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Article

On the qualitative analysis of the boundary value problem of the Ψ-Caputo implicit fractional pantograph differential equation

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Abstract: In this manuscript, the primary objective is to analyze a Ψ-Caputo fractional pantograph implicit differential equation using the Ψ-Caputo fractional derivative. We employ a newly developed method based on fixed-point theorems to explore the existence and uniqueness of the solution to our proposed problem. Furthermore, we investigate the stability of the proposed problem. Finally, we provide an example that illustrates the application of our newly obtained results, confirming their practical significance.

Keywords: fractional differential equation; Ψ-Caputo fractional derivative; fixed point results; stability

1. Introduction

The branch of mathematics known as fractional calculus deals with integrals and derivatives of non-integer orders. Numerous renowned mathematicians, such as Abel, Letnikov, Laplace, Riemann, Liouville, and Fourier, have shown significant interest in this field of study. For a comprehensive review of the fundamental concepts and physical or geometric interpretations of fractional calculus, see [1–3].

Fractional differential equations have attracted considerable attention from researchers in various fields, including chemistry, physics, biology, engineering, and economics, in recent decades due to their broad applicability, and more (refer, for instance, to [4–8]). Numerous scholars have employed diverse techniques within fixed-point theory to yield intriguing findings concerning the existence and uniqueness of solutions across a range of initial and boundary value predicaments. For further elaboration, we direct the reader to [9–16].

Fractional-order derivatives and integrals are defined in several ways, including the Hadamard, Riemann-Liouville, and Caputo variants. As researchers try to maintain specific characteristics of these operators, these definitions are generally different from one another, with a few exceptions. Almeida [17] proposed the Ψ-Caputo fractional derivative, which employs a kernel function. This work expands upon the contributions of various renowned researchers, including those highlighted in [1, 18, 19]. Subsequently, the Ψ-Caputo fractional derivative has garnered significant attention, with several studies exploring its applications. We refer to reviewing some of these works, such as [20–22]. Additional articles employ various definitions of established fractional derivatives to investigate theoretical inquiries regarding the existence, uniqueness, and Hyers-Ulam stability of solutions within fractional differential equations. These investigations are elaborated upon in the following works: [23–28].

The pantograph delay equation has become a valuable instrument for understanding various contemporary topics across fields such as quantum mechanics, probability theory, number theory, control systems, and electrodynamics. Considerable investigation has been conducted to explore the characteristics of this particular category of fractional differential equations, employing both analytical and numerical methodologies. The outcomes of this study have been documented in published works, as referenced in [29–31].

The standard formulation of the pantograph equation is presented as follows:

$$\begin{cases} \nu'(\mathfrak{r}) = c_1\nu(\mathfrak{r}) + c_2\nu(\delta\mathfrak{r}), \ 0 \leq \mathfrak{r} \leq \mathfrak{p} \\ \nu(0) = \nu_0 \end{cases} \tag{1}$$

such that $0 < \delta < 1$.

Balachandran and Kiruthika [29] investigated the use of the Caputo operator for the fractional adaptation of the pantograph equation, which is formulated as follows:

$$\begin{cases} {}^{C}\mathbb{D}^{\rho}\nu(\mathfrak{r}) = f(\mathfrak{r}, \nu(\mathfrak{r}), \nu(\delta\mathfrak{r})), \quad 0 \leq \mathfrak{r} \leq \mathfrak{p} \\ \quad \nu(0) = \nu_0 \end{cases} \tag{2}$$

where $0 < \rho < 1$ and $0 < \delta < 1$.

Motivated by previous work, we present the Ψ-Caputo fractional pantograph implicit differential equation. Here, we apply a fixed-point theorem to examine the existence, uniqueness, and stability of solutions to the given problem:

$$\begin{cases} {}^{C}\mathbb{D}^{\varsigma^\star,\Psi}_{\mathfrak{q}^+}({}^{C}\mathbb{D}^{\rho,\Psi}_{\mathfrak{q}^+} + \delta)\nu(\mathfrak{r}) = f(\mathfrak{r}, \nu(\mathfrak{r}), \nu(\delta\mathfrak{r}), {}^{C}\mathbb{D}^{\varsigma^\star,\Psi}_{\mathfrak{q}^+}\nu(\mathfrak{r})), \ \mathfrak{r} \in \mathfrak{J} = [\mathfrak{q}, \mathfrak{p}] \\ \nu(\mathfrak{q}) = 0, \ \ \nu(\mathfrak{p}) + \delta I^{\rho,\Psi}_{\mathfrak{q}^+}\nu(\mathfrak{p}) = 0, \ {}^{C}\mathbb{D}^{\rho,\Psi}_{\mathfrak{q}^+}\nu(\eta) + \delta\nu(\eta) = 0, \ \ \eta \in (\mathfrak{q}, \mathfrak{p}) \end{cases} \tag{3}$$

where ${}^{C}\mathbb{D}^{\rho,\Psi}_{\mathfrak{q}^+}$ and ${}^{C}\mathbb{D}^{\varsigma^\star,\Psi}_{\mathfrak{q}^+}$ represent the Ψ-Caputo fractional derivatives with respective orders ρ and $\varsigma^\star$, where $\rho \in (0, 1]$ and $\varsigma^\star \in (1, 2]$. The Ψ-Riemann-Liouville integral is denoted by $I^{\rho,\Psi}_{\mathfrak{q}^+}$. For $\delta \in (0, 1)$, the function $f : \mathfrak{J} \times \mathbb{R}^3 \to \mathbb{R}$ is continuous.

Preliminaries

The definitions and findings in this section are needed later.

The notation $\mathfrak{C} = C(\mathfrak{J}, \mathbb{R})$ represents the Banach space of continuous functions. The norm on this space is defined as:

$$\|\nu\| = \sup_{\mathfrak{r} \in \mathfrak{J}} |\nu(\mathfrak{r})|$$

Let $\mathfrak{U} = \{\nu : \nu(\mathfrak{r}), \nu(\delta\mathfrak{r}), {}^{C}\mathbb{D}^{\varsigma^\star,\Psi}_{\mathfrak{q}^+}\nu \in \mathfrak{C}\}$. The norm on the Banach space $\mathfrak{U}$ is defined as follows:

$$\begin{aligned}||\nu||_{\mathfrak{U}} &= ||\nu(\mathfrak{r})|| + ||\nu(\delta\mathfrak{r})|| + ||{}^{C}\mathbb{D}^{\varsigma^\star,\Psi}_{\mathfrak{q}^+}\nu(\mathfrak{r})|| \\ &= \sup_{\mathfrak{r}\in\mathfrak{I}}|\nu(\mathfrak{r})| + \sup_{\mathfrak{r}\in\mathfrak{I}}|\nu(\delta\mathfrak{r})| + \sup_{\mathfrak{r}\in\mathfrak{I}}|{}^{C}\mathbb{D}^{\varsigma^\star,\Psi}_{\mathfrak{q}^+}\nu(\mathfrak{r})|\end{aligned}$$

Definition 1. [1] *The fractional integral of order $\rho > 0$ for the Ψ-Riemann-Liouville operator, applied to an integrable function $\nu : \mathfrak{I} \to \mathbb{R}$, is defined as:*

$$I^{\rho,\Psi}_{\mathfrak{q}^+}\nu(\mathfrak{r}) = \frac{1}{\Gamma(\rho)}\int_{\mathfrak{q}}^{\mathfrak{r}} \Psi'(\wp)(\Psi(\mathfrak{r}) - \Psi(\wp))^{\rho-1}\nu(\wp)d\wp \tag{4}$$

where the Gamma function is denoted by Γ.

