ the cone of nonnegative functions $x(t)$ of the space C satisfying the condition

$$\min_{t\in[0,1]} x(t) \ge \frac{1}{1-\mu+\alpha} t\|x\|_C,$$

where $\alpha = \int_0^1 g(s)ds$.

It is easy to verify that the operator A acts in the space of non-negative continuous functions, is invariant under the cone $\widetilde{K}$, and is completely continuous due to the Arzela-Ascoli theorem.

In what follows, to prove the existence of at least one positive solution to problems (1)–(3), we will need the well-known Go-Krasnoselsky theorem[8].

Theorem 1. *Let X be a Banach space, and let $P \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \bar{\Omega}_1 \subset \Omega_2$, and let $\mathcal{A}: P \to P$ be a completely continuous operator such that either*

(i) $\|\mathcal{A}u\| \leq \|u\|,\ u \in P \cap \partial\Omega_1,\ \|\mathcal{A}u\| \geq \|u\|,\ u \in P \cap \partial\Omega_2$, *or*
(ii) $\|\mathcal{A}u\| \geq \|u\|,\ u \in P \cap \partial\Omega_1,\ \|\mathcal{A}u\| \leq \|u\|,\ u \in P \cap \partial\Omega_2$.

Let $\mathcal{A}$ has a fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Let us introduce the notation

$$\Omega_r = \{u \in \widetilde{K}: \|u\|_C < r\},\quad \Omega_R = \{u \in \widetilde{K}: \|u\|_C < R\},$$
$$\partial\Omega_r = \{u \in \widetilde{K}: \|u\|_C = r\},\quad \partial\Omega_R = \{u \in \widetilde{K}: \|u\|_C = R\},$$
$$\Omega = \overline{\Omega}_R \backslash \Omega_r,$$

where $0 < r < R$.

In addition, we need the following notation

$$f_0 = \lim_{u \to 0^+} \max_{t \in [0,1]} \frac{f(t,u)}{u},$$
$$f_\infty = \lim_{u \to +\infty} \max_{t \in [0,1]} \frac{f(t,u)}{u}.$$

Theorem 2. *Let us assume that inequality (6) and the conditions*

1) $p > q > 1$;
2) $f_\infty = \infty$;
3) $\min\limits_{t \in [0,1]} (T\theta)(t) > 0,\ \ \theta(t) = t.$

Then the boundary value problems (1)–(3) has at least one positive solution.

Proof. Let us check the fulfillment of condition (i) of Theorem 1. To do this, we show the existence of a number $r > 0$ such that for $x \in \widetilde{K} \cap \partial\Omega_r$

$$\|Ax\|_C \leq \|x\|_C \tag{7}$$

Indeed, due to (6) and property 2 of the function $G(t,s)$, for $x \in \widetilde{K} \cap \partial\Omega_r$ we have

$$\begin{aligned}
(Ax)(t) &= \int_0^1 G(t,s) f\big(s,(Tx)(s)\big)ds + \tfrac{t}{1-\mu}\int_0^1 g(\tau)\left[\int_0^1 G(\tau,s) f\big(s,(Tx)(s)\big)ds\right]d\tau \\
&\leq b\int_0^1 s(Tx)^{\frac{p}{q}}(s)ds + \tfrac{b\alpha}{1-\mu}\int_0^1 s(Tx)^{\frac{p}{q}}(s)ds \leq \left(b + \tfrac{b\alpha}{1-\mu}\right)\int_0^1 s(Tx)^{\frac{p}{q}}(s)ds \\
&\leq \left(b + \tfrac{b\alpha}{1-\mu}\right)\left(\tfrac{1}{q'+1}\right)^{\frac{1}{q'}}\|Tx\|_{L_p}^{\frac{p}{q}} \leq \left(b + \tfrac{b\alpha}{1-\mu}\right)\left(\tfrac{1}{q'+1}\right)^{\frac{1}{q'}}\tau^{\frac{p}{q}}\|x\|_C^{\frac{p}{q}} \\
&= \left(b + \tfrac{b\alpha}{1-\mu}\right)\left(\tfrac{1}{q'+1}\right)^{\frac{1}{q'}}\tau^{\frac{p}{q}}\|x\|_C^{\frac{p}{q}-1}\|x\|_C = \frac{b(1-\mu+\alpha)}{(1-\mu)(q'+1)^{\frac{1}{q'}}} r^{\frac{p}{q}-1}\|x\|_C,
\end{aligned}$$

where τ is the norm of the operator T, $\frac{1}{q'} + \frac{1}{q} = 1$.

Now choosing for r any positive number such that

$$r \leq \left(\frac{(1-\mu)(q'+1)^{\frac{1}{q'}}}{b(1-\mu+\alpha)\tau^{\frac{p}{q}}} \right)^{\frac{q}{p-q}},$$

obviously ensure the fulfillment of (7).

Let us now find a number $R > 0$ such that for $x \in \widetilde{K} \cap \partial\Omega_R$

$$\|Ax\|_C \geq \|x\|_C \tag{8}$$

By virtue of condition (2) of the theorem, there exists a number $L > 0$ such that

$$\max_{t\in[0,1]} f(t,u) \geq \delta u, \qquad u \geq L \tag{9}$$

where δ satisfies the condition $\delta \geq \frac{(1-\mu)(1-\mu+\alpha)}{\int_0^1 s(T\theta)(s)ds} > 0$.

