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Article

From computer algebra to gravitational waves

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Abstract: The first finite length differential sequence has been introduced by Janet (1920). Thanks to the first book of Pommaret (1978), the Janet algorithm has been extended by Blinkov, Gerdt, Quadrat, Robertz, Seiler and others who introduced Janet and Pommaret bases in computer algebra. Also, new intrinsic tools have been developed by Spencer in the study of Lie pseudogroups or by Kashiwara in differential homological algebra. The achievement has been to define *differential extension modules* through the systematic use of *differential double duality*. Roughly, if $D = K[d]$ is the non-commutative ring of differential operators with coefficients in a differential field K, let $\mathcal{D}$ be a linear differential operator with coefficients in K. A *direct problem* is to find the generating *compatibility conditions* (CC) in the form of a differential operator $\mathcal{D}_1$ such that $\mathcal{D}\xi = \eta$ implies $\mathcal{D}_1\eta = 0$ and so on. Taking the adjoint operators, we have $ad(\mathcal{D}) \circ ad(\mathcal{D}_1) = ad(\mathcal{D}_1 \circ \mathcal{D}) = 0$ but $ad(\mathcal{D})$ may not generate *all* the CC of $ad(\mathcal{D}_1)$. If M is the D-module defined by $\mathcal{D}$ and N is the D-module defined by $ad(\mathcal{D})$ with torsion submodule $t(N)$, then $t(N) = ext^1(M)$ "*measures*" this gap that only depends on M and *not* on the way to define it. Also, $R = hom_K(M, K)$ is a differential module for the Spencer operator $d : R \to T^* \otimes R$, first introduced by Macaulay with his inverse systems (1916). When $\mathcal{D} : T \to S_2T^* : \xi \to \mathcal{L}(\xi)\omega = \Omega$ is the $Killing$ operator for the Minkowski metric ω with perturbation Ω, then N is the differential module defined by the $Cauchy = ad(Killing)$ operator and $t(N) = ext^1(M) = 0$ because the Spencer sequence is isomorphic to the tensor product of the Poincaré sequence by a Lie algebra. The $Cauchy$ operator can be thus parametrized by stress functions having nothing to do with Ω, like the $Airy$ function for plane elasticity. This result is thus pointing out the *terrible confusion* done by Einstein (1915) while "*adapting*" to space-time the work done by Beltrami (1892) for space only. both of them using the same $Einstein$ operator but ignoring it was self-adjoint in the framework of differential double duality (1995). Though unpleasant it is, we shall prove that *the mathematical foundations of General Relativity are not coherent with these new results* which are also illustrated by many other explicit examples.

Keywords: differential sequence; spencer operator; differential modules; differential duality; extension modules; control theory

MSC CLASSIFICATION: 13D02; 16E30; 20J05; 35N10; 47A09; 83C22; 83C35; 93B05

1. Introduction

In the fall of 1969 I decided to become a visiting student of D. C. Spencer in Princeton university, being attracted by learning his work at the source for future applications to physics. By chance, Spencer gave me his own key of the mathematical library, opened day and night and well furnished with french mathematical literature. However, if on one side I discovered that the intrinsic homological procedure developed by Spencer was exactly what I was dreaming about and decided to “bet” my life on it, on the other side it has been a very bad moment when I discovered that Spencer and collaborators, being proud to be “pure” mathematicians, *were totally unable to compute any explicit example*, the reason for which I had never found any one of them in their papers or books. The reader needs less than five minutes to discover that the introductory examples in the book [1] have not a single link with the core of this book. It is at this precise moment that I discovered, during a night in the library, the work of M. Janet written in 1920 [2] that provided me the “Janet tabular” and the way to mix up a combinatoric approach with an intrinsic framework. But I also discovered the work of E. Vessiot on Lie pseudogroups (1903), that is groups of transformations solutions of systems of ordinary or partial differential equations, still unknown after more than a century and its application to the *Differential Galois theory* (DGT) (1904). I understood at once why the “*structure constant*” appearing in the ”constant curvature condition” existing for Riemannian structures, had strictly nothing to do with the well known structure constants of a lie algebra, a result not known by Spencer. In the meantime, J. A. Wheeler (1911–2008), a close friend of Spencer (1912–2001) in Princeton did propose in 1970 a \$ 1000 challenge for deciding about the existence of a “*potential*” for Einstein equations in vacuum, by analogy with what is existing in Electromagnetism (EM) for Maxwell equations.

This led me to my first GB book appeared in 1978 [3] which has been translated into Russian by MIR in 1983 with a successful distribution in what was called East of Europe because it was new and cheap. This has been the origin of my first private contacts with V. Gerdt (Lectures in Moscow, Doubna and Iaroslav, 14–28/10/1995) before he introduced with Y. Blinkov what they called “Pommaret bases” [4–11] . The problem is that the computer algebra community did not understand that Spencer wanted to apply his methods for studying Lie pseudogroups, not at all for dealing with computers (See the Introduction of my first Kluwer book of 1994 [12] for the first computer study of an example provided by Janet and the references [147-150] for the various results obtained in 1983). During many years, I tried to convince Gerdt and the people of Aachen who were regularly inviting me that the important side for applications is the intrinsic one, even if “*intrinsicness is competing with complexity in computer algebra*” but vainly and I gave it up after they supervised the thesis published in the reference [5] of [13] that must be compared to [3] that they did quote but did not read. It also happened that I had the chance to meet Janet many times as he was living in Paris only a few blocks away from my parents and I can claim that his goal has always been to construct differential sequences along the footnote of his 1920 paper [2]. The main purpose of this paper is to revisit the definition of Pommaret bases in the light of the only existing “*canonical*” differential sequences, respectively called “*Janet sequence*” or “*Spencer sequence*” and to explain why gravitational waves are not coherent with the results obtained.

Let me say a few words about this point. Indeed, as we shall see on many examples later on, it is a FACT that most people believe that if one has a linear differential operator $\mathcal{D} : \xi \to \eta$ and one wants to solve the system $\mathcal{D}\xi = \eta$, there must exist *compatibility conditions* (CC) of the form $\mathcal{D}_1\eta = 0$ and so on. Such a procedure will be called "*step by step*". However, introducing the vector bundles $\wedge^r T^*$ of r-forms with standard multi-index $I = i_1 < ... < i_r$ over a manifold X with local coordinates $(x^i \mid i = 1, 2..., n)$, one may construct "*as a whole*" what is called the *Poincaré sequence* (in France!) for the first order "*exterior derivative* " operator d, namely:

$$\boxed{0 \rightarrow \Theta \rightarrow \wedge^0 T^* \xrightarrow{d} \wedge^1 T^* \xrightarrow{d} \wedge^2 T^* \xrightarrow{d} ... \xrightarrow{d} \wedge^n T^* \rightarrow 0}$$

generalizing the well known $(grad, curl, div)$ sequence existing in vector geometry when $n = 3$ and allowing to describe the parametrization $dA = F$ of the first set of Maxwell equations $dF = 0$. Indeed, one may use the *unique formula* that can be found in any textbook:

$$dx^I = dx^{i_1} \wedge ... \wedge dx^{i_r} \Rightarrow d : \wedge^r T^* \rightarrow \wedge^{r+1} T^* : \omega = \omega_I dx^I \rightarrow d\omega = \partial_i \omega_I dx^i \wedge dx^I$$

as a way to obtain a differential sequence because $\partial_{ij}\omega_I dx^i \wedge dx^j = 0 \Rightarrow d \circ d = 0$. We invite the reader to look at the diagrams in ([3], p. 185 + p. 391) in order to understand why the new methods we shall use in classical or conformal Riemannian geometry have NEVER been used in General Relativity (GR). In particular, studying the Lanczos problems in 2001, I discovered that the $Beltrami = ad(Riemann)$ operator can be parametrized by the $Lanczos = ad(Bianchi)$ operator in the adjoint sequence. As a byproduct, the main purpose of this paper is to explain the confusion done between the $Cauchy = ad(Killing)$ operator and the $Bianchi$ operator in the geometrical and physical long exact dual differential sequences of operators acting on tensors, giving order of operators and number of components:

$$\boxed{\begin{array}{ccccccc} n & \underset{1}{\xrightarrow{Killing}} & \frac{n(n+1)}{2} & \underset{2}{\xrightarrow{Riemann}} & \frac{n^2(n^2-1)}{12} & \underset{1}{\xrightarrow{Bianchi}} & \frac{n^2(n^2-1)(n-2)}{24} \\ & & & & & & \\ n & \underset{1}{\xleftarrow{Cauchy}} & \frac{n(n+1)}{2} & \underset{2}{\xleftarrow{Beltrami}} & \frac{n^2(n^2-1)}{12} & \underset{1}{\xleftarrow{Lanczos}} & \frac{n^2(n^2-1)(n-2)}{24} \end{array}}$$

We end this historical part of the Introduction saying that NO PROGRESS was made during the next 25 years, until I gave a negative answer in 1995, using new concepts introduced for control theory (See chapter VII of [3] and the examples following it), contrary to what the GR community was believing. Wheeler sent me back a letter with a one-dollar bill attached, refusing to admit this result and the impossibility to parametrize the Einstein equations in vacuum can thus only be found in books on control theory (Springer LNCIS 256, 2000 and 311, 2005). Also, I don't know any other reference on the application of "*differential double duality*" to mathematical physics, the main difficulty being that the adjoint of an involutive operator may not be involutive *at all* for both OD and PD equations as we shall see on many illustrating examples, the best and simplest one being surely the double pendulums described in Example 7.

We start recalling standard notations of differential geometry. For that, let $(E, F, ...)$ are vector bundles over a manifold X of dimension n with sections $(\xi, \eta, ...)$, in particular me may introduce the tangent bundle T and cotangent bundle T^*. We shall denote by $J_q(E)$ the q-jet bundle of E with sections ξ_q transforming like the

q-derivatives $j_q(\xi)$. If $\Phi : J_q(E) \to E'$ is a bundle morphism, we shall consider the system $R_q = ker(\Phi) \subset J_q(E)$ of order q on E. The r-prolongation $\rho_r(R_q) = J_r(R_q) \cap J_{q+r}(E) \subset J_r(J_q(E))$, obtained by differentiating formally r times the given ordinary (OD) or partial (PD) defining equations of R_q, will be the kernel of the composite morphism $\rho_r(\Phi) : J_{q++r}(E) \to J_r(J_q(E)) \stackrel{J_r(\Phi)}{\to} J_r(E')$. The symbol $g_{q+r} = R_{q+r} \cap S_{q+r}T^* \otimes E \subset J_{q+r}(E)$ of R_{q+r} is the r-prolongation of the symbol g_q of R_q and the kernel of the composite morphism $\sigma_r(\Phi) : S_{q+r}T^* \otimes E \to S_rT^* \otimes E'$ obtained by restriction. The Spencer operator $d : R_{q+1} \to T^* \otimes R_q : \xi_{q+1} \to j_1(\xi_q) - \xi_{q+1}$ is obtained by using the fact that $R_{q+1} = J_1(R_q) \cap J_{q+1}(E)$ and that $J_1(R_q)$ is an affine vector bundle over R_q modelled on $T^* \otimes R_q$. We shall always suppose that Φ is an epimorphism and introduce the vector bundle $F_0 = J_q(E)/R_q$. The system R_q is said to be *formally integrable* (FI) if $r+1$ prolongations do not bring new equations of order $q+r$ other than the ones obtained after only r prolongations, for any $r \geq 0$, that is all the equations of order $q+r$ can be obtained by differentiating r times *only* the given equations of order q for any $r \geq 0$. The system is said to be *involutive* if it is FI and the symbol g_q is involutive, a purely algebraic property as we shall see [14, 15]. In that case, the successive CC operators can only be *at most* $\mathcal{D}_1, ..., \mathcal{D}_n$ which are first order and involutive operators.