Definition 2. [1] *For a function ν of order ρ, where $\rho \in (\mathfrak{m}-1, \mathfrak{m})$ and $\mathfrak{m} \in \mathbb{N}$, the Ψ-Riemann-Liouville fractional derivative is defined as follows:*

$$\mathbb{D}^{\rho,\Psi}_{a+}\nu(\mathfrak{r}) = \left(\frac{1}{\Psi'(\mathfrak{r})}\frac{d}{dt}\right)^{\mathfrak{m}} I^{\mathfrak{m}-\rho,\Psi}_{a+}\nu(\mathfrak{r}) \tag{5}$$

where $\mathfrak{m} = [\rho] + 1$.

Definition 3. [17] *A function ν of fractional order ρ, where $\rho \in (\mathfrak{m}-1, \mathfrak{m})$ and $\Psi \in C^{\mathfrak{m}}(\mathfrak{I}, \Re)$, has the following definition for its Ψ- Caputo fractional derivative:*

$${}^{C}\mathbb{D}^{\rho,\Psi}_{\mathfrak{q}^+}\nu(\mathfrak{r}) = I^{\mathfrak{m}-\rho,\Psi}_{\mathfrak{q}^+}\left(\frac{1}{\Psi'(\mathfrak{r})}\frac{d}{dt}\right)^{\mathfrak{m}}\nu(\mathfrak{r})$$

where $\mathfrak{m} = [\rho] + 1$ for $\rho \notin \mathbb{N}$, $\mathfrak{m} = \rho$ for $\rho \in \mathbb{N}$.

In **Figures 1** and **2**, the dynamical behavior of the Ψ-Caputo fractional derivative can be observed on given functions $\nu(\mathfrak{r}) = \frac{\mathfrak{r}^2}{4}$, $\nu(\mathfrak{r}) = \frac{\mathfrak{r}^6}{8}$ respectively.

Lemma 1. [1] *If $\nu \in C(\mathfrak{I}, \Re)$ and $\rho, \varsigma^\star > 0$, then we get*

$$I^{\rho,\Psi}_{\mathfrak{q}^+}I^{\varsigma^\star,\Psi}_{\mathfrak{q}^+}\nu(\mathfrak{r}) = I^{\rho+\varsigma^\star,\Psi}_{\mathfrak{q}^+}\nu(\mathfrak{r}), \quad \mathfrak{r} \in \mathfrak{I}$$

Lemma 2. [22] *Let's say $\rho > 0$. Assuming that $x \in C(\mathfrak{I}, \Re)$, then*

$${}^{C}\mathbb{D}^{\rho,\Psi}_{\mathfrak{q}^+}I^{\rho,\Psi}_{\mathfrak{q}^+}\nu(\mathfrak{r}) = \nu(\mathfrak{r}), \quad \mathfrak{r} \in \mathfrak{I}$$

and if $x \in C^{n-1}(\mathfrak{I}, \Re)$, then

$$I^{\rho,\Psi}_{\mathfrak{q}^+}{}^{C}\mathbb{D}^{\rho,\Psi}_{\mathfrak{q}^+}\nu(\mathfrak{r}) = \nu(\mathfrak{r}) - \sum_{k=0}^{n-1}\frac{x^{(k)}_{\Psi}a}{k!}[\Psi(\mathfrak{r}) - \Psi(\mathfrak{p})]^k, \quad \mathfrak{r} \in \mathfrak{I}$$

Lemma 3. [1,17] *Let $\Upsilon(\mathfrak{r}) = (\Psi(\mathfrak{r}) - \Psi(\mathfrak{p}))$ and the following conditions: $\mathfrak{r} > a$, $\rho \geq 0$, $\varsigma^\star > 0$, then*

$$(a) : I^{\rho,\Psi}_{\mathfrak{q}^+}(\Upsilon(\mathfrak{r}))^{\varsigma^\star-1} = \frac{\Gamma(\varsigma^\star)}{\Gamma(\varsigma^\star+\rho)}(\Upsilon(\mathfrak{r}))^{\varsigma^\star+\rho-1},$$

$$(b) :{}^{C}\mathbb{D}^{\rho,\Psi}_{\mathfrak{q}^+}(\Upsilon(\mathfrak{r}))^{\varsigma^\star-1} = \frac{\Gamma(\varsigma^\star)}{\Gamma(\varsigma^\star-\rho)}(\Upsilon(\mathfrak{r}))^{\varsigma^\star-\rho-1},$$
$$(c) :{}^{C}\mathbb{D}^{\rho,\Psi}_{\mathfrak{q}^+}(\Upsilon(\mathfrak{r}))^{k} = 0, \quad \text{for all } k=0,\ldots,n-1, n\in\mathbb{N}.$$

Theorem 1. [32] *Let $\mathfrak{B}$ be a non-empty closed subset of a Banach space $\mathfrak{U}$, and let $\mathfrak{M}:\mathfrak{B}\to\mathfrak{B}$ be a contraction. Then, $\mathfrak{M}$ has a unique fixed point.*

Theorem 2. [33] *Let $\mathfrak{B}$ be a non-empty, closed, convex subset of a Banach space $\mathfrak{U}$. If $\mathfrak{M}:\mathfrak{B}\to\mathfrak{B}$ is a compact operator, then $\mathfrak{M}$ has at least one fixed point.*

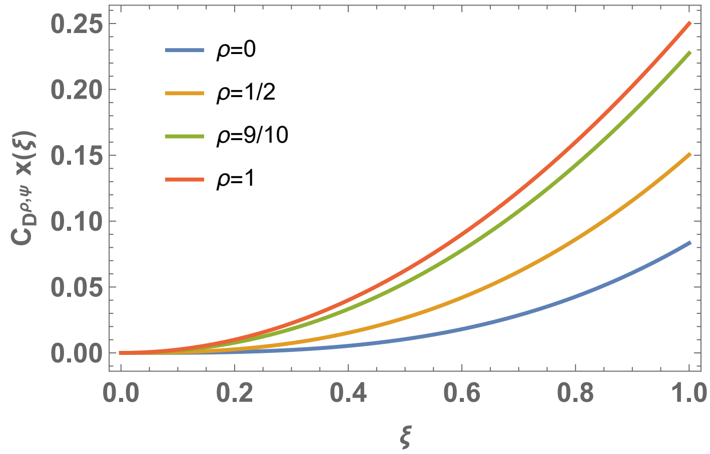

Figure 1. $\nu(\mathfrak{r}) = \frac{\mathfrak{r}^2}{4}$.

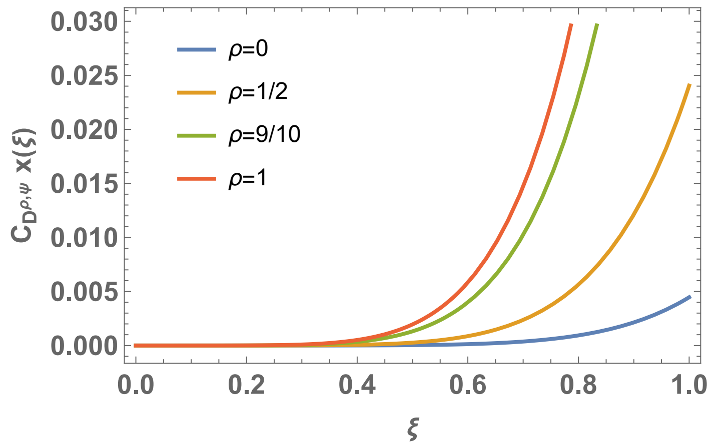

Figure 2. $\nu(\mathfrak{r}) = \frac{\mathfrak{r}^6}{8}$.