Choosing $R = max\left\{\frac{L(1-\mu+\alpha)}{\min\limits_{t\in[0,1]}(T\theta)(t)}, 2r\right\}$, for $x \in \widetilde{K} \cap \partial\Omega_R$ we get

$$\min_{t\in[0,1]}(Tx)(t) \geq \frac{1}{1-\mu+\alpha}\|x\|_C(T\theta)(t) \geq \frac{1}{1-\mu+\alpha} R \min_{t\in[0,1]}(T\theta)(t) \geq L.$$

By virtue of (9) and the corresponding properties $G(t,s)$, we have

$$(Ax)(t) = \int_0^1 G(t,s)f\big(s,(Tx)(s)\big)ds + \frac{t}{1-\mu}\int_0^1 g(\tau)\left[\int_0^1 G(\tau,s)f\big(s,(Tx)(s)\big)ds\right]d\tau$$
$$\geq t\delta\int_0^1 s(Tx)(s)ds + \frac{t}{1-\mu}\mu\delta\int_0^1 s(Tx)(s)ds = \left(\delta + \frac{\mu}{1-\mu}\delta\right)t\int_0^1 s(Tx)(s)ds$$
$$\geq \frac{\delta}{1-\mu}\cdot\frac{t}{(1-\mu+\alpha)}\|x\|_C\int_0^1 s(T\theta)(s)ds = \frac{\delta}{(1-\mu)(1-\mu+\alpha)}t\int_0^1 s(T\theta)(s)dst\cdot\|x\|_C\,.$$

Having normalized both sides of the last inequality, taking into account the restrictions on δ, we arrive at the required relation (8).

Therefore, a completely continuous operator A has at least one fixed point in $\widetilde{K} \cap \Omega$ such that $r \leq \|x\|_c \leq R$, which in turn is equivalent to the existence of at least one positive solution of the boundary problem (1)–(3) with the above property.

Remark 1. *In the case* $0 < \frac{p}{q} < 1$ *and* $f_0 = \infty$ *we have the fulfillment of condition (ii) of Theorem 1, which guarantees the existence, at least one positive solution to problems (1)–(3).*

Example 1. Consider the following problem

$$x''(t) + e^{-t}\left(\int_0^1 x(s)ds\right)^2 = 0, \quad 0 < t < 1 \tag{10}$$

$$x(0) = 0 \tag{11}$$

$$x'(1) = \int_0^1 sx(s)ds. \tag{12}$$

where $f(t,u) = e^{-t}u^2$, $g(t) = t$. In the future, for convenience and simplicity by calculation, we set $p = 4$, $q = 2$. The first two conditions of Theorem 2 are obvious. It is easy to verify the validity of the third condition for the linear integral operator $T: C \to L_4$ defined by equality

$$(Tx)(t) = \int_0^1 x(s)ds$$
$$(T\theta)(t) = \int_0^1 sds = \frac{1}{2} > 0.$$

To find the numbers r and R, we use the corresponding inequalities given in the course of the proof of Theorem 2. In particular, we choose r from the condition

$$0 < r \leq \left(\frac{(1-\mu)(q'+1)^{\frac{1}{q'}}}{b(1-\mu+\alpha)\tau^{\frac{p}{q}}} \right)^{\frac{q}{p-q}} \tag{13}$$

Let us find the components of inequality (13)

$$\mu = \int_0^1 sg(s)ds = \int_0^1 s^2 ds = \tfrac{1}{3},$$
$$\alpha = \int_0^1 g(s)ds = \int_0^1 sds = \tfrac{1}{2}.$$

Taking $b = 1$, taking into account the fact that $\tau = 1$ and $q' = 2$, we finally obtain

$$0 < r \leq \tfrac{4\sqrt{3}}{7}.$$

The value of R is calculated, respectively, by the formula

$$R = \max\left\{ \frac{L(1-\mu+\alpha)}{\min\limits_{t\in[0,1]}(T\theta)(t)}, 2r \right\}.$$

To determine L, we use the inequality (9)

$$u^2 \geq \delta u,\ u \geq L, \tag{14}$$

where $\delta \geq \frac{(1-\mu)(1-\mu+\alpha)}{\int_0^1 s(T\theta)(s)ds}$.

After performing simple calculations, we have

$$\int_0^1 s(T\theta)(s)ds = \tfrac{1}{2}\int_0^1 sds = \tfrac{1}{4}.$$

It is easy to see that in inequality (14), as L, one can take any number $\delta \geq \frac{28}{9} \approx 3.1$, for example, $L = 4$. Finally, we get

$$R = max\left\{\tfrac{28}{9}, 2r\right\} = \tfrac{28}{9}.$$

Thus, according to Theorem 2, problems (10)–(12) have at least one positive solution such that $r \leq \|x\|_c \leq R$, where r and R are fixed updated above.

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Conflict of interest

The author declares no conflict of interest.

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