When R_q is not involutive, a standard *prolongation/projection* (PP) procedure allows in general to find integers r, s such that the image $R^{(s)}_{q+r}$ of the projection at order $q+r$ of the prolongation $\rho_{r+s}(R_q) = J_{r+s}(R_q) \cap J_{q+r+s}(E) \subset J_{r+s}(J_q(E))$ is involutive with $\rho_t(R^{(s)}_{q+r}) = R^{(s)}_{q+r+t}, \forall t \geq 0$ but it may highly depend on the parameters.

The next problem is to define the CC operator $\mathcal{D}_1 : F_0 \to F_1 : \eta \to \zeta$ in such a way that the CC of $\mathcal{D}\xi = \eta$ is of the form $\mathcal{D}_1\eta = 0$. As shown in many books [16, 17], such a problem may be quite difficult because the order of the generating CC may be quite high. Proceeding in this way, we may construct the CC $\mathcal{D}_2 : F_1 \to F_2$ of $\mathcal{D}_1$ and so on. The difficulty, shown on the motivating examples, is that " *jumps* " in the successive orders may appear, even on elementary examples. Now, if the map Φ depends on constant (or variable) parameters $(a, b, c, ...)$, then the study of the two previous problems becomes much harder because the ranks of the matrices $\rho_r(\Phi)$ and/or $\sigma_r(\Phi)$ may also highly depend on the parameters as we shall see. Such a question is particularly delicate in the study of the Kerr, Schwarzschild and Minkowski metrics while computing the dimensions of the inclusions $R^{(3)}_1 \subset R^{(2)}_1 \subset R^{(1)}_1 = R_1 \subset J_1(T)$ for the respective Killing operators as the numbers of generating second order CC and the numbers of generating third order CC may change drastically [18]. Other striking motivating examples are also presented.

Example 1. *Let $n = 2, m = 1$ and introduce the trivial vector bundle E with local coordinates (x^1, x^2, ξ) for a section over the base manifold X with local coordinates (x^1, x^2). Let us consider the linear second order system $R_2 \subset J_2(E)$ defined by the two linearly independent equations $d_{22}\xi = 0,\ d_{12}\xi + ad_1\xi = 0$ where a is an arbitrary constant parameter. Using crossed derivatives, we get the second order system $R^{(1)}_2 \subset R_2$ defined by the PD equations $d_{22}\xi = 0, d_{12}\xi + ad_1\xi = 0, a^2d_1\xi = 0$ which is easily seen not to be formally integrable. Hence we have two possibilities:*

- $a = 0$: *We obtain the following second order homogeneous involutive system:*

$$R_2^{(1)} = R_2 \subset J_2(E) \qquad \left\{ \begin{array}{lcl} d_{22}\xi & = & \eta^2 \\ d_{12}\xi & = & \eta^1 \end{array} \right. \boxed{\begin{array}{cc} 1 & 2 \\ 1 & \bullet \end{array}}$$

with only one first order CC operator $d_2\eta^1 - d_1\eta^2 = \zeta$, *leading to the Janet sequence:*

$$\boxed{0 \to \Theta \to \xi \underset{2}{\xrightarrow{\mathcal{D}}} \eta \underset{1}{\xrightarrow{\mathcal{D}_1}} \zeta \to 0}$$

Multiplying by λ *and integrating by parts, we get the operator* $ad(\mathcal{D}_1)$ *which is described by* $-d_2\lambda = \mu^1, -d_1\lambda = \mu^2$.
Multiplying the first equation by μ^1, *the second by* μ^2, *summing and integrating by parts, we notice that* $ad(\mathcal{D})$, *described by* $d_{12}\mu^1 + d_{22}\mu^2 = \nu$, *is of order* 2 *and does not therefore generates the CC of* $ad(\mathcal{D}_1)$ *which is of order* 1, *namely* $d_1\mu^1 + d_2\mu^2 = \nu'$ *as below:*

$$\boxed{\begin{array}{ccccc} \nu & \overset{ad(\mathcal{D})}{\longleftarrow} & \mu & \overset{ad(\mathcal{D}_1)}{\longleftarrow} & \lambda \\ & \swarrow & & & \\ \nu' & & & & \end{array}}$$

and ν' *is a torsion element of the differential module defined by* $ad(\mathcal{D})$ *because* $d_2\nu' = \nu$. *As we shall see in the third section, if* M_1 *is the differential module defined by* $\mathcal{D}_1$, *then we know that* $ext^1(M_1) \neq 0$ *when* $a = 0$ *because* $t(N)$ *is generated by* ν'.

- $a \neq 0$: *We obtain the second order system* $R_2^{(1)}$ *defined by* $d_{22}\xi = 0, d_{12}\xi = 0, d_1\xi = 0$ *with a strict inclusion* $R_2^{(1)} \subset R_2$ *because* $3 < 4$. *We may define* $\eta = d_1\eta^2 - d_2\eta^1 + a\eta^1$ *and obtain the involutive and finite type system in* δ*-regular coordinates:*

$$R_2^{(2)} \subset J_2(E) \qquad \left\{ \begin{array}{lcl} d_{22}\xi & = & \eta^2 \\ d_{12}\xi & = & \eta^1 - \frac{1}{a}\eta \\ d_{11}\xi & = & \frac{1}{a^2}d_1\eta \\ d_1\xi & = & \frac{1}{a^2}\eta \end{array} \right. \boxed{\begin{array}{cc} 1 & 2 \\ 1 & \bullet \\ 1 & \bullet \\ \bullet & \bullet \end{array}}$$

Counting the dimensions, we have the following strict inclusions by comparing the dimensions:

$$R_2^{(2)} \subset R_2^{1)} \subset R_2 \subset J_2(E), \qquad 2 < 3 < 4 < 6$$

The symbol g_{2+r} is involutive with $dim(g_{2+r}) = 1, \forall r \geq 0$ and we have $dim(R_{2+r}) = 4, \forall r \geq 0$. Indeed, using jet notation, the 4 parametric jets of R_2 are $(\xi, \xi_1, \xi_2, \xi_{11})$. The 4 parametric jets of R_3 are now $(\xi, \xi_2, \xi_{11}, \xi_{111})$ and so on. Accordingly, the dimension of g_{2+r} is 1 because the only parametric jet is $\xi_{1....1}$. We have the short exact sequence $0 \to g_{r+2} \to R_{r+2} \to R_{r+1}^{(1)} \to 0$. As $R_2^{(1)}$ does not depend any longer on the parameter, the general solution is easily seen to be of the form $\xi = cx^2 + d$ and is thus only depending on two arbitrary constants, contrary to what could be imagined from this result but in a coherent way with the fact that $dim(R^{(2)}_{2+r}) = 2, \forall r \geq 0$.

After differentiating twice, we could be waiting for CC of order 3. However, we obtain the 4 CC:

$$d_2\eta^1 - \frac{1}{a}d_2\eta - d_1\eta^2 = 0,\ \frac{1}{a^2}d_{12}\eta - d_1\eta^1 + \frac{1}{a}d_1\eta = 0,\ \frac{1}{a^2}d_2\eta - \eta^1 + \frac{1}{a}\eta = 0,\ \frac{1}{a^2}(d_1\eta - d_1\eta) = 0$$

The last CC that we shall call "*identity to zero* " must not be taking into account. The second CC is just the derivative with respect to x^1 of the third CC which amounts to

$$(d_{12}\eta^2 - d_{22}\eta^1 + ad_2\eta^1) - a^2\eta^1 + a(d_1\eta^2 - d_2\eta^1 + a\eta^1) = 0 \Leftrightarrow d_{12}\eta^2 - d_{22}\eta^1 + ad_1\eta^2 = 0$$

which is a second order CC amounting to the first. Hence we get the only generating CC operator $\mathcal{D}_1 : (\eta^1, \eta^2) \to d_{12}\eta^2 - d_{22}\eta^1 + ad_1\eta^2 = \zeta$ which is thus formally surjective.

For helping the reader, we recall that basic elementary combinatorics arguments are giving $dim(S_qT^*) = q + 1$ while $dim(J_q(E)) = (q + 1)(q + 2)/2$ because $n = 2$ and $m = dim(E) = 1$. Hence, the number of generating CC of order 1 is zero and the number of generating CC of strict order 2 is $dim(\rho_1(R'_1)) - dim(R'_2) = 12 - (15 - 4) = 12 - 11 = 1$ in a coherent way.

Setting $F_1 = Q_2$ with $dim(Q_2) = 1$, we obtain the commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & g_5 & \to & S_5T^* \otimes E & \to & \boxed{S_3T^* \otimes F_0} & \to & T^* \otimes F_1 & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & R_5 & \to & J_5(E) & \to & J_3(F_0) & \to & J_1(F_1) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & R_4 & \to & J_4(E) & \to & J_2(F_0) & \to & F_1 & \to 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 & & 0 &
\end{array}$$

with dimensions:

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 1 & \rightarrow & 6 & \rightarrow & \boxed{8} & \rightarrow & 2 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 4 & \rightarrow & 21 & \rightarrow & 20 & \rightarrow & 3 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 4 & \rightarrow & 15 & \rightarrow & 12 & \rightarrow & 1 & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}
$$

The upper symbol sequence is not exact at $S_3T^* \otimes F_0$ even though the two other sequences are exact on the jet level. As a byproduct we have the exact sequences $\forall r \geq 0$:

$$0 \rightarrow R_{r+4} \rightarrow J_{r+4}(E) \rightarrow J_{r+2}(F_0) \rightarrow J_r(F_1) \rightarrow 0$$

Such a result can be checked directly through the identity:

$$4 - (r+5)(r+6)/2 + 2(r+3)(r+4)/2 - (r+1)(r+2)/2 = 0$$

We obtain therefore the formally exact sequence we were looking for with $1 - 2 + 1 = 0$:

$$0 \rightarrow \Theta \rightarrow E \underset{2}{\xrightarrow{\mathcal{D}}} F_0 \underset{2}{\xrightarrow{\mathcal{D}_1}} F_1 \rightarrow 0$$

The very surprising fact is that, in this case, $ad(\mathcal{D})$ *generates the CC of* $ad(\mathcal{D}_1)$. Indeed, multiplying by the Lagrange multiplier test function λ and integrating by parts, we obtain the second order operator $\lambda \rightarrow (-d_{22}\lambda = \mu^1, d_{12}\lambda - ad_1\lambda = \mu^2)$ and thus $-a^2 d_1\lambda = d_1\mu^1 + d_2\mu^2 + a\mu^2$. Substituting, we finally get the only second order CC operator $d_{12}\mu^1 + d_{22}\mu^2 - ad_1\mu^1 = 0$. As we shall see in the third section, we have *now* $ext^1(M_1) = 0$ when $a \neq 0$ and the adjoint sequences:

$$
\begin{array}{ccccccccc}
 & & \xi & \xrightarrow{\mathcal{D}} & \eta & \xrightarrow{\mathcal{D}_1} & \zeta & \rightarrow & 0 \\
 & & & & & & & & \\
0 & \longleftarrow & \nu & \xleftarrow{ad(\mathcal{D})} & \mu & \xleftarrow{ad(\mathcal{D}_1)} & \lambda & &
\end{array}
$$

but the Janet sequence with $1 - 4 + 4 - 1 = 0$ is, thanks to the last Janet tabular [3]:

$$\boxed{0 \rightarrow \Theta \rightarrow 1 \underset{2}{\xrightarrow{\mathcal{D}}} 4 \underset{1}{\xrightarrow{\mathcal{D}_1}} 4 \underset{1}{\xrightarrow{\mathcal{D}_2}} 1 \rightarrow 0}$$

In the differential module framework over the commutative ring $D = K[d_1, d_2]$ of differential operators with coefficients in the trivially differential field $K = \mathbb{Q}(a)$, we have the free resolution:

$$\boxed{0 \to D \underset{2}{\xrightarrow{\mathcal{D}_1}} D^2 \underset{2}{\xrightarrow{\mathcal{D}}} D \to M \to 0}$$

of the differential module M with Euler-Poincaré characteristic $rk_D(M) = 1-2+1 = 0$ (See the next two sections for more details and definitions).