2. Main results

We investigate the existence and uniqueness of the solution for Problem (3).

Lemma 4. *Assuming $\varsigma^\star\in(1,2]$ and $\rho\in(0,1]$, the problem*

$$\begin{cases} {}^{C}\mathbb{D}^{\varsigma^\star,\Psi}_{\mathfrak{q}^+}({}^{C}\mathbb{D}^{\rho,\Psi}_{\mathfrak{q}^+}+\delta)\nu(\mathfrak{r}) = \nu(\mathfrak{r}), \quad \mathfrak{r}\in(a,b),\\ \nu(\mathfrak{q})=0,\\ \nu(\mathfrak{p})+\delta I^{\rho,\Psi}_{\mathfrak{q}^+}\nu(\mathfrak{p})=0,\\ {}^{C}\mathbb{D}^{\rho,\Psi}_{\mathfrak{q}^+}\nu(\eta)+\delta\nu(\eta)=0, \qquad \eta\in(a,b)\end{cases} \tag{6}$$

has a solution given by:

$$\nu(\mathfrak{r}) + \delta I_{\mathfrak{q}^+}^{\rho,\Psi}\nu(\mathfrak{r}) = I_{\mathfrak{q}^+}^{\rho+\varsigma^\star,\Psi}\nu(\mathfrak{r}) + \vartheta(\mathfrak{r}) I_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\eta) + \varpi(\mathfrak{r}) I_{\mathfrak{q}^+}^{\rho+\varsigma^\star,\Psi}\nu(\mathfrak{p}) \tag{7}$$

where

$$\vartheta(\mathfrak{r}) = \frac{(\Psi(\mathfrak{r})-\Psi(0))^\rho}{\Gamma(\rho+1)}\left(\frac{(\Psi(\mathfrak{r})-\Psi(0))|\nu_3|}{(\rho+1)|\nabla|} - \frac{1}{|\nabla|}\right)$$
$$\varpi(\mathfrak{r}) = \frac{(\Psi(\mathfrak{r})-\Psi(0))^\rho}{\Gamma(\rho+1)}\left(\frac{|\nu_1|}{|\nabla|} - \frac{(\Psi(\mathfrak{r})-\Psi(0))|\nu_2|}{(\rho+1)|\nabla|}\right)$$

with

$$\nu_1 = \frac{(\Psi(\mathfrak{r})-\Psi(0))^{\rho+1}}{\Gamma(\rho+2)}, \quad \nu_2 = \frac{(\Psi(\mathfrak{r})-\Psi(0))^{\rho}}{\Gamma(\rho+1)}, \quad \nu_3 = (\Psi(\mathfrak{r})-\Psi(0)),$$
$$\nabla = \nu_2 - \nu_1\nu_3 \neq 0$$

Proof. When we apply the integrator operator $I_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}$ and utilize Lemma (3), we obtain:

$$({}^C\mathbb{D}_{\mathfrak{q}^+}^{\rho} + \delta)\nu(\mathfrak{r}) = \mathfrak{c}_0 + \mathfrak{c}_1(\Psi(\mathfrak{r})-\Psi(0)) + I_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\mathfrak{r}), \quad \mathfrak{r}\in(0,b] \tag{8}$$

Next, by applying the operator $I_{\mathfrak{q}^+}^{\rho,\Psi}$ once more and incorporating the outcomes of Lemma (3), we arrive at the general solution representation for problem (7).

$$\nu(\mathfrak{r}) = I_{\mathfrak{q}^+}^{\rho+\varsigma^\star,\Psi}\nu(\mathfrak{r}) - \delta I_{\mathfrak{q}^+}^{\rho}\nu(\mathfrak{r}) + \mathfrak{c}_1\frac{(\Psi(\mathfrak{r})-\Psi(0))^{\rho+1}}{\Gamma(\rho+2)} + \mathfrak{c}_2\frac{(\Psi(\mathfrak{r})-\Psi(0))^{\rho}}{\Gamma(\rho+1)} + \mathfrak{c}_2 \tag{9}$$

Here, let $\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{c}_2 \in \Re$. By incorporating the boundary conditions specified in problem (7) along with the equation mentioned above, we can deduce that

$$\mathfrak{c}_2 = 0, \quad \text{and} \quad \mathfrak{c}_0\frac{(\Psi(\mathfrak{r})-\Psi(0))^{\rho+1}}{\Gamma(\rho+2)} + \mathfrak{c}_1\frac{(\Psi(\mathfrak{r})-\Psi(0))^{\rho}}{\Gamma(\rho+1)} + I_{\mathfrak{q}^+}^{\rho+\varsigma^\star,\Psi}\nu(\mathfrak{p}) = 0 \tag{10}$$

Furthermore, we derive:

$$\mathfrak{c}_0 + \mathfrak{c}_1(\Psi(\eta)-\Psi(0)) + I_{\mathfrak{q}^+}^{\varsigma^\star}\nu(\eta) = 0 \tag{11}$$

We can express Equations (11) and (12) as follows:

$$\mathfrak{c}_0\nu_1 + \mathfrak{c}_1\nu_2 = 0$$
$$\mathfrak{c}_0 + \mathfrak{c}_1\nu_3 = 0$$

After resolving the final two equations for $\mathfrak{c}_0$ and $\mathfrak{c}_1$ we arrive at the following solution:

$$\mathfrak{c}_0 = \frac{x_3}{\nabla} I_{\mathfrak{q}^+}^{\rho+\varsigma^\star,\Psi} - \frac{x_2}{\nabla} I_{\mathfrak{q}^+}^{\varsigma^\star}\nu(\eta)$$
$$\mathfrak{c}_1 = \frac{x_1}{\nabla} I_{\mathfrak{q}^+}^{\varsigma^\star}\nu(\eta) - \frac{1}{\nabla} I_{\mathfrak{q}^+}^{\rho+\varsigma^\star,\Psi}\nu(\mathfrak{p})$$

After substituting $\mathfrak{c}_0$ and $\mathfrak{c}_1$ into Equation (10), we obtain the desired solution representation as shown in Equation (8). Additionally, leveraging the outcomes from Lemmas (3), it becomes evident that Equation (8) effectively addresses problem (7). This concludes the proof. □

The functions ϑ and ϖ exhibit continuity over the interval $\mathfrak{I}$ and adhere to the following properties.

Remark 1. *The following characteristics of the continuous functions ϑ and ϖ over the interval $\mathfrak{I}$ are as follows:*

- $\vartheta^* = \max_{0\leq \mathfrak{r} \leq b} |\vartheta(\mathfrak{r})|$,
- $\varpi^* = \max_{0<\mathfrak{r}<b} |\varpi(\mathfrak{r})|$.