The purpose of the present paper is to revisit these works by using new homological techniques [19–21]. As a matter of fact, they do not agree with the previous ones for the third order CC involved, an unpleasant situation. In any case, we have written this paper in such a way that we are only using elementary combinatorics and diagram chasing. However, an equally important second purpose is to notice that important concepts such as *differential extension modules* have been introduced in *differential homological algebra* [22] and are known, thanks to a quite difficult theorem [20], to be the only intrinsic results that could be obtained independently of the differential sequence that could be used, provided that one is using another system on E with the same solutions. Equivalently, this amounts to say, in a few words but a more advanced language, if *we are keeping an isomorphic differential module but changing its presentation*.

Of course, *in general* as we just saw, the extension modules may highly depend on the parameters. However, as we shall see, there are even simple academic systems depending on parameters but such that *a convenient equivalent system, say involutive with the same solutions, may no longer depend on the parameters* and the extension modules do not depend on the parameters because it is known that they do not depend on the differential sequence used for their definition. This will be *exactly* the situation met in the study of the Minkowski, Schwarzschild and Kerr metrics while studying the respective Killing operators [18] .

Also, in a totally independent way, still not acknowledged after more than a century, E. Vessiot has shown that certain operators may depend on geometric objects satisfying non-linear *structure equations* that are depending on certain *Vessiot structure constants* c. The simplest example is the condition of constant Riemannian curvature [3, 12] which is necessary in order that the Killing system becomes FI but the case of classical or unimodular contact structures is similar [13] . In such situations, the extension modules may of course depend on these constants.

2. Differential systems

If X is a manifold of dimension n with local coordinates $(x) = (x^1, ..., x^n)$, we denote as usual by $T = T(X)$ the *tangent bundle* of X, by $T^* = T^*(X)$ the *cotangent bundle*, by $\wedge^r T^*$ the *bundle of r-forms* and by S_qT^* the *bundle of q-symmetric tensors*. More generally, let E be a *vector bundle* over X with local coordinates (x^i, y^k) for $i = 1, ..., n$ and $k = 1, ..., m$ simply denoted by (x, y), *projection* $\pi : E \to X : (x, y) \to (x)$ and changes of local coordinate $\bar{x} = \varphi(x), \bar{y} = A(x)y$. We shall denote by E^* the vector bundle obtained by inverting the matrix A of the changes of coordinates, exactly like T^* is obtained from T. We denote by $f : X \to E : (x) \to (x, y = f(x))$ a global *section* of E, that is a map such that $\pi \circ f = id_X$ but local sections over an open set $U \subset X$ may also be considered when needed. Under a change of coordinates, a section transforms like $\bar{f}(\varphi(x)) = A(x)f(x)$ and the changes of the derivatives can also be obtained with more work. We shall denote by $J_q(E)$ the *q-jet bundle* of E with local coordinates $(x^i, y^k, y^k_i, y^k_{ij}, ...) = (x, y_q)$ called *jet coordinates* and sections $f_q : (x) \rightarrow (x, f^k(x), f^k_i(x), f^k_{ij}(x), ...) = (x, f_q(x))$ transforming like the sections $j_q(f) : (x) \rightarrow (x, f^k(x), \partial_i f^k(x), \partial_{ij} f^k(x), ...) = (x, j_q(f)(x))$ where both f_q and $j_q(f)$ are over the section f of E. For any $q \geq 0$, $J_q(E)$ is a vector bundle over X with projection π_q while $J_{q+r}(E)$ is a vector bundle over $J_q(E)$ with projection $\pi^{q+r}_q, \forall r \geq 0$.

Let $\mu = (\mu_1, ..., \mu_n)$ be a multi-index with *length* $|\mu| = \mu_1 + ... + \mu_n$, *class* i if $\mu_1 = ... = \mu_{i-1} = 0, \mu_i \neq 0$ and $\mu + 1_i = (\mu_1, ..., \mu_{i-1}, \mu_i + 1, \mu_{i+1}, ..., \mu_n)$. We set $y_q = \{y^k_\mu | 1 \leq k \leq m, 0 \leq |\mu| \leq q\}$ with $y^k_\mu = y^k$ when $|\mu| = 0$. If E is a vector bundle over X and $J_q(E)$ is the *q-jet bundle* of E, then both sections $f_q \in J_q(E)$ and $j_q(f) \in J_q(E)$ are over the section $f \in E$. There is a natural way to distinguish them by introducing the *Spencer* operator $d : J_{q+1}(E) \rightarrow T^* \otimes J_q(E)$ with components $(df_{q+1})^k_{\mu,i}(x) = \partial_i f^k_\mu(x) - f^k_{\mu+1_i}(x)$. The kernel of d consists of sections such that $f_{q+1} = j_1(f_q) = j_2(f_{q-1}) = ... = j_{q+1}(f)$. Finally, if $R_q \subset J_q(E)$ is a *system* of order q on E locally defined by linear equations $\Phi^\tau(x, y_q) \equiv a^{\tau\mu}_k(x) y^k_\mu = 0$ and local coordinates (x, z) for the parametric jets up to order q, the *r-prolongation* $R_{q+r} = \rho_r(R_q) = J_r(R_q) \cap J_{q+r}(E) \subset J_r(J_q(E))$ is locally defined when $r = 1$ by the linear equations $\Phi^\tau(x, y_q) = 0, d_i\Phi^\tau(x, y_{q+1}) \equiv a^{\tau\mu}_k(x) y^k_{\mu+1_i} + \partial_i a^{\tau\mu}_k(x) y^k_\mu = 0$ and has *symbol* $g_{q+r} = R_{q+r} \cap S_{q+r}T^* \otimes E \subset J_{q+r}(E)$ if one looks at the *top order terms*. If $f_{q+1} \in R_{q+1}$ is over $f_q \in R_q$, differentiating the identity $a^{\tau\mu}_k(x) f^k_\mu(x) \equiv 0$ with respect to x^i and subtracting the identity $a^{\tau\mu}_k(x) f^k_{\mu+1_i}(x) + \partial_i a^{\tau\mu}_k(x) f^k_\mu(x) \equiv 0$, we obtain the identity $a^{\tau\mu}_k(x)(\partial_i f^k_\mu(x) - f^k_{\mu+1_i}(x)) \equiv 0$ and thus the restriction $d : R_{q+1} \rightarrow T^* \otimes R_q$. More generally, we have the restriction:

$$d : \wedge^s T^* \otimes R_{q+1} \rightarrow \wedge^{s+1} T^* \otimes R_q : (f^k_{\mu,I}(x) dx^I) \rightarrow ((\partial_i f^k_{\mu,I}(x) - f^k_{\mu+1_i,I}(x)) dx^i \wedge dx^I)$$

with standard multi-index notation for exterior forms and one can easily check that $d \circ d = 0$. The restriction of $-d$ to the symbol is called the *Spencer* map δ in the sequences:

$$\wedge^{s-1} T^* \otimes g_{q+1} \xrightarrow{\delta} \wedge^s T^* \otimes g_q \xrightarrow{\delta} \wedge^{s+1} T^* \otimes g_{q-1}$$

with $\delta \circ \delta = 0$ leading to the purely algebraic δ-*cohomology* $H^s_{q+r}(g_q)$ at $\wedge^s T^* \otimes g_q$ [3,12,16].

Definition 1. *If $R_q \subset J_q(E)$ is a system of order q on E, then $R_{q+r} = \rho_r(R_q) = J_r(R_q) \cap J_{q+r}(E) \subset J_r(J_q(E))$ is called the r-prolongation of R_q. In actual practice, if the system is defined by PDE $\Phi^\tau \equiv a^{\tau\mu}_k(x) y^k_\mu = 0$ the first prolongation is defined by adding the PDE $d_i\Phi^\tau \equiv a^{\tau\mu}_k(x) y^{\mu+1_i} + \partial_i a^{\tau mu}_k(x) y^k_\mu = 0$. Accordingly, $f_q \in R_q \Leftrightarrow a^{\tau\mu}_k(x) f^k_\mu(x) = 0$ and $f_{q+1} \in R_{q+1} \Leftrightarrow a^{\tau\mu}_k(x) f^k_{\mu+1_i}(x) + \partial_i a^{\tau\mu}_k(x) f^k_\mu(x) = 0$ as identities on X. Differentiating the first relation with respect to x^i and substracting the second, we finally obtain:*

$$a_k^{\tau\mu}(x)(\partial_i f_\mu^k(x) - f_{\mu+1_i}^k(x)) = 0 \Rightarrow df_{q+1} \in T^* \otimes R_q$$

and the Spencer operator restricts to $d : \mathcal{R}_{q+1} \to T^* \otimes R_q$*. We set* $R_{q+r}^{(s)} = \pi_{q+r}^{q+r+s}(R_{q+r+s})$*.*

Definition 2. *The symbol of* R_q *is the family* $g_q = R_q \cap S_q T^* \otimes E$ *of vector spaces over* X*. The symbol* g_{q+r} *of* $\mathcal{R}_{q+r}$ *only depends on* g_q *by a direct prolongation procedure. We may define the vector bundle* F_0 *over* $\mathcal{R}_q$ *by the short exact sequence* $0 \to R_q \to J_q(E) \to F_0 \to 0$ *and we have the exact induced sequence* $0 \to g_q \to S_q T^* \otimes E \to F_0$*.*

When $\mid \mu \mid= q$*, we obtain:*

$$g_q = \{v_\mu^k \in S_q T^* \otimes E \mid a_k^{\tau\mu}(x) v_\mu^k = 0\}, \mid \mu \mid= q$$

$$\Rightarrow g_{q+r} = \rho_r(g_q) = \{v_{\mu+\nu}^k \in S_{q+r} T^* \otimes E \mid a_k^{\tau\mu}(x) v_{\mu+\nu}^k = 0\}, \mid \mu \mid= q, \mid \nu \mid= r$$

In general, neither g_q nor g_{q+r} are vector bundles over X as can be seen in the simple example $xy_x - y = 0 \Rightarrow xy_{xx} = 0$.

On $\wedge^s T^*$ we may introduce the usual bases $\{dx^I = dx^{i_1} \wedge ... \wedge dx^{i_s}\}$ where we have set $I = (i_1 < ... < i_s)$. In a purely algebraic setting, one has:

Proposition 1. *There exists a map* $\delta : \wedge^s T^* \otimes S_{q+1} T^* \otimes E \to \wedge^{s+1} T^* \otimes S_q T^* \otimes E$ *which restricts to* $\delta : \wedge^s T^* \otimes g_{q+1} \to \wedge^{s+1} T^* \otimes g_q$ *and* $\delta^2 = \delta \circ \delta = 0$*.*

Proof. Let us introduce the family of s-forms $\omega = \{\omega_\mu^k = v_{\mu,I}^k dx^I\}$ and set $(\delta\omega)_\mu^k = dx^i \wedge \omega_{\mu+1_i}^k$. We obtain at once $(\delta^2\omega)_\mu^k = dx^i \wedge dx^j \wedge \omega_{\mu+1_i+1_j}^k = 0$ and $a_k^{\tau\mu}(\delta\omega)_\mu^k = dx^i \wedge (a_k^{\tau\mu}\omega_{\mu+1_i}^k) = 0$. □

The kernel of each δ in the first case is equal to the image of the preceding δ but this may no longer be true in the restricted case and we set:

Definition 3. *Let* $B_{q+r}^s(g_q) \subseteq Z_{q+r}^s(g_q)$ *and* $H_{q+r}^s(g_q) = Z_{q+r}^s(g_q)/B_{q+r}^s(g_q)$ *with* $H^s(g_q) = H_q^s(g_q)$ *be the coboundary space* $im(\delta)$*, cocycle space* $ker(\delta)$ *and cohomology space at* $\wedge^s T^* \otimes g_{q+r}$ *of the restricted* δ*-sequence which only depend on* g_q *and may not be vector bundles. The symbol* g_q *is said to be s-acyclic if* $H_{q+r}^1 = ... = H_{q+r}^s = 0, \forall r \geq 0$*, involutive if it is n-acyclic and finite type if* $g_{q+r} = 0$ *becomes trivially involutive for r large enough. In particular, if* g_q *is involutive and finite type, then* $g_q = 0$*. Finally,* $S_q T^* \otimes E$ *is involutive for any* $q \geq 0$ *if we set* $S_0 T^* \otimes E = E$*.*

Having in mind the example of $xy_x - y = 0 \Rightarrow xy_{xx} = 0$ with rank changing at $x = 0$, we have:

Proposition 2. *If* g_q *is* 2*-acyclic and* g_{q+1} *is a vector bundle, then* g_{q+r} *is a vector bundle* $\forall r \geq 1$*.*

Proof. We may define the vector bundle F_1 by the following ker/coker long exact sequence:

$$0 \to g_{q+1} \to S_{q+1} T^* \otimes E \to T^* \otimes F_0 \to F_1 \to 0$$

and we obtain by induction on r the following commutative and exact diagram of vector bundles:

$$\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \rightarrow & g_{q+r+1} & \rightarrow & S_{q+r+1}T^* \otimes E & \rightarrow & S_{r+1}T^* \otimes F_0 & \rightarrow & S_rT^* \otimes F_1 \\
 & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\
0 \rightarrow & T^* \otimes g_{q+r} & \rightarrow & T^* \otimes S_{q+r}T^* \otimes E & \rightarrow & T^* \otimes S_rT^* \otimes F_0 & \rightarrow & T^* \otimes S_{r-1}T^* \otimes F_1 \\
 & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
0 \rightarrow & \wedge^2T^* \otimes g_{q+r-1} & \rightarrow & \wedge^2T^* \otimes S_{q+r-1}T^* \otimes E & \rightarrow & \wedge^2T^* \otimes S_{r-1}T^* \otimes F_0 & & \\
 & \downarrow \delta & & \downarrow \delta & & & & \\
 & \wedge^3T^* \otimes S_{q+r-2}T^* \otimes E & = & \wedge^3T^* \otimes S_{q+r-2}T^* \otimes E & & & &
\end{array}$$

A chase proves that the upper sequence is exact at $S_{r+1}T^* \otimes F_0$ whenever g_q is 2-acyclic by extending the diagram. Defining g'_{r+1} by the exact sequence:

$$0 \rightarrow g'_{r+1} \rightarrow S_{r+1}T^* \otimes F_0 \rightarrow S_rT^* \otimes F_1$$

the proposition finally follows by upper-semicontinuity from the relation:

$$dim(g_{q+r+1}) + dim(g'_{r+1}) = m\ dim(S_{q+r+1}T^*)$$

□

Lemma 1. *If g_q is involutive and g_{q+1} is a vector bundle, then g_q is also a vector bundle. In this case, changing linearly the local coordinates if necessary, we may look at the maximum number β of equations that can be solved with respect to $v^k_{n...n}$ and the intrinsic number $\alpha = m - \beta$ indicates the number of y that can be given arbitrarily.*

Using the exactness of the preceding diagram and chasing in the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & g_{q+r+1} & \rightarrow & S_{q+r+1}T^* \otimes E & \rightarrow & S_{r+1}T^* \otimes F_0 & \rightarrow & S_rT^* \otimes F_1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R_{q+r+1} & \rightarrow & J_{q+r+1}(E) & \rightarrow & J_{r+1}(F_0) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R_{q+r} & \rightarrow & J_{q+r}(E) & \rightarrow & J_r(F_0) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

we have (See [12], pp. 95–98 for details):

Theorem 1. *If $R_q \subset J_q(E)$ is a system of order q on E such that g_{q+1} is a vector bundle and g_q is 2-acyclic, then there is a commutative diagram:*

$$\begin{array}{ccccccc}
0 & \rightarrow & R^{(1)}_{q+r} & \rightarrow & R_{q+r} & \xrightarrow{\kappa_r} & S_rT^* \otimes F_1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & J_r(R^{(1)}_q) & \rightarrow & J_r(R_q) & \xrightarrow{J_r(\kappa)} & J_r(F_1)
\end{array}$$

where κ_r is called the r-curvature and $\kappa = \kappa_0$ is simply called the curvature of R_q.

We notice that $R_{q+r+1} = \rho_r(R_{q+1})$ and $R_{q+r} = \rho_r(R_q)$ in the following commutative diagram:

$$
\begin{array}{ccc}
R_{q+r+1} & \stackrel{\pi^{q+r+1}_{q+1}}{\longrightarrow} & R_{q+1} \\
\downarrow \pi^{q+r+1}_{q+r} & & \downarrow \pi^{q+1}_{q} \\
R^{(1)}_{q+r} & \stackrel{\pi^{q+r}_{q}}{\longrightarrow} & R^{(1)}_{q} \\
\cap & & \cap \\
R_{q+r} & \stackrel{\pi^{q+r}_{q}}{\longrightarrow} & R_q
\end{array}
$$

We also have $R^{(1)}_{q+r} \subseteq \rho_r(R^{(1)}_q)$ because we have successively:

$$
\begin{array}{rcl}
R^{(1)}_{q+r} = \pi^{q+r+1}_{q+r}(R_{q+r+1}) & = & \pi^{q+r+1}_{q+r}(J_r(R_{q+1}) \cap J_{q+r+1}(E) \\
 & \subseteq & J_r(\pi^{q+1}_q)(J_r(R_{q+1})) \cap J_{q+r}(E) \\
 & = & J_r(R^{(1)}_q) \cap J_{q+r}(E) \\
 & = & \rho_r(R^{(1)}_q)
\end{array}
$$

while chasing in the following commutative 3-dimensional diagram:

$$
\begin{array}{ccccccc}
 & & J_r(R_{q+1}) & & \longrightarrow & & J_r(J_{q+1}(E)) \\
 & \nearrow & \downarrow & & & \nearrow & \\
R_{q+r+1} & & \longrightarrow & J_{q+r+1}(E) & & \downarrow & \\
\downarrow & & J_r(R_q) & & \longrightarrow & & J_r(J_q(E)) \\
 & \nearrow & & \downarrow & & \nearrow & \\
R_{q+r} & & \longrightarrow & J_{q+r}(E) & & &
\end{array}
$$

with a well defined map $J_r(\pi^{q+1}_q) : J_r(J_{q+1}(E)) \rightarrow J_r(J_q(E))$.

We finally obtain the following crucial Theorem [3, 12] which is crucial for any application:

Theorem 2. *Let $R_q \subset J_q(E)$ be a system of order q on E such that R_{q+1} is a vector sub-bundle of $J_{q+1}(E)$. If g_q is 2-acyclic and g_{q+1} is a vector bundle, then we have $R^{(1)}_{q+r} = \rho_r(R^{(1)}_q)$ for all $r \geq 0$.*

Definition 4. *A system $R_q \subset J_q(E)$ is said to be formally integrable if $\pi^{q+r+1}_{q+r} : R_{q+r+1} \rightarrow R_{q+r}$ is an epimorphism of vector bundles $\forall r \geq 1$ and involutive if it is formally integrable with an involutive symbol g_q. We have the following useful test [3,12]:*

Corollary 1. *Let $R_q \subset J_q(E)$ be a system of order q on E such that R_{q+1} is a vector sub-bundle of $J_{q+1}(E)$. If g_q is 2-acyclic (involutive) and if the map $\pi^{q+1}_q : R_{q+1} \rightarrow R_q$ is an epimorphism of vector bundles, then R_q is formally integrable (involutive). Such a result can be easily extended to nonlinear systems [12].*

The next procedure providing a *Pommaret basis* and where *one may have to change linearly the independent variables if necessary*, is intrinsic even though it must be checked in a particular coordinate system called δ*-regular* [3, 12]. For exampe, the system $y_{12} = 0, y_{11} = 0$ is not δ-regular as no y_{22} may appear unless we exchange x^1 with x^2 in order to obtain the new system $y_{22} = 0, y_{12} = 0$ as in Example 1.

- *Equations of class n*: Solve the maximum number β^n_q of equations with respect to the jets of order q and class n. Then call $(x^1, ..., x^n)$ *multiplicative variables*.
- *Equations of class $i \geq 1$*: Solve the maximum number β^i_q of *remaining* equations

with respect to the jets of order q and class i. Then call $(x^1, ..., x^i)$ *multiplicative variables* and $(x^{i+1}, ..., x^n)$ *non-multiplicative variables*.

- *Remaining equations equations of order* $\leq \ q-1$: Call $(x^1, ..., x^n)$ *non-multiplicative variables*.

In actual practice, we shall use a *Janet tabular* where the multiplicative "variables" are in upper left position while the non-multiplicative variables are represented by dots in lower right position. ccording to the previous results, a system of PD equations is *involutive* if its first prolongation can be obtained by prolonging its equations only with respect to the corresponding multiplicative variables. In that case, we may introduce the *characters* $\alpha_q^i = m\frac{(q+n-i-1)!}{(q-1)!((n-i)!} - \beta_q^i$ for $i = 1, ..., n$ with $\alpha_q^1 \geq ... \geq \alpha_q^n \geq 0$ and we have $dim(g_q) = \alpha_q^1 + ... + \alpha_q^n$ while $dim(g_{q+1}) = \alpha_q^1 + ... + n\alpha_q^n$.

We now recall the main results and definitions that are absolutely needed for the applications.

With canonical epimorphism $\Phi_0 = \Phi : J_q(E) \Rightarrow J_q(E)/R_q = F = F_0$, the various prolongations are described by the following commutative and exact "*introductory diagram* " in which we set $R'_r = im(\rho_r(\Phi)) \subset J_r(F_0)$ with $R'_0 = F_0$, $Q_r = coker(\rho_r(\Phi))$ and $h_{r+1} = coker(\sigma_{r+1}(\Phi))$:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & g_{q+r+1} & \rightarrow & S_{q+r+1}T^* \otimes E & \stackrel{\sigma_{r+1}(\Phi)}{\rightarrow} & S_{r+1}T^* \otimes F_0 & \rightarrow & h_{r+1} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R_{q+r+1} & \rightarrow & J_{q+r+1}(E) & \stackrel{\rho_{r+1}(\Phi)}{\rightarrow} & J_{r+1}(F_0) & \rightarrow & Q_{r+1} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R_{q+r} & \rightarrow & J_{q+r}(E) & \stackrel{\rho_r(\Phi)}{\rightarrow} & J_r(F_0) & \rightarrow & Q_r & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}$$

Chasing along the diagonal of this diagram while applying the standard "*snake*" lemma, we notice that $im(\sigma_r(\Phi)) \subseteq g'_r$ and obtain the useful "*long exact connecting sequence* ":

$$\boxed{0 \rightarrow g_{q+r+1} \rightarrow R_{q+r+1} \rightarrow R_{q+r} \rightarrow h_{r+1} \rightarrow Q_{r+1} \rightarrow Q_r \rightarrow 0}$$

which is thus connecting in a tricky way FI (*lower left*) with CC (*upper right*).