We show that a solution to Problem (3) exists iff the operator $\mathfrak{M}$ has a fixed point. The operator $\mathfrak{M} : \mathfrak{U} \to \mathfrak{U}$ is defined as follows:

$$\begin{aligned} \mathfrak{M}\nu(\mathfrak{r}) = & \int_{\mathfrak{q}}^{\mathfrak{r}} \mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(\mathfrak{r},\wp) f(\wp,\nu(\wp),\nu(\delta\wp), {}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp))d\wp - \delta \int_{\mathfrak{q}}^{\mathfrak{r}} \mathfrak{M}_{\Psi}^{\rho}(\mathfrak{r},\wp)\nu(\wp)d\wp \\ & + \vartheta(\mathfrak{r}) \int_{\mathfrak{q}}^{\eta} \mathfrak{M}_{\Psi}^{\varsigma^\star}(\eta,\wp) f(\wp,\nu(\wp),\nu(\delta\wp), {}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp))d\wp \\ & + \varpi(\mathfrak{r}) \int_{\mathfrak{q}}^{\mathfrak{p}} \mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(b,\wp) f(\wp,\nu(\wp),\nu(\delta\wp), {}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp))d\wp \end{aligned} \tag{12}$$

We propose the following assumptions:

(II_1) : Let f be a continuous function.

(II_2) : When $\mathfrak{r} \in \mathfrak{I}$ and $\nu_i, \upsilon_i \in \Re$, there exists a $\mathbf{K}_f > 0$ such that:

$$|f(\mathfrak{r},\nu_1,\nu_2,\nu_3) - f(\mathfrak{r},\upsilon_1,\upsilon_2,\upsilon_3)| \leq \mathbf{K}_f(|\nu_1-\upsilon_1| + |\nu_2-\upsilon_2| + |\nu_3-\upsilon_3|)$$

For simplicity

$$\begin{aligned} \Omega = \Bigg(& \frac{\mathbf{K}_f(\Psi(\mathfrak{p})-\Psi(\mathfrak{q})^{\rho+\varsigma^\star+1}}{\Gamma(\rho+\varsigma^\star+2)} + |\delta|\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q})^{\rho+1}}{\Gamma(\rho+2)} \\ & + \frac{\mathbf{K}_f\varpi^*(\Psi(\mathfrak{p})-\Psi(\mathfrak{q})^{\rho+\varsigma^\star+1}}{\Gamma(\rho+\varsigma^\star+2)} + \frac{\mathbf{K}_f\vartheta^*(\Psi(\eta)-\Psi(\mathfrak{q})^{\rho+1}}{\Gamma(\varsigma^\star+2)}\Bigg) \end{aligned} \tag{13}$$

and

$$\mathfrak{M}_{\Psi}^{\Upsilon}(\mathfrak{r},\wp) = \frac{\Psi'(\wp)(\Psi(\mathfrak{r})-\Psi(\wp))^{\Upsilon}}{\Gamma(\Upsilon+1)}, \quad \Upsilon > 0 \tag{14}$$

Also note that

$$\int_{\mathfrak{q}}^{\mathfrak{r}} \mathfrak{M}_{\Psi}^{\Upsilon}(\mathfrak{r},\wp)\, d\wp \leq \frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q})^{\Upsilon+1}}{\Gamma(\Upsilon+2)}$$

Theorem 3. *If conditions (II_1)-(II_2) are satisfied and $\Omega < 1$, then problem (3) has a unique solution.*

Proof. Let us define the operator $\mathfrak{M}$ as given in Equation (13). Define the set $\mathfrak{B}\epsilon = \{\nu \in \mathfrak{U} : ||\nu||_{\mathfrak{U}} \leq \epsilon\}$. We will demonstrate that $\mathfrak{M}$ exhibits contraction. Given any $\nu, \upsilon \in \mathfrak{B}_\epsilon$, for every $\mathfrak{r} \in \mathfrak{I}$ we obtain

$$
\begin{aligned}
&|\mathfrak{M}\nu(\mathfrak{r})-\mathfrak{M}y(\mathfrak{r})|\\
&=\int_{\mathfrak{q}}^{\mathfrak{r}}\mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(\mathfrak{r},\wp)|f(\wp,\nu(\wp),\nu(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp))-f(\wp,\upsilon(\wp),\upsilon(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\upsilon(\wp))|d\wp\\
&+|\delta|\int_{\mathfrak{q}}^{\mathfrak{r}}\mathfrak{M}_{\Psi}^{\rho}(\mathfrak{r},\wp)|\nu(\wp)-\upsilon(\wp)|d\wp\\
&+\varpi^*\int_{\mathfrak{q}}^{\mathfrak{r}}\mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(b,\wp)|f(\wp,\nu(\wp),\nu(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp))-f(\wp,\upsilon(\wp),\upsilon(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\upsilon(\wp))|d\wp\\
&+\vartheta^*\int_{\mathfrak{q}}^{\eta}\mathfrak{M}_{\Psi}^{\varsigma^\star}(\eta,\wp)|f(\wp,\nu(\wp),\nu(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp))-f(\wp,\upsilon(\wp),\upsilon(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\upsilon(\wp))|d\wp\\
&\le\int_{\mathfrak{q}}^{\mathfrak{r}}\mathbf{K}_f\mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(\mathfrak{r},\wp)\big(|\nu(\wp)-\upsilon(\wp)|+|\nu(\delta\wp)-\upsilon(\delta\wp)|+|{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\gamma,\Psi}(\wp)-{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\gamma,\Psi}\upsilon(\wp)|\big)d\wp\\
&+|\delta|\int_{\mathfrak{q}}^{\mathfrak{r}}\mathfrak{M}_{\Psi}^{\rho}(\mathfrak{r},\wp)|\nu(\wp)-\upsilon(\wp)|d\wp\\
&+\varpi^*\int_{\mathfrak{q}}^{\mathfrak{r}}\mathbf{K}_f\mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(b,\wp)\big(|\nu(\wp)-\upsilon(\wp)|+|\nu(\delta\wp)-\upsilon(\delta\wp)|+|{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\gamma,\Psi}(\wp)-{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\gamma,\Psi}\upsilon(\wp)|\big)d\wp\\
&+\vartheta^*\int_{\mathfrak{q}}^{\eta}\mathbf{K}_f\mathfrak{M}_{\Psi}^{\varsigma^\star}(\eta,s)\big(|\nu(\wp)-\upsilon(\wp)|+|\nu(\delta\wp)-\upsilon(\delta\wp)|+|{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\gamma,\Psi}(\wp)-{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\gamma,\Psi}\upsilon(\wp)|\big)d\wp\\
&=||\nu-\upsilon||_{\mathfrak{U}}\left(\int_{\mathfrak{q}}^{\mathfrak{r}}(\mathbf{K}_f\mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(\mathfrak{r},\wp)+|\delta|\mathfrak{M}_{\Psi}^{\rho}(\mathfrak{r},\wp)+\varpi^*\mathbf{K}_f\mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(b,\wp))d\wp+\int_{\mathfrak{q}}^{\eta}\vartheta^*\mathbf{K}_f\mathfrak{M}_{\Psi}^{\varsigma^\star}(\eta,\wp)d\wp\right)
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\|\mathfrak{M}\nu-\mathfrak{M}\upsilon\|\le&\bigg(\frac{\mathbf{K}_f(\Psi(\mathfrak{p})-\Psi(\mathfrak{q})^{\rho+\varsigma^\star+1}}{\Gamma(\rho+\varsigma^\star+2)}+|\delta|\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q})^{\rho+1}}{\Gamma(\rho+2)}\\
&+\frac{\mathbf{K}_f\varpi^*(\Psi(\mathfrak{p})-\Psi(\mathfrak{q})^{\rho+\varsigma^\star+1}}{\Gamma(\rho+\varsigma^\star+2)}+\frac{\mathbf{K}_f\vartheta^*(\Psi(\eta)-\Psi(\mathfrak{q})^{\rho+1}}{\Gamma(\varsigma^\star+2)}\bigg)\|\nu-\upsilon\|_{\mathfrak{U}}\\
&=\Omega\|\nu-\upsilon\|
\end{aligned}
$$