A key step in the procedure for constructing differential sequences will be to use the following (difficult) theorems and corollary (See [3,12] for more details).

Theorem 3. *There is a finite Prolongation/Projection (PP) algorithm providing two integers* $r, s \geq 0$ *by successive increase of each of them such that the new system* $R^{(s)}_{q+r} = \pi^{q+r+s}_{q+r}(R_{q+r+s})$ *has the same solutions as* R_q *but is FI with a* 2*-acyclic or involutive symbol and first order CC. The maximum order of* $\mathcal{D}_1$ *is thus equal to* $r+s+1$ *as we used* $r+s$ *prolongations but it may be lower because certain CC may generate the higher order ones as will be seen in the motivating examples. Without this procedure, nothing can be said about the CC.*

Definition 5. *A differential sequence is said to be formally exact if it is exact on the jet level composition of the prolongations involved. A formally exact sequence is said to be strictly exact if all the operators/systems involved are FI (See again [3,12] and [18] for more details). A strictly exact sequence is called canonical if all the operators/systems are involutive.*

When $d : J_{q+1}(E) \to T^* \otimes J_q(E) : f_{q+1} \to j_1(f_q) - f_{q+1}$ is the Spencer operator, we have:

Proposition 3. *If $R_q \subset J_q(E)$ and $R_{q+1} \subset J_{q+1}(E)$ are two systems of respective orders q and $q+1$, then $R_{q+1} \subset \rho_1(R_q)$ if and onlty if $\pi_q^{q+1}(R_{q+1}) \subset R_q$ and $dR_{q+1} \subset T^* \otimes R_q$.*

Definition 6. *Let us "cut" the preceding introductory diagram by means of a central vertical line and define $R'_r = im(\rho_r(\Phi)) \subseteq J_r(F_0)$ with $R'_0 = F_0$. Chasing in this diagram, we notice that $\pi_r^{r+1} : J_{r+1}(F_0) \to J_r(F_0)$ induces an epimorphism $\pi_r^{r+1} : R'_{r+1} \to R'_r, \forall r \geq 0$ but the kernel of this epimorphism is not $im(\sigma_{r+1}(\Phi))$ unless R_q is FI.*

Theorem 4. *$R'_{r+1} \subseteq \rho_1(R'_r)$ and $dim(\rho_1(R'_r)) - dim(R'_{r+1})$ is the number of new generating CC of order $r+1$.*

Corollary 2. *The system $R'_r \subset J_r(F_0)$ becomes FI with a 2-acyclic or involutive symbol and $R'_{r+1} = \rho_1(R'_r) \subset J_{r+1}(F_0)$ when r is large enough.*

Using the preceding intrinsic results, one may always suppose that we may start with an involutive system $R_q \subset J_q(E)$, that is formaly integrable with an involutive symbol $g_q \subset S_qT^* \otimes E$. Then, using the $Spencer$ operator, one can constuct *STEP BY STEP* or *AS A WHOLE*, the three differential sequences related by the following *fundamental diagram I* below, namely the $Spencer$ sequence, the $Janet$ sequence and the central *hybrid* sequence which is at the same time the Janet sequence for the trivially involutive operator j_q and the Spencer sequence for the first order system $J_{q+1}(E) \subset J_1(J_q(E))$. With more details, when R_q is involutive, the operator $\mathcal{D} : E \xrightarrow{j_q} J_q(E) \xrightarrow{\Phi} J_q(E)/R_q = F_0$ of order q is said to be *involutive*. Introducing the $Janet$ bundles $F_r = \wedge^r T^* \otimes J_q(E)/(\wedge^r T^* \otimes R_q + \delta(S_{q+1}T^* \otimes E))$, we obtain the *linear Janet sequence* induced by the Spencer operator that has been introduced in [3,12]:

$$\boxed{0 \longrightarrow \Theta \longrightarrow E \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} ... \xrightarrow{\mathcal{D}_n} F_n \longrightarrow 0}$$

where each other operator is first order involutive and generates the CC of the preceding one.

Similarly, introducing the $Spencer$ bundles $C_r = \wedge^r T^* \otimes R_q/\delta(\wedge^{r-1}T^* \otimes g_{q+1})$ we obtain the *linear Spencer sequence* induced by the Spencer operator [3,12]:

$$\boxed{0 \longrightarrow \Theta \xrightarrow{j_q} C_0 \xrightarrow{D_1} C_1 \xrightarrow{D_2} ... \xrightarrow{D_n} C_n \longrightarrow 0}$$

These two sequences are related by the following commutative *Fundamental Diagram I*:

$$
\boxed{\begin{array}{ccccccccccccccc}
 & & & & & & 0 & & 0 & & & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & \Theta & \stackrel{j_q}{\rightarrow} & C_0 & \stackrel{D_1}{\rightarrow} & C_1 & \stackrel{D_2}{\rightarrow} ... & \stackrel{D_{n-1}}{\rightarrow} & & C_{n-1} & \stackrel{D_n}{\rightarrow} & C_n & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & E & \stackrel{j_q}{\rightarrow} & C_0(E) & \stackrel{D_1}{\rightarrow} & C_1(E) & \stackrel{D_2}{\rightarrow} ... & \stackrel{D_{n-1}}{\rightarrow} & & C_{n-1}(E) & \stackrel{D_n}{\rightarrow} & C_n(E) & \rightarrow & 0 \\
 & & & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & & & \downarrow \Phi_{n-1} & & \downarrow \Phi_n & & \\
0 \rightarrow & & \Theta & \rightarrow & E & \stackrel{\mathcal{D}}{\rightarrow} & F_0 & \stackrel{\mathcal{D}_1}{\rightarrow} & F_1 & \stackrel{\mathcal{D}_2}{\rightarrow} ... & \stackrel{\mathcal{D}_{n-1}}{\rightarrow} & & F_{n-1} & \stackrel{\mathcal{D}_n}{\rightarrow} & F_n & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & & & & 0 & & 0 & &
\end{array}}
$$

In this diagram with exact columns, we have:

$$\boxed{0 \rightarrow \wedge^r T^* \otimes R_q + \delta(\wedge^{r-1} T^* \otimes S_{q+1} T^* \otimes E) \rightarrow \wedge^r T^* \otimes J_q(E) \rightarrow F_r \rightarrow 0}$$

$$\boxed{0 \rightarrow \delta(\wedge^{r-1} T^* \otimes g_{q+1}) \rightarrow \wedge^r T^* \otimes R_q \rightarrow C_r \rightarrow 0}$$

We finally recall that, when $R_q \subset J_q(E)$ is an involutive system, then the Spencer sequence is nothing else than the Janet sequence for the first order system $R_{q+1} \subset J_1(R_q)$. Such a (difficult) result can be obtained by using inductively a snake chase in the following commutative and exact diagram, starting with $\Phi_0 = \Phi$ with $R_q \subset J_q(E))$ for $r = 0$ and we have for $r = 1$ the commutative and exact diagram allowing to define C_1:

$$
\boxed{\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & g_{q+1} & \rightarrow & T^* \otimes R_q & \rightarrow & C_1 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \rightarrow & R_{q+1} & \rightarrow & J_1(R_q) & \rightarrow & C_1 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R_q & = & R_q & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}}
$$

More generally, one has $R_{q+r} \subset J_r(R_q)$ for $r \geq 1$ and the procedure is ending when $r = n$ because all the δ-sequences are exact at $\wedge^s T^{\cdot *} \otimes g_{q+r}$ for any $0 \leq s \leq n, r \geq 0$:

$$
\begin{array}{cccccccccccccc}
 & & & & & & 0 & & 0 & & & & 0 & \\
 & & & & & & \downarrow & & \downarrow & & & & \downarrow & \\
 & & 0 & \rightarrow & R_{q+r} & \rightarrow & J_r(R_q) & \rightarrow & J_{r-1}(C_1) & \rightarrow & ... & \rightarrow & C_r & \rightarrow 0 \\
 & & & & & & \downarrow & & \downarrow & & & & \downarrow & \\
 & & 0 & \rightarrow & J_{q+r}(E) & \rightarrow & J_r(J_q(E)) & \rightarrow & J_{r-1}(C_1(E)) & \rightarrow & ... & \rightarrow & C_r(E) & \rightarrow 0 \\
 & & & & \| & & \downarrow J_r(\Phi_0) & & \downarrow J_{r-1}(\Phi_1) & & & & \downarrow \Phi_r & \\
0 \rightarrow & & R_{q+r} & \rightarrow & J_{q+r}(E) & \rightarrow & J_r(F_0) & \rightarrow & J_{r-1}(F_1) & \rightarrow & ... & \rightarrow & F_r & \rightarrow 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & \\
 & & & & 0 & & 0 & & 0 & & & & 0 &
\end{array}
$$

All the following motivating example are taken from standard papers on the relations existing between Janet bases and Pommaret bases but the comparison needs no comment.

Example 2. *With polynomial variables* (x, y, z) *and ground field* $k = \mathbb{Q}$*, let us consider the ideal* $\mathfrak{a} = (z^2 - y^2 - 2x^2, xz + xy, yz + y^2 + x^2) \subset k[x, y, z]$ *with ordering* $x \prec y \prec z$*. Of course, such an ideal is not prime because* $x(y + z) \in \mathfrak{a}$*. Moreover, adding twice the third polynomial to the first, we obtain* $(y + z)^2 \in \mathfrak{a}$ *and we get the prime ideal* $rad(\mathfrak{a}) = \sqrt{\mathfrak{a}} = (y + z, x) = \mathfrak{p}$*. Dropping the present notation while passing to jet notations in the linear PDE framework, we get the following linear homogeneous system of order two* $R_2 \subset J_2(E)$ *with* $dim(R_2) = dim(J_2(E)) - 3 = 10 - 3 = 7$*:*

$$\left\{ \begin{array}{lcl} y_{33} - y_{22} - 2y_{11} & = & 0 \\ y_{23} + y_{22} + y_{11} & = & 0 \\ y_{13} + y_{12} & = & 0 \end{array} \right. \begin{array}{|ccc|} \hline 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & \bullet & \bullet \\ \hline \end{array}$$

Using the last bottom two non-multiplicative variables, we let the reader check that *the system is not involutive in this system of coordinates* ... which does not mean it is not involutive by choosing convenient δ-regular coordinates, even if we already know that R_2 is surely formally integrable (FI) because it is homogeneous of order two. One cannot proceed ahead for computing the characters, even if we know that $\alpha_2^3 = 0$ because the differential module admits the torsion element $z = y_3 + y_2$ with $d_1 z = 0$ and the first equation is solved with respect to y_{33}. Looking at the previous prime ideal, we may add twice the second equation to the first and choose the new variables:

$$\bar{x}^3 = x^3 + x^2, \;\; \bar{x}^2 = x^1, \;\; \bar{x}^1 = x^2$$

in order to obtain now the new equivalent following system $R_2 \subset J_2(E)$ after taking out the bar for simplicity:

$$\left\{ \begin{array}{lcl} y_{33} & = & 0 \\ y_{23} & = & 0 \\ y_{22} + y_{13} & = & 0 \end{array} \right. \begin{array}{|ccc|} \hline 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \\ \hline \end{array}$$

As $dim(E) = 1$, its symbol g_2 is involutive with $dim(g_2) = 3$ with parametric jets (y_{11}, y_{12}, y_{13}). The three characters are thus $\alpha_2^3 = 0, \alpha_2^2 = 0, \alpha_2^1 = 3$ and we check that $dim(g_2) = \alpha_2^1 + \alpha_2^2 + \alpha_2^3 = 3$. We have also $dim(g_3) = \alpha_2^1 + 2\alpha_2^2 + 3\alpha_2^3 = \alpha_2^1 = 3$ and so on with $dim(g_{2+r}) = 3, \forall r \geq 0$.