Consequently, $\mathfrak{M}$ is a contraction since $\Omega<1$. Thus, according to Theorem (1), Problem (3) has a unique solution, which concludes the proof. □

Furthermore, we will use the following assumptions:

(Π_3): There exists a non-decreasing function $\phi\in C(\mathfrak{I},\mathbb{R})$ such that

$$
|f(\wp,\nu(\wp),\nu(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp)|\le\phi(\wp)(||\nu||_{\mathfrak{U}})
$$

and

$$
\phi^*=\sup_{\wp\in\mathfrak{I}}|\phi(\wp)|
$$

For simplicity

$$
\Delta=\phi^*\left(\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q})^{\rho+\varsigma^\star+1}}{\Gamma(\rho+\varsigma^\star+2)}(\varpi^*+1)+\vartheta^*\frac{(\Psi(\eta)-\Psi(\mathfrak{p}))^{\varsigma^\star+1}}{\Gamma(\varsigma^\star+2)}\right)+|\delta|\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q}))^{\rho}}{\Gamma(\rho+2)}
$$

Theorem 4. *Assuming that conditions (Π_1)-(Π_3) are met, if $\Delta<1$, then Problem (3) has at least one solution.*

Proof. Define the set

$$\mathfrak{B}\epsilon = \{\nu \in \mathfrak{U} : ||\nu||_{\mathfrak{U}} \leq \epsilon\}$$

Clearly the subset $\mathfrak{B}_\epsilon$ is evidently convex, closed, and bounded. We will show that $\mathfrak{M}$ meets the requirements outlined in Theorem (2) by providing a proof in three distinct steps.

Step 1: First, we demonstrate that $\mathfrak{M}\mathfrak{B}_\epsilon \subset \mathfrak{B}_\epsilon$. Consider $\nu \in \mathfrak{B}_\epsilon$. For each $\mathfrak{r} \in \mathfrak{I}$, we have:

$$\begin{aligned}
|\mathfrak{M}\nu(\mathfrak{r})| &\leq \int_{\mathfrak{q}}^{\mathfrak{r}} \mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(\mathfrak{r},\wp)|f(\wp,\nu(\wp),\nu(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp))|d\wp - \delta\int_{\mathfrak{q}}^{\mathfrak{r}} \mathfrak{M}_{\Psi}^{\rho}(\mathfrak{r},\wp)|\nu(\wp)|d\wp \\
&+ \vartheta^* \int_{\mathfrak{q}}^{\eta} \mathfrak{M}_{\Psi}^{\varsigma^\star}(\eta,s)|f(\wp,\nu(\wp),\nu(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp))|d\wp \\
&+ \varpi^* \int_{\mathfrak{q}}^{\mathfrak{p}} \mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(b,\wp)|f(\wp,\nu(\wp),\nu(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp))|d\wp \\
&\leq \int_{\mathfrak{q}}^{\mathfrak{r}} \mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(\mathfrak{r},\wp)\phi(\wp)\|\nu\|_{\mathfrak{U}}d\wp + \delta\int_{\mathfrak{q}}^{\mathfrak{r}} \mathfrak{M}_{\Psi}^{\rho}(\mathfrak{r},\wp)|\nu(\wp)|d\wp \\
&+ \vartheta^* \int_{\mathfrak{q}}^{\mathfrak{r}} \mathfrak{M}_{\Psi}^{\varsigma^\star}(\mathfrak{r},\wp)\phi(\wp)\|\nu\|_{\mathfrak{U}}d\wp + \varpi^*\int_{\mathfrak{q}}^{\mathfrak{r}} \mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(b,\wp)\phi(\wp)\|\nu\|_{\mathfrak{U}}d\wp \\
&\leq \phi^*\epsilon\int_{\mathfrak{q}}^{\mathfrak{r}} \mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(\mathfrak{r},\wp)(1)d\wp + \delta\epsilon\int_{\mathfrak{q}}^{\mathfrak{r}} \mathfrak{M}_{\Psi}^{\rho}(\mathfrak{r},\wp)(1)d\wp \\
&+ \vartheta^*\phi^*\epsilon\int_{\mathfrak{q}}^{\mathfrak{r}} \mathfrak{M}_{\Psi}^{\varsigma^\star}(\mathfrak{r},\wp)(1)d\wp + \varpi^*\phi^*\epsilon\int_{\mathfrak{q}}^{\mathfrak{r}} \mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(b,\wp)(1)d\wp \\
&\leq \phi^*\epsilon\left(\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q})^{\rho+\varsigma^\star+1}}{\Gamma(\rho+\varsigma^\star+2)}(\varpi^*+1) + \vartheta^*\frac{(\Psi(\eta)-\Psi(\mathfrak{p}))^{\varsigma^\star+1}}{\Gamma(\varsigma^\star+2)}\right) + |\delta|\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q})^{\rho}}{\Gamma(\rho+2)}\epsilon \\
&= \epsilon\left(\phi^*\left(\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q})^{\rho+\varsigma^\star+1}}{\Gamma(\rho+\varsigma^\star+2)}(\varpi^*+1) + \vartheta^*\frac{(\Psi(\eta)-\Psi(\mathfrak{p}))^{\varsigma^\star+1}}{\Gamma(\varsigma^\star+2)}\right) + |\delta|\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q})^{\rho+1}}{\Gamma(\rho+2)}\right) \\
&\leq \epsilon
\end{aligned}$$

Thus, we get $||\mathfrak{M}\nu|| \leq \epsilon$, which implies that $\mathfrak{M}\mathfrak{B}_\epsilon \subset \mathfrak{B}_\epsilon$.

Step 2: We prove the continuity of $\mathfrak{M}$.

In the set $\mathfrak{U}$, let's consider the sequence ν_n converging to ν. Consequently, for any $\mathfrak{r} \in \mathfrak{I}$,

$$
\begin{aligned}
&\|\mathfrak{M}\nu_n(\mathfrak{r})-\mathfrak{M}\nu(\mathfrak{r})\| \\
&\leq \int_{\mathfrak{q}}^{\mathfrak{r}} \mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(\mathfrak{r},\wp)\left|f(\wp,\nu_n(\wp),\nu_n(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu_n(\wp))-f(\wp,\nu(\wp),\nu(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp))\right|d\wp \\
&-\delta\int_{\mathfrak{q}}^{\mathfrak{r}} \mathfrak{M}_{\Psi}^{\rho}(\mathfrak{r},\wp)\left|\nu_n(\wp)-\nu(\wp)\right|d\wp \\
&+\vartheta^*\int_{\mathfrak{q}}^{\eta} \mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(\eta,\wp)\left|f(\wp,\nu_n(\wp),\nu_n(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu_n(\wp))-f(\wp,\nu(\wp),\nu(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp))\right|d\wp \\
&+\varpi^*\int_{\mathfrak{q}}^{\mathfrak{p}} \mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(b,\wp)\left|f(\wp,\nu_n(\wp),\nu_n(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu_n(\wp))-f(\wp,\nu(\wp),\nu(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp))\right|d\wp \\
&\leq\left(\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q})^{\rho+\varsigma^\star+1}}{\Gamma(\rho+\varsigma^\star+2)}(\varpi^*+1)+\vartheta^*\frac{(\Psi(\eta)-\Psi(\mathfrak{p}))^{\varsigma^\star+1}}{\Gamma(\varsigma^\star+2)}\right)||\nu_n-x|| \\
&+\delta\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q})^{\rho+1}}{\Gamma(\rho+2)}\|\nu_n-x\| \\
&\leq\left(\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q})^{\rho+\varsigma^\star+1}}{\Gamma(\rho+\varsigma^\star+2)}(\varpi^*+1)+\vartheta^*\frac{(\Psi(\eta)-\Psi(\mathfrak{p}))^{\varsigma^\star+1}}{\Gamma(\varsigma^\star+2)}+|\delta|\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q})^{\rho+1}}{\Gamma(\rho+2)}\right)\|\nu_n-\nu\|
\end{aligned}
$$

Thus, according to the Lebesgue dominated convergence theorem, the norm of $||\mathfrak{M}\nu_n(\mathfrak{r})-\mathfrak{M}\nu(\mathfrak{r})||\to 0$. Hence, $\mathfrak{M}$ is continuous.

Step 3: This clearly shows the uniform boundedness of the operator $\mathfrak{M}$. Now, we present the equicontinuity of $\mathfrak{M}$. For this, we suppose $\mathfrak{r}_1,\mathfrak{r}_2\in\mathfrak{J}$ such that $\mathfrak{r}_1<\mathfrak{r}_2$. We have

$$
\begin{aligned}
\|\mathfrak{M}\nu(\mathfrak{r}_2)-\mathfrak{M}\nu(\mathfrak{r}_1)\| \leq& \int_{\mathfrak{q}}^{\mathfrak{r}}\Big(\mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(\mathfrak{r}_2,\wp)-\mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(\mathfrak{r}_1,\wp)\Big)|f(\wp,\nu(\wp),\nu(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp))|d\wp \\
&+\delta\int_{a}^{\mathfrak{r}_1}\Big(\mathfrak{M}_{\Psi}^{\varsigma^\star}(\mathfrak{r}_2,\wp)-\mathfrak{M}_{\Psi}^{\varsigma^\star}(\mathfrak{r}_1,\wp)\Big)|\nu(\wp)|d\wp \\
&+|\vartheta(\mathfrak{r}_2)-\vartheta(\mathfrak{r}_1)|\int_{\mathfrak{q}}^{\eta}\mathfrak{M}_{\Psi}^{\varsigma^\star}(\eta,\wp)|f(\wp,\nu(\wp),\nu(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp))|d\wp \\
&+|\varpi(\mathfrak{r}_2)-\varpi(\mathfrak{r}_1)|\int_{\mathfrak{q}}^{\mathfrak{r}}\mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(b,\wp)|f(\wp,\nu(\wp),\nu(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp))|d\wp \\
\leq& \frac{\phi(\wp)||\nu||_{\mathfrak{U}}(\Psi(\mathfrak{r}_2)-\Psi(\mathfrak{r}_1))^{\rho+\varsigma^\star+1}}{\Gamma(\rho+\varsigma^\star+2)}+\frac{\delta\epsilon(\Psi(\mathfrak{r}_2)-\Psi(\mathfrak{r}_1))^{\rho+1}}{\Gamma(\rho+2)} \\
&+|\varpi(\mathfrak{r}_2)-\varpi(\mathfrak{r}_1)|\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q})^{\rho+\varsigma^\star+1}\phi(\wp)||\nu||_{\mathfrak{U}}}{\Gamma(\rho+\varsigma^\star+2)} \\
&+|\vartheta(\mathfrak{r}_2)-\vartheta(\mathfrak{r}_1)|\frac{(\Psi(\eta)-\Psi(\mathfrak{p}))^{\varsigma^\star+1}\phi(\wp)||\nu||_{\mathfrak{U}}}{\Gamma(\varsigma^\star+2)}
\end{aligned}
$$

Then $\|\mathfrak{M}\nu(\mathfrak{r}_2)-\mathfrak{M}\nu(\mathfrak{r}_1)\|\to 0\ as\ \mathfrak{r}_2\to\mathfrak{r}_1$. Therefore, $\mathfrak{M}$ is equicontinuous and, as a result, relatively compact. By the Arzelà-Ascoli theorem, the operator $\mathfrak{M}$ is relatively compact on $\mathfrak{B}_\epsilon$. Therefore, applying Theorem (2) confirms that Problem (3) has at least one solution, thus concluding the proof. □

3. Stability

The purpose of this section is to analyze the stability of Problem (3).

Assume $\varepsilon > 0$. Next, we examine the following inequality:

$$\left|\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\left(\mathbb{D}_{\mathfrak{q}^+}^{\rho,\Psi}+\delta\right)\nu(\mathfrak{r})-f(\mathfrak{r},\nu(\mathfrak{r}),\nu(\delta\mathfrak{r}),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\mathfrak{r}))\right|\leq\epsilon,\quad\mathfrak{r}\in\mathfrak{J}\tag{15}$$

Definition 4. *For each $\epsilon > 0$ and any solution $\overline{\nu} \in \mathfrak{C}$ to inequalities (15), there exists a corresponding solution $\nu \in \mathfrak{C}$ to Equation (3). Under these conditions, problem (3) demonstrates Hyers-Ulam stability, if $\mathfrak{v} > 0$ such that*

$$|\overline{\nu}(\mathfrak{r})-\nu(\mathfrak{r})|\leq\epsilon\mathfrak{v},\ \ \mathfrak{r}\in\mathfrak{J}$$

Definition 5. *If there exists a function $\varphi \in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\varphi(0) = 0$, then Problem (3) is said to exhibit generalized Hyers-Ulam stability. In particular, for any solution $\overline{\nu} \in \mathfrak{C}$ of inequality (15) and for every $\epsilon > 0$, the following holds true: there exists a solution $\nu \in \mathfrak{C}$ to Problem (3).*

$$|\overline{\nu}(\mathfrak{r})-\nu(\mathfrak{r})|\leq\varphi(\epsilon),\ \ \mathfrak{r}\in\mathfrak{J}$$

Remark 2. *A function represented as $\overline{\nu}$ in the set $\mathfrak{C}$ fulfills inequality (15) iff there is a function ν in the set $\mathfrak{C}$ such that*

- $|\mathbf{F}(\mathfrak{r})| \leq \epsilon,\ \mathfrak{r} \in [a,b]$.
- ${}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\left({}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\rho,\Psi}\overline{\nu}(\mathfrak{r})-\delta\overline{\nu}(\mathfrak{r})\right)=f(\mathfrak{r},\overline{\nu}(\mathfrak{r}),\overline{\nu}(\delta\mathfrak{r}),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\overline{\nu}(\mathfrak{r}))+\mathbf{F}(\mathfrak{r}),\ \mathfrak{r}\in\mathfrak{J}.$

Theorem 5. *If conditions (II_1)-(II_2) hold, problem (3) demonstrates Hyers-Ulam stability and is consequently generalized Hyers-Ulam stable.*