As R_2 is an involutive system, the corresponding formally exact Janet sequence can be written as follows with only two differentially independent CC for $\mathcal{D}$:

$$0 \longrightarrow \Theta \longrightarrow E \xrightarrow[2]{\mathcal{D}} F_0 \xrightarrow[1]{\mathcal{D}_1} F_1 \longrightarrow 0$$

and thus $dim(E) = 1, dim(F_0) = 3, dim(F_1) = 2$. We write down the following useful Spencer δ-sequences which are exact because g_2 is involutive while providing the dimensions:

$$0 \longrightarrow g_3 \xrightarrow{\delta} T^* \otimes g_2 \xrightarrow{\delta} \wedge^2 T^* \otimes T^* \otimes E \Rightarrow 0 \longrightarrow 3 \xrightarrow{\delta} 9 \xrightarrow{\delta} 9$$

$$0 \longrightarrow g_4 \xrightarrow{\delta} T^* \otimes g_3 \xrightarrow{\delta} \wedge^2 T^* \otimes g_2 \xrightarrow{\delta} \wedge^3 T^* \otimes T^* \otimes E \Rightarrow 0 \longrightarrow 3 \xrightarrow{\delta} 9 \xrightarrow{\delta} 9 \xrightarrow{\delta} 3 \longrightarrow 0$$

$$0 \longrightarrow g_5 \xrightarrow{\delta} T^* \otimes g_4 \xrightarrow{\delta} \wedge^2 T^* \otimes g_3 \xrightarrow{\delta} \wedge^3 T^* \otimes g_2 \longrightarrow 0 \Rightarrow 0 \longrightarrow 3 \xrightarrow{\delta} 9 \xrightarrow{\delta} 9 \xrightarrow{\delta} 3 \longrightarrow 0$$

$$0 \longrightarrow g_6 \xrightarrow{\delta} T^* \otimes g_5 \xrightarrow{\delta} \wedge^2 T^* \otimes g_4 \xrightarrow{\delta} \wedge^3 T^* \otimes g_3 \longrightarrow 0 \Rightarrow 0 \longrightarrow 3 \xrightarrow{\delta} 9 \xrightarrow{\delta} 9 \xrightarrow{\delta} 3 \longrightarrow 0$$

We obtain therefore easily the Spencer bundles $C_r = \wedge^r T^* \otimes R_2/\delta(\wedge^{r-1} T^* \otimes g_3)$. Hence, taking into account the exactness of the previous δ- sequences, we get successively:

$$C_0 = R_2, C_1 = T^* \otimes R_2/\delta(g_3), C_2 = \wedge^2 T^* \otimes R_2/\delta(T^* \otimes g_3), C_3 = \wedge^3 T^* \otimes R_2/\delta(\wedge^2 T^* \otimes g_3)$$

and thus $dim(C_0) = 7, dim(C_1) = (3 \times 7) - 3 = 18, dim(C_2) = (3 \times 7) - 6 = 15, dim(C_3) = 7 - 3 = 4$.

$$\begin{array}{ccccccccccccc}
 & & & & 0 & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \longrightarrow & \Theta & \xrightarrow{j_2} & C_0 & \xrightarrow{D_1} & C_1 & \xrightarrow{D_2} & C_2 & \xrightarrow{D_3} & C_3 & \longrightarrow 0 \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \longrightarrow & E & \xrightarrow{j_2} & C_0(E) & \xrightarrow{D_1} & C_1(E) & \xrightarrow{D_2} & C_2(E) & \xrightarrow{D_3} & C_3(E) & \longrightarrow 0 \\
 & & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & \downarrow \Phi_2 & & \downarrow \Phi_3 & \\
0 \longrightarrow & \Theta & \longrightarrow & E & \xrightarrow{\mathcal{D}} & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 & \xrightarrow{\mathcal{D}_2} & F_2 & \xrightarrow{\mathcal{D}_3} & F_3 & \longrightarrow 0 \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & & & & & 0 & & 0 & & 0 & & 0 &
\end{array}$$

$$\begin{array}{ccccccccccccc}
 & & & & & 0 & & 0 & & 0 & & 0 & \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \longrightarrow & \Theta & \xrightarrow{j_2} & 7 & \xrightarrow{D_1} & 18 & \xrightarrow{D_2} & 15 & \xrightarrow{D_3} & 4 & \longrightarrow 0 \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \longrightarrow & 1 & \xrightarrow{j_2} & 10 & \xrightarrow{D_1} & 20 & \xrightarrow{D_2} & 15 & \xrightarrow{D_3} & 4 & \longrightarrow 0 \\
 & & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & \downarrow & & \downarrow & \\
0 \longrightarrow & \Theta & \longrightarrow & 1 & \xrightarrow{\mathcal{D}} & 3 & \xrightarrow{\mathcal{D}_1} & 2 & \longrightarrow & 0 & & 0 & \\
 & & & & & \downarrow & & \downarrow & & & & & \\
 & & & & & 0 & & 0 & & & & &
\end{array}$$

The morphisms Φ_1, Φ_2, Φ_3 in the vertical short exact sequences are inductively induced from the morphism $\Phi_0 = \Phi$ in the first short exact vertical sequence on the left. The central horizontal sequence can be called " *hybrid sequence* " because it is at the same time a Spencer sequence for the first order system $J_3(E) \subset J_1(J_2(E))$ over $J_2(E)$ and a formally exact Janet sequence for the involutive injective operator $j_2 : E \rightarrow J_2(E)$. It can be constructed step by step, starting with the short exact sequence: $0 \longrightarrow J_3(E) \longrightarrow J_1(J_2(T)) \longrightarrow C_1(E) \longrightarrow 0$ or, equivalently, the short exact symbol sequence: $0 \longrightarrow S_3 T^* \otimes E \longrightarrow T^* \otimes J_2(E) \longrightarrow C_1(E) \longrightarrow 0$, kernel of the projection onto $J_2(T)$ of these affine vector bundles. We also invite the reader, as an exercise, to construct it as a whole by introducing the Spencer bundles $C_r(E) = \wedge^r T^* \otimes J_2(E)/\delta(\wedge^{r-1} T^* \otimes S_3 T^* \otimes E)$. We notice that, *in this particular case*, the Janet sequence may be quite simpler than the Spencer sequence, contrary to what could happen in other examples. We recall that they are *absolutely* needed for studying the group of conformal transformations as in [23].

Example 3. *With polynomial variables* (x, y, z) *and ground field* $k = \mathbb{Q}$*, let us consider the ideal* $\mathfrak{a} = (z^2, y^2, z + x) \Rightarrow \sqrt{\mathfrak{a}} = (x, y, z) = \mathfrak{m}$ *a maximal ideal. Dropping the present notation while passing to jet notations in the linear PDE framework, we get the following linear homogeneous system of order two* $R_2 \subset J_2(E)$ *with* $dim(R_2) = dim(J_2(E)) - 3 = 10 - 3 = 7$ *defined by* $y_{33} = 0, y_{22} = 0, y_3 + y_1 = 0$ *which is not involutive because it is neither formally integrable and the symbol* g_2 *defined by* $y_{33} = 0, y_{22} = 0$ *with* $y = 0, y_1 = 0, y_2 = 0$ *is surely not involutive . We may thus use at least one prolongation, with the hope that the symbol* g_3 *of* $R_3 = \rho_1(R_2)$ *becomes involutive. We have the Janet tabular for* g_3*:*

$$
\left\{
\begin{array}{lcl|ccc|}
\cline{4-6}
y_{333} & = & 0 & 1 & 2 & 3 \\
y_{233} & = & 0 & 1 & 2 & \bullet \\
y_{223} & = & 0 & 1 & 2 & \bullet \\
y_{222} & = & 0 & 1 & 2 & \bullet \\
y_{133} & = & 0 & 1 & \bullet & \bullet \\
y_{122} & = & 0 & 1 & \bullet & \bullet \\
\cline{4-6}
\end{array}
\right.
$$

Using the non-multiplicative variables, we let the reader check that g_3 *is indeed involutive in this system of coordinates* which is thus δ-regular. We already know that R_2 and thus R_3 are surely *not* formally integrable (FI) because of the only PD equation $y_3 + y_1 = 0$. Using the important Theorems 2 and 3, we are sure that $\rho_r(R_3^{(1)}) = R_{r+3}^{(1)}, \forall r \geq 0$. However, one cannot proceed ahead for computing the characters, even if we know that $\alpha_2^3 = 0$ because the first equation is solved with respect to y_{33}. We have $\alpha_3^3 = 0, \alpha_3^2 = 0, \alpha_3^1 = 6 - 2 = 4$ because we have 6 jet coordinates of class 1, namely $(y_{111}, y_{112}, y_{113}, y_{122}, y_{123}, y_{133})$.

We may thus construct $R_3^{(1)}$ with $dim(R_3^{(1)}) = 20 - 16 = 4$ with parametric jets (y, y_1, y_2, y_{12}):

$$\left\{\begin{array}{lcl|ccc|}
y_{333} & = & 0 & 1 & 2 & 3 \\
y_{233} & = & 0 & 1 & 2 & \bullet \\
y_{223} & = & 0 & 1 & 2 & \bullet \\
y_{222} & = & 0 & 1 & 2 & \bullet \\
y_{133} & = & 0 & 1 & \bullet & \bullet \\
y_{123} & = & 0 & 1 & \bullet & \bullet \\
y_{122} & = & 0 & 1 & \bullet & \bullet \\
y_{113} & = & 0 & 1 & \bullet & \bullet \\
y_{112} & = & 0 & 1 & \bullet & \bullet \\
y_{111} & = & 0 & 1 & \bullet & \bullet \\
y_{33} & = & 0 & \bullet & \bullet & \bullet \\
y_{23}+y_{12} & = & 0 & \bullet & \bullet & \bullet \\
y_{22} & = & 0 & \bullet & \bullet & \bullet \\
y_{13} & = & 0 & \bullet & \bullet & \bullet \\
y_{11} & = & 0 & \bullet & \bullet & \bullet \\
y_{3}+y_{1} & = & 0 & \bullet & \bullet & \bullet
\end{array}\right.$$

Strikingly, we discover that the symbol of $R_3^{(1)}$ is $g_3^{(1)} = 0$, providing a finite type involutive system. Such a result could have been found directly through an explicit integration of the initial system which is providing the general solution in the form $y = a(x^2x^3 - x^1x^2) + bx^2 + c(x^3 - x^1) + d$ with 4 arbitrary constants (a, b, c, d). We are thus in position to exhibit the two corresponding Janet and Spencer differential sequences, *without any reference to other technical tools*. *In this particular case indeed*, the Spencer sequence is just isomorphic to the tensor product of the Poincaré differential sequence for the exterior derivative $(grad, curl, div)$ by $\mathbb{R}^4$, a result *absolutely not evident at first sight*, contrary to the Janet sequence which is quite more "elaborate". According to [3, 12], the respective dimensions of the Janet bundles is known at once from the last Janet tabular by counting the number $3 + (2 \times 6) + (3 \times 6) = 33$ of single $\bullet$, the number $6 + (3 \times 6) = 24$ of possible double $\bullet\bullet$ and finally the number 6 of triple $\bullet\bullet\bullet$.

$$\begin{array}{ccccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \Theta & \xrightarrow{j_3} & 4 & \xrightarrow{D_1} & 12 & \xrightarrow{D_2} & 12 & \xrightarrow{D_3} & 4 & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & 1 & \xrightarrow{j_3} & 20 & \xrightarrow{D_1} & 45 & \xrightarrow{D_2} & 36 & \xrightarrow{D_3} & 10 & \longrightarrow & 0 \\
 & & & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & \downarrow \Phi_2 & & \downarrow \Phi_3 & & \\
0 \longrightarrow & & \Theta & \longrightarrow & 1 & \xrightarrow{\mathcal{D}} & 16 & \xrightarrow{\mathcal{D}_1} & 33 & \xrightarrow{\mathcal{D}_2} & 24 & \xrightarrow{\mathcal{D}_3} & 6 & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

As for the corresponding resolution of the differential module involved, it becomes:

$$0 \to D^6 \to D^{24} \to D^{33} \to D^{16} \to D \xrightarrow{p} M \to 0$$

with Euler-Poincaré characteristic $rk_D(M) = 1 - 16 + 33 - 24 + 6 = 0$ in a coherent way.