Proof. Suppose $\epsilon > 0$. Assume that $\overline{\nu} \in \mathfrak{C}$ is a solution satisfying inequality (15). Let $\nu \in \mathfrak{C}$ denote its unique solution. For each $\mathfrak{r} \in [a,b]$, we have:

$$\begin{aligned}
&|\bar{\nu}(\mathfrak{r})-\nu(\mathfrak{r})|\\
&\leq\int_{\mathfrak{q}}^{\mathfrak{r}}\mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(\mathfrak{r},\wp)|f(\wp,\bar{\nu}(\wp),\bar{\nu}(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\bar{\nu}(\wp))-f(\wp,\nu(\wp),\nu(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp))|d\wp\\
&+|\delta|\int_{\mathfrak{q}}^{\mathfrak{r}}\mathfrak{M}_{\Psi}^{\rho}(\mathfrak{r},\wp)|\bar{\nu}(\wp)-\nu(\wp)|d\wp\\
&+\varpi^*\int_{\mathfrak{q}}^{\mathfrak{r}}\mathfrak{M}_{\Psi}^{\rho+\varsigma^\star}(b,\wp)|f(\wp,\bar{\nu}(\wp),\bar{\nu}(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\bar{\nu}(\wp))-f(\wp,\nu(\wp),\nu(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp))|d\wp\\
&+\vartheta^*\int_{\mathfrak{q}}^{\eta}\mathfrak{M}_{\Psi}^{\varsigma^\star}(\eta,\wp)|f(\wp,\bar{\nu}(\wp),\bar{\nu}(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\bar{\nu}(\wp))-f(\wp,\nu(\wp),\nu(\delta\wp),{}^{C}\mathbb{D}_{\mathfrak{q}^+}^{\varsigma^\star,\Psi}\nu(\wp))|d\wp\\
&\leq\left(\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q}))^{\rho+\varsigma^\star}}{\Gamma(\rho+\varsigma^\star+1)}(\varpi^*+1)+\vartheta^*\frac{(\Psi(\eta)-\Psi(\mathfrak{p}))^{\varsigma^\star}}{\Gamma(\varsigma^\star+1)}+|\delta|\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q}))^{\varsigma^\star}}{\Gamma(\varsigma^\star+1)}\right)||\overline{\nu}-\nu||_{\mathfrak{U}}
\end{aligned}$$

applying remark (2), we get

$$||\bar{\nu}(\mathfrak{r})-\nu(\mathfrak{r})||\leq\left(\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q}))^{\rho+\varsigma^\star+1}}{\Gamma(\rho+\varsigma^\star+2)}(\varpi^*+1)+\vartheta^*\frac{(\Psi(\eta)-\Psi(\mathfrak{p}))^{\varsigma^\star+1}}{\Gamma(\varsigma^\star+2)}+|\delta|\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q}))^{\varsigma^\star+1}}{\Gamma(\varsigma^\star+2)}\right)\epsilon$$

We set

$$\mathfrak{v}=\left(\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q}))^{\rho+\varsigma^\star+1}}{\Gamma(\rho+\varsigma^\star+2)}(\varpi^*+1)+\vartheta^*\frac{(\Psi(\eta)-\Psi(\mathfrak{p}))^{\varsigma^\star+1}}{\Gamma(\varsigma^\star+2)}+|\delta|\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q}))^{\varsigma^\star+1}}{\Gamma(\varsigma^\star+2)}\right)$$

then the condition for Hyers-Ulam stability holds. In general, when considering $\varphi(\epsilon) = \left(\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q}))^{\rho+\varsigma^\star+1}}{\Gamma(\rho+\varsigma^\star+2)}(\varpi^*+1)+\vartheta^*\frac{(\Psi(\eta)-\Psi(\mathfrak{p}))^{\varsigma^\star+1}}{\Gamma(\varsigma^\star+2)}+|\delta|\frac{(\Psi(\mathfrak{p})-\Psi(\mathfrak{q}))^{\varsigma^\star+1}}{\Gamma(\varsigma^\star+2)}\right)\epsilon$ with $\varphi(0)=0$, it has been observed that the criterion for generalized Hyers-Ulam stability is also met. This confirms the conclusion of the proof.

4. Application

This section provides an example to demonstrate how to apply and verify the conclusions drawn earlier. To illustrate the application of the theoretical concepts discussed in this work, we outline a specific scenario.

Example

Let's discuss the boundary value problem presented below:

$$\begin{cases} {}^C\mathbb{D}^{\frac{5}{4},\Psi}_{\mathfrak{q}^+}\left({}^C\mathbb{D}^{\frac{1}{5},\Psi}_{\mathfrak{q}^+}+\frac{1}{3}\right)\nu(\mathfrak{r}) = \frac{1}{19+\cos(\mathfrak{r})}\left(\frac{|\nu(\mathfrak{r})|+2}{|\nu(\mathfrak{r})|+1}+\frac{\nu(\frac{1}{3}\mathfrak{r})}{1+\nu(\frac{1}{3}\mathfrak{r})}+{}^C\mathbb{D}^{\frac{5}{4},\Psi}_{\mathfrak{q}^+}\nu(\mathfrak{r})\right) \\ \nu(0)=0,\ \nu(1)+\frac{1}{3}I^{\frac{1}{5},\Psi}_{\mathfrak{q}^+}+x(1)=0,\ {}^C\mathbb{D}^{\frac{1}{5},\Psi}_{\mathfrak{q}^+}x\left(\frac{1}{2}\right)+\frac{1}{3}x\left(\frac{1}{2}\right)=0,\ \eta\in(0,1) \end{cases} \tag{16}$$

Clearly f is continuous,

$$f(\mathfrak{r},\nu(\mathfrak{r}),\nu(\delta\mathfrak{r}),{}^C\mathbb{D}^{\varsigma^\star,\Psi}_{\mathfrak{q}^+}\nu(\mathfrak{r})) = \frac{1}{19+\cos(\mathfrak{r})}\left(\frac{|\nu(\mathfrak{r})|+2}{|\nu(\mathfrak{r})|+1}+\frac{x(\frac{1}{3}\mathfrak{r})}{1+x(\frac{1}{3}\mathfrak{r})}+{}^C\mathbb{D}^{\frac{5}{4},\Psi}_{\mathfrak{q}^+}\nu(\mathfrak{r})\right)$$

where

$$\mathfrak{J}=[0,1]\quad \varsigma^\star=\frac{5}{4},\quad \rho=\frac{1}{5},\quad \mathfrak{p}=1,\quad \mathfrak{q}=0,\quad \Psi(\mathfrak{r})=\mathfrak{r},\quad \delta=\frac{1}{3},\quad \eta=\frac{1}{2}$$

Now

$$\begin{aligned} |f(\mathfrak{r},\nu_1,\nu_2,\nu_3)-f(\mathfrak{r},\upsilon_1,\upsilon_2,\upsilon_3)| &\leq \frac{1}{20}\left(\left|\frac{|\nu_1|+2}{|\nu_1|+1}-\frac{|\upsilon_1|+2}{|\upsilon_1|+1}\right|+\left|\frac{\nu_2}{1+\nu_2}-\frac{\upsilon_2}{1+\upsilon_2}\right|+|\nu_3-\upsilon_3|\right) \\ &\leq \frac{1}{20}\left(|\nu_1-\upsilon_1|+|\nu_2-\upsilon_2|+|\nu_3-\upsilon_3|\right) \end{aligned}$$

So we have $\mathbf{K}_f=\frac{1}{20}$, we find that $\nabla=\nu_2-\nu_1\nu_3\approx 0.1815\neq 0$.

Thus, the conditions $(\mathrm{II}_1)-(\mathrm{II}_3)$ hold with $\mathbf{K}_f=\frac{1}{20}$. Furthermore, using the provided data, it is simple to determine $\Omega\approx 0.435193<1$. By Theorem (3) has a unique solution to problem (17). Additionally, $\Delta=0.418032<1$. Consequently, according to Theorem (4), problem (17) possesses at lest one solution. Additionally, Theorem (5) verifies that problem (17) exhibits both Hyers-Ulam stability and generalized Hyers-Ulam stability.

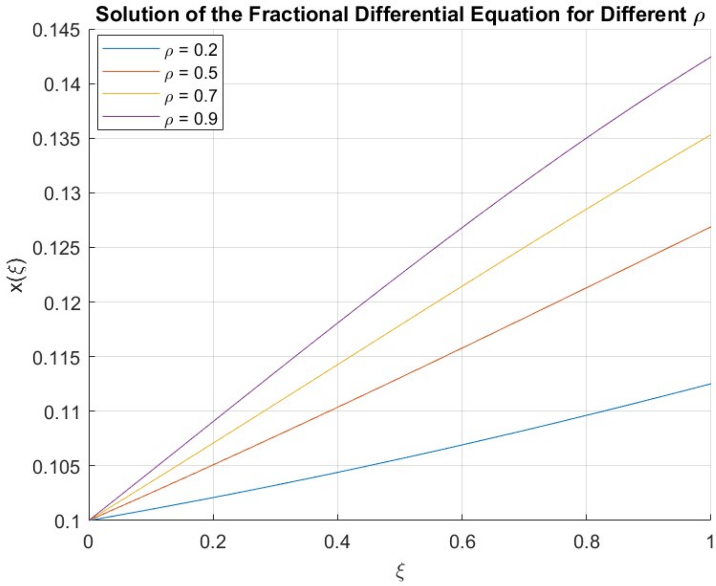

Figure 3. Graph represents the dynamical behavior of the solutions to the fractional differential equation.

5. Conclusion

In this research work, we investigated the existence, uniqueness, and stability of solutions for Problem (3). By employing fixed point theorems, we explored the corresponding theoretical insights. To our knowledge, this methodology has not been previously applied to such problems. Our goal is to enhance the literature by offering a thorough exploration of diverse dynamical processes and their practical applications within fractal environments. In future work, the fractional boundary value problem could be extended to include additional types of fractional derivatives, such as the ABC (Atangana-Baleanu in the Caputo sense) fractional derivative and the mABC (modified Atangana-Baleanu in the Caputo sense) fractional derivative.

Author contributions:

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References

1. Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. In: North-Holland Mathematics Studies. Elsevier; 2006. Volume 204.
2. Podlubny I. Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Elsevier; 1998.
3. Samko SG, Kilbas AA, Marichev OI. Fractional integrals and derivatives: Theory and applications. Gordon and Breach Science Publishers;1993.
4. Fallahgoul H, Focardi S, Fabozzi F. Fractional calculus and fractional processes with applications to financial economics: Theory and application. Academic Press; 2016.
5. Hilfer R. Applications of fractional calculus in physics. World scientific; 2000.
6. Mainardi F. Fractional calculus and waves in linear viscoelasticity: An introduction to mathematical models. World Scientific; 2022.
7. Sabatier J, Agrawal OP, Machado JAT. Advances in fractional calculus. Springer Dordrecht; 2007.
8. Tarasov VE. Fractional dynamics: Applications of fractional calculus to dynamics of particles, fields and media.

Springer; 2011.
9. Argyros IK. The theory and applications of iteration methods. CRC Press; 2022.
10. Ahmed I, Kumam P, Jarad F, et al. On Hilfer generalized proportional fractional derivative. Advances in Difference Equations. 2020; 1–18.
11. Ahmed I, Kumam P, Jarad F, et al. Stability analysis for boundary value problems with generalized nonlocal condition via Hilfer-Katugampola fractional derivative. Advances in Difference Equations. 2020; 1–18.
12. Borisut P, Kumam P, Ahmed I, Sitthithakerngkiet K. Nonlinear Caputo fractional derivative with nonlocal Riemann-Liouville fractional integral condition via fixed point theorems. Symmetry. 2019; 11(6): 829.
13. Argyros IK, George S, Shakhno S, et al. Asymptotically Newton-Type Methods without Inverses for Solving Equations. Mathematics. 2024; 12(7): 1069.
14. Jamal N, Sarwar M, Mlaiki N, Aloqaily A. Solution of linear correlated fuzzy differential equations in the linear correlated fuzzy spaces. AIMS Mathematics. 2024; 9(2): 2695–2721.
15. Khan H, Alzabut J, Shah A, et al. On fractal-fractional waterborne disease model: A study on theoretical and numerical aspects of solutions via simulations. Fractals. 2023; 31(04): 2340055.
16. Khan H, Chen W, Sun H. Analysis of positive solution and Hyers-Ulam stability for a class of singular fractional differential equations with -Laplacian in Banach space. Mathematical Methods in the Applied Sciences. 2018; 41(9): 3430–3440.
17. Almeida R. A Caputo fractional derivative of a function with respect to another function. Communications in Nonlinear Science and Numerical Simulation. 2017; 44: 460–481.
18. Jarad F. Abdeljawad T, Baleanu D. Caputo-type modification of the Hadamard fractional derivatives. Advances in Difference Equations. 2012; 1–8.
19. Luchko Y, and Trujillo J. Caputo-type modification of the Erdélyi-Kober fractional derivative. Fractional Calculus and Applied Analysis. 2007; 10(3): 249–267.
20. Abdo MS, Panchal SK, Saeed AM. Fractional boundary value problem with ψ-Caputo fractional derivative. Proceedings-Mathematical Sciences. 2019; 129(5): 65.
21. Almeida R. Fractional differential equations with mixed boundary conditions. Bulletin of the Malaysian Mathematical Sciences Society. 2019; 42: 1687–1697.
22. Almeida R, Malinowska AB, Monteiro MTT. Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications. Mathematical Methods in the Applied Sciences. 2018; 41(1): 336–352.
23. Abbas S, Benchohra M, Lagreg JE, et al. Existence and Ulam stability for fractional differential equations of Hilfer-Hadamard type. Advances in Difference Equations. 2017; 1–14.
24. Ahmad B, Alsaedi A, Salem S. On a nonlocal integral boundary value problem of nonlinear Langevin equation with different fractional orders. Advances in Difference Equations. 2019; 1–14.
25. Agarwal R, O'Regan D, Hristova S. Stability of Caputo fractional differential equations by Lyapunov functions. Applications of Mathematics. 2015; 60: 653–676.
26. Abbas MI. Existence results and the Ulam stability for fractional differential equations with hybrid proportional-Caputo derivatives. Journal of Nonlinear Functional Analysis. 2020; 1–14.
27. Sene N. Stability analysis of the fractional differential equations with the Caputo-Fabrizio fractional derivative. Journal of Fractional Calculus and Applications. 2020; 11(2): 160–172.
28. Lu Z, Zhu Y, Lu Q. Stability analysis of nonlinear uncertain fractional differential equations with Caputo derivative. Fractals. 2021; 29(03): 2150057.
29. Balachandran K, Kiruthika S, Trujillo J. Existence of solutions of nonlinear fractional pantograph equations. Acta Mathematica Scientia. 2013; 33(3): 712–720.
30. Bhalekar S, and Patade J. Series solution of the pantograph equation and its properties. Fractal and Fractional. 2017; 1(1): 16.
31. Liu MZ, Li D. Properties of analytic solution and numerical solution of multi-pantograph equation. Applied Mathematics and Computation. 2004; 155(3): 853–871.
32. Granas A, Dugundji J. Fixed Point Theory. In: Springer Monographs in Mathematics. Springer; 2003.
33. Smart DR. Fixed point theorems. In: Cambridge Tracts in Mathematics. Cambridge University Press; 1980.