It is finally important to notice that, in the fundamental diagram I, $R_3^{(1)}$ *cannot* be replaced by $\pi_2^3(R_3^{(1)}) = R_2^{(2)}$ after replacing j_3 by j_2 even though $dim(R_2^{(2)}) = dim(R_3^{(1)}) = 4$:

$$\left\{ \begin{array}{lcl|ccc} y_{33} & = & 0 & 1 & 2 & 3 \\ y_{23} + y_{12} & = & 0 & 1 & 2 & \bullet \\ y_{22} & = & 0 & 1 & 2 & \bullet \\ y_{13} & = & 0 & 1 & \bullet & \bullet \\ y_{11} & = & 0 & 1 & \bullet & \bullet \\ y_3 + y_1 & = & 0 & \bullet & \bullet & \bullet \end{array} \right.$$

because $y_{112} = 0$ is missing when using the Janet tabular, the true reason for which the definition of Pommaret bases existing today in the literature is far from being intrinsic as we said in the Introduction (See [24], Example 6, p. 119 for a similar situation showing out the importance of Spencer δ-acyclicity). In the present situation, as we already said, the Spencer sequence is nothing else than the Janet sequence for the involutive first order system $R_3^{(1)} \subset J_1(R_2^{(2)})$. Replacing the four parametric jets (y, y_1, y_2, y_{12}) of $R_2^{(2)}$ by the new four unknowns (z^1, z^2, z^3, z^4), we obtain the first order system:

$$\left\{ \begin{array}{lcl|ccc} z_3^1 + z^2 & = & 0 & 1 & 2 & 3 \\ z_3^2 & = & 0 & 1 & 2 & 3 \\ z_3^3 + z^4 & = & 0 & 1 & 2 & 3 \\ z_3^4 & = & 0 & 1 & 2 & 3 \\ z_2^1 - z^3 & = & 0 & 1 & 2 & \bullet \\ z_2^2 - z^4 & = & 0 & 1 & 2 & \bullet \\ z_2^3 & = & 0 & 1 & 2 & \bullet \\ z_2^4 & = & 0 & 1 & 2 & \bullet \\ z_1^1 - z^2 & = & 0 & 1 & \bullet & \bullet \\ z_1^2 & = & 0 & 1 & \bullet & \bullet \\ z_1^3 - z^4 & = & 0 & 1 & \bullet & \bullet \\ z_1^4 & = & 0 & 1 & \bullet & \bullet \end{array} \right.$$

The Janet tabular for this "*lowered system*" with 12 equations has $4 + (2 \times 4) = 12$ single $\bullet$ and 4 double $\bullet$ in a coherent way with the previous Spencer sequence and the corresponding resolution is The corresponding resolution of the differential module M' is thus:

$$0 \to D^4 \to D^{12} \to D^{12} \to D^4 \xrightarrow{p} M' \to$$

with Euler-Poincaré characteristic $rk_D(M') = 4 - 12 + 12 - 4 = 0$ in a coherent way.

Example 4. *We study in an intrinsic way an example proposed by V. Gerdt in* 2000. *With the notations of the previous examples, let us consider the ideal* $\mathfrak{a} = (x^2y -$

$z, xy^2 - y) \subset \mathbb{Q}[x, y, z]$. *We transform it into the third order system defined by the two corresponding PD equations, namely* $(y_{112} - y_3 = 0, y_{122} - y_2 = 0)$. *This system is neither involutive, nor even formally integrable. Using crossed derivatives, we obtain two new second order PD equations* $(y_{23} - y_{12} = 0, y_{33} - y_{13} = 0)$ *because* $y_{33} = y_{1123} = y_{1112} = y_{13}$ *after two prolongations that we may differentiate once more in order to get the two new third order PD equations* $(y_{223} - y_2 = 0, y_{233} - y_3 = 0)$. *Accordingly, as* $y_{233} = y_{123}$, *we have thus obtained an equivalent third order system* $R_3 \subset J_3(E)$ *defined by the following PD equations:*

$$\left\{\begin{array}{lcl|ccc|} \cline{4-6} y_{333} - y_{113} & = & 0 & 1 & 2 & 3 \\ y_{233} - y_3 & = & 0 & 1 & 2 & \bullet \\ y_{223} - y_2 & = & 0 & 1 & 2 & \bullet \\ y_{133} - y_{113} & = & 0 & 1 & \bullet & \bullet \\ y_{123} - y_3 & = & 0 & 1 & \bullet & \bullet \\ y_{122} - y_2 & = & 0 & 1 & \bullet & \bullet \\ y_{112} - y_3 & = & 0 & 1 & \bullet & \bullet \\ y_{33} - y_{13} & = & 0 & \bullet & \bullet & \bullet \\ y_{23} - y_{12} & = & 0 & \bullet & \bullet & \bullet \\ \cline{4-6} \end{array}\right.$$

This system may not be involutive because $d_2(y_{122} - y_2)$ cannot vanish but we may modify the ordering by changing coordinates. As the character α_3^2 must be zero, we must in any case make the class 2 full by introducing y_{222} as leading term. The easiest possibility is to make the change of coordinates $(\bar{x}^1 = x^1, \bar{x}^2 = x^2 + x^1, \bar{x}^3 = x^3)$. Suppressing the bar, we obtain the new equations:

$$\left\{\begin{array}{lcl|ccc|} \cline{4-6} y_{333} - y_{113} + y_2 - 2y_3 & = & 0 & 1 & 2 & 3 \\ y_{233} - y_3 & = & 0 & 1 & 2 & \bullet \\ y_{223} - y_2 & = & 0 & 1 & 2 & \bullet \\ y_{222} - y_{112} + y_3 - 2y_2 & = & 0 & 1 & 2 & \bullet \\ y_{133} - y_{113} + y_2 - y_3 & = & 0 & 1 & \bullet & \bullet \\ y_{123} + y_2 - y_3 & = & 0 & 1 & \bullet & \bullet \\ y_{122} + y_{112} + y_2 - y_3 & = & 0 & 1 & \bullet & \bullet \\ y_{33} - y_{23} - y_{13} & = & 0 & \bullet & \bullet & \bullet \\ y_{23} - y_{22} - y_{12} & = & 0 & \bullet & \bullet & \bullet \\ \cline{4-6} \end{array}\right.$$

The system has an involutive symbol if we set $y_2 = 0, y_3 = 0$ and is formally integrable, thus involutive in this new coordinates. *As it is an intrinsic property, it is thus involutive in any coordinate system.* It follows that we have at once the Janet sequence by counting 15 single $\bullet$, $3 + (2 \times 3) = 9$ double $\bullet\bullet$ and 2 triple $\bullet\bullet\bullet$. We have thus $dim(E) = 1, dim(F_0) = 9, dim(F_1) = 15, dim(F_2) = 9, dim(F_3) = 2$. The fundamental diagram I becomes:

$$
\begin{array}{ccccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \Theta & \xrightarrow{j_3} & 11 & \xrightarrow{D_1} & 30 & \xrightarrow{D_2} & 27 & \xrightarrow{D_3} & 8 & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & 1 & \xrightarrow{j_3} & 20 & \xrightarrow{D_1} & 45 & \xrightarrow{D_2} & 36 & \xrightarrow{D_3} & 10 & \longrightarrow & 0 \\
 & & & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & \downarrow \Phi_2 & & \downarrow \Phi_3 & & \\
0 \longrightarrow & & \Theta & \longrightarrow & 1 & \xrightarrow{\mathcal{D}} & 9 & \xrightarrow{\mathcal{D}_1} & 15 & \xrightarrow{\mathcal{D}_2} & 9 & \xrightarrow{\mathcal{D}_3} & 2 & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

It is *much more difficult* to check the dimensions of the Spencer bundles. Calling again $R_3 \subset J_3(E)$ this involutive system, the characters of its symbol g_3 are $\alpha_3^3 = 0, \alpha_3^2 = 0, \alpha_3^1 = 6 - 3 = 3$. We obtain $dim(g_4) = (1 \times 3) + (2 \times 0) + (3 \times 0) = 3$ and more generally $dim(g_{r+3}) = 3, \forall r \geq 0$.
Using the exact δ-sequence:

$$0 \rightarrow g_5 \xrightarrow{\delta} T^* \otimes g_4 \xrightarrow{\delta} \wedge^2 T^* \otimes g_3 \xrightarrow{\delta} \wedge^3 T^* \otimes S_2 T^* \otimes E$$

we obtain for example $C_2 : \wedge^2 T^* \otimes R_3 / \delta(T^* \otimes g_4) \Rightarrow dim(C_2) = (3 \times 11) - ((3 \times 3) - 3) = 33 - 6 = 27$ in a coherent way. Once more, *no classical method could provide these results*.

Example 5. *(Macaulay) Among the best examples we know that justify our comments on Pommaret bases, we shall revisit one which is trivially FI but is among the few rare elementary explicit examples of a 2-acyclic symbol which is not involutive, apart from the symbol of the conformal Killing operator for a non-degenerate metric that we shall consider later on [25]. Let us consider the homogeneous ideal $\mathfrak{a} = (z^2, yz - x^2, y^2) \subset k[x, y, z]$ with $rad(\mathfrak{a}) = (x, y, z) = \mathfrak{m}$ a maximal and thus zero-dimensional prime ideal. We may transform it into an homogeneous second order system of PD equations $R_2 \subset J_2(E)$ defined by $(y_{33} = 9, y_{23} - y_{11} = 0, y_{22} = 0)$ but the reader may treat as well the system $(y_{33} - y_{11} = 0, y_{23} = 0, y_{22} - y_{11} = 0)$. Of course, this system is FI because it is homogeneous but we let the reader check on the Janet tabular that g_2 is not involutive though the coordinate system is surely δ-regular because be have full class 3 and full class 2. All the third order jets vanish but $y_{123} - y_{111} = 0$ leading to $dim(g_3) = 1 \Rightarrow dim(R_3) = 8 = 2^3$ [25]. Finally $g_4' = 0 \Rightarrow dim(R_4) = 8$ and we could believe that we do not need any PP procedure as R_4 is an ivolutive system because g_4 is trivially involutive and R_2 is finite type like the Killing system. Moreover, as we have constant coefficients, the three brackets and their only Jacobi identity do provide the following sequence which is of course quite far from being a Janet sequence as it only involves second order operators. Also, It is important to notice that the knowledge of the first second order operator does not provide any way to obtain the third without passing through the second, contrary to the situation existing in the Janet sequence:*

$$\boxed{0 \to \Theta \to 1 \underset{2}{\xrightarrow{\mathcal{D}}} 3 \underset{2}{\xrightarrow{\mathcal{D}_1}} 3 \underset{2}{\xrightarrow{\mathcal{D}_2}} 1 \to 0}$$

Moreover, such a procedure is rather “experimental” and must be coherent with the theorem saying that the order of generating CC is one plus the number of prolongations needed to reach a 2-acyclic symbol, that is g_3 *must be* 2-acyclic. Equivalently the δ-sequence:

$$0 \to \wedge^2 T^* \otimes g_3 \xrightarrow{\delta} \wedge^3 T^* \otimes g_2 \to 0$$

must be exact. We let the reader prove that the corresponding 3×3 matrix has maximum rank.

It remains to work out the corresponding Janet and Spencer sequences and there is a first delicate point to overcome. Indeed, as $g_4 = 0$, we have thus $R_4 \simeq R_3$ and it could be tempting to start with R_3 and thus to replace j_4 by j_3 in the fundamental diagram I. It should lead to a dead end because $C_0 = R_q \subset J_q(E)$ *must be* an involutive system ([24], Example 3.14, p 119-126). Using j_4 and the fact that $g_4 = 0 \Rightarrow g_5 = 0$, we have $C_r = \wedge^r T^* \otimes R_4, \forall r = 0, 1, 2, 3$.The interest of this approach, having is to obtain the Janet sequence without any explicit calculation from the formula $dim(C_r(E)) = dim(C_r) + dim(F_r)$ in the diagram:

$$\begin{array}{ccccccccccccc}
 & & & & 0 & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \longrightarrow & \Theta & \xrightarrow{j_4} & 8 & \xrightarrow{D_1} & 24 & \xrightarrow{D_2} & 24 & \xrightarrow{D_3} & 8 & \longrightarrow 0 \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \longrightarrow & 1 & \xrightarrow{j_4} & 35 & \xrightarrow{D_1} & 84 & \xrightarrow{D_2} & 70 & \xrightarrow{D_3} & 20 & \longrightarrow 0 \\
 & & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & \downarrow \Phi_2 & & \downarrow \Phi_3 & \\
0 \longrightarrow & \Theta & \longrightarrow & 1 & \underset{4}{\xrightarrow{\mathcal{D}}} & 27 & \xrightarrow{\mathcal{D}_1} & 60 & \xrightarrow{\mathcal{D}_2} & 46 & \xrightarrow{\mathcal{D}_3} & 12 & \longrightarrow 0 \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & & & & & 0 & & 0 & & 0 & & 0 &
\end{array}$$

3. Differential duality

For example, the fact that the Cauchy operator is the adjoint of the Killing operator for the Euclidean metric is in any textbook of continuum mechanics in the chapter ”variational calculus” and the parametrization problem has been quoted by many famous authors, as we said in the Abstract, but only from a computational point of view. However it is still not known that the adjoint of the 20 components of the Bianchi operator has been introduced by C. Lanczos as we explained with details in [26]. However, the main trouble is that these two problems have *never* ben treated in an intrinsic way and, in particular, changes of coordinates have *never* been considered. The same situation can be met for Maxwell equations [27, 28]. We start explaining its link existing between differential duality and the concept of “integration by parts ” [22,29].

For this, let K be a differential field with n derivations d and $D = K[d]$ be the non-commutative ring of differential operators with coefficients in K. If $\xi \stackrel{\mathcal{D}}{\to} \eta$: $m \to p$ is an operator with coefficients in K going from m functions to p functions of n variables, let us introduce p test functions μ called "Lagrange multipliers ". When $m = n = p = 1$, one has $\mu(d^2\eta) = (d^2\mu)\eta + d(-(d\mu)\eta + \mu d\eta) \Rightarrow ad(d^2) = d^2$ and d^2 is self-adjoint. As we shall see, such a result can be extended to operators $P, Q \in D$ with $ad(PQ) = ad(Q)ad(P)$ in D. However, the operator $\nu \stackrel{ad(\mathcal{D})}{\leftarrow} \mu : m \leftarrow p$ is going BACKWARDS, a FACT missed successively by Airy, Beltrami, Maxwell and Einstein. The main problem for computer algebra as we shall see in many examples, in particular for the double pendulums is that an operator can be VERY simple while its adjoint can be VERY complicate and " vice versa ". Also, if both ξ and η have physical meanings, then μ and ν may have COMPLETELY different meanings. Finally if $\mathcal{D}\xi = \eta$ has generating compatibility conditions (CC) $\mathcal{D}_1\eta = 0$, then $ad(\mathcal{D}) \circ ad(\mathcal{D}_1) = ad(\mathcal{D}_1 \circ \mathcal{D}) = 0$ but $ad(\mathcal{D})$ may not generate ALL the CC of $ad(\mathcal{D}_1)$, the " gap " being measured by the differential EXTENSION MODULE $ext^1(M)$ when M is the D-module defined by $\mathcal{D}$, independently of its presentation. One of the most important situation is that of the Poincaré sequence for the exterior derivative d. In this case, it is known the dual sequence is also, *up to sign*, a Poincaré sequence for $ad(d)$ because $ad(d) \circ ad(d) = ad(d \circ d) = 0$. For example, when $n = 3$, we have $ad(grad) = -div, ad(curl) = curl, ad(div) = -grad$. Now, according to the famous three theorems of S. Lie, whenever one is considering a Lie group of transformations $G \times X \to X : (a, x) \to y = f(x, a)$ when G is a Lie group with p parameters $(a^1, ..., a^p)$, any infinitesimal transformation of this group can be generated by a finite number of infinitesimal transformations $\{\theta^i_\tau(x)\partial_i \mid \tau = 1, ..., p\}$ with constant coefficients λ^τ, that is with $\partial_i\lambda^\tau = 0$. With q large enough, introducing the Lie algebra $\mathcal{G} = T_e(G)$, tangent space to G at the identity $e \in G$, we may obtain as in [3] an isomorphism

$$\boxed{\wedge^0 T^* \otimes \mathcal{G} \to R_q \subset J_q(T) : \lambda^\tau(x) \to \xi^k_\mu(x) = \lambda^\tau(x)\partial_\mu\theta^k_\tau(x))}$$

alowing to establish an isomorphism between the tensor product by $\mathcal{G}$ of the Poincaré sequence and the Spencer sequence by means of the Spencer operator as we have:

$$\boxed{(d\xi_{q+1})^k_{\mu,i}(x) = \partial_i\xi^k_\mu(x) - \xi^k_{\mu+1_i} = \partial_i\lambda^k_\tau(x)\partial_\mu\theta^k_\tau(x)}$$

Roughly, if M is the D-module defined by the Killing operator $\mathcal{D}$ for a non-degenerate metric, N is the D-module defined by the Cauchy operator $ad(\mathcal{D})$ with torsion submodule $t(N)$, then $ext^i(M) = 0, \forall i = 1, \ldots, n-1$ and $t(N) = ext^1(M) = 0$ as a way to obtain ALL the results provided in the beginning of the introduction ... by means of a single formula as the extension modules do not depend on the presentation, that is on the underlying differential sequence used!

Though this is far from being evident, we have explained in [28,29] that the link existing between integration by parts and differential duality is the key technical lemma:

Lemma 2. *When $y^k = f^k(x)$ is invertible with $\Delta(x) = det(\partial_i f^k(x)) \neq 0$ and inverse $x = g(y)$, then we have n identities $\frac{\partial}{\partial y^k}(\frac{1}{\Delta}\partial_i f^k(g(y))) = 0$.*

Proposition 4. *The Cauchy operator is the adjoint of the Killing operator in arbitrary dimension, up to sign.*

Proof. Let X be a manifold of dimension n with local coordinates $(x^1, ..., x^n)$, tangent bundle T and cotangent bundle T^*. If $\omega \in S_2T^*$ is a metric with $det(\omega) \neq 0$, we my introduce the standard Lie derivative in order to define the first order Killing operator:

$$\mathcal{D} : \xi \in T \rightarrow \Omega = (\Omega_{ij} = \omega_{rj}(x)\partial_i\xi^r + \omega_{ir}(x)\partial_j\xi^r + \xi^r\partial_r\omega_{ij}(x)) \in S_2T^*$$

Here start the problems because, in our opinion at least, a systematic use of the adjoint operator has never been used in mathematical physics, in particular in control theory and even in continuum mechanics apart through a variational procedure. As will be seen later on, the purely intrinsic definition of the adjoint can only be done in the theory of differential modules by means of the so-called *side changing functor*. From a purely differential geometric point of view, the idea is to associate to any vector bundle E over X a new vector bundle $ad(E) = \wedge^nT^* \otimes E^*$ where E^* is obtained from E by patching local coordinates while inverting the transition matrices, exactly like T^* is obtained from T. It follows that the stress tensor $\sigma = (\sigma^{ij}) \in ad(S_2T^*) = \wedge^nT^* \otimes S_2T$ is *not* a tensor but a tensor density, that is transforms like a tensor up to a certain power of the Jacobian matrix. When $n = 4$, the fact that such an object is called stress-energy tensor does not change anything as it cannot be related to the Einstein tensor which is a true *tensor* indeed. Of course, it is always possible in GR to use $(det(\omega))^{\frac{1}{2}}$ but, as we shall see, the study of contact structures *must* be done without any reference to a background metric. In any case, we may define as usual:

$$ad(\mathcal{D}) : \wedge^nT^* \otimes S_2T \rightarrow \wedge^nT^* \otimes T : \sigma \rightarrow \varphi$$

Multiplying Ω_{ij} by σ^{ij} and integrating by parts, the factor of $-2\,\omega_{kr}\,\xi^r$ is easly seen to be:

$$\nabla_i\sigma^{ik} = \partial_i\sigma^{ik} + \gamma^k_{ij}\sigma^{ij} = \varphi^k$$

with well known Christoffel symbols $\gamma^k_{ij} = \frac{1}{2}\omega^{kr}(\partial_i\omega_{rj} + \partial_j\omega_{ir} - \partial_r\omega_{ij})$.

However, if the stress should be a tensor, we should get for the covariant derivative:

$$\nabla_r\sigma^{ij} = \partial_r\sigma^{ij} + \gamma^i_{rs}\sigma^{sj} + \gamma^j_{rs}\sigma^{is} \Rightarrow \nabla_i\sigma^{ik} = \partial_i\sigma^{ik} + \gamma^r_{ri}\sigma^{ik} + \gamma^k_{ij}\sigma^{ij}$$

The difficulty is to prove that we do not have a contradiction because σ is a tensor density.

If we have an invertible transformation like in the lemma, we have successively by using it:

$$\tau^{kl}(f(x)) = \frac{1}{\Delta}\partial_if^k(x)\partial_jf^l(x)\sigma^{ij}(x)$$

$$\frac{\partial\tau^{kl}}{\partial y^k} = \frac{\partial}{\partial y^k}((\frac{1}{\Delta}\partial_if^k) + \frac{1}{\Delta}\partial_if^k\frac{\partial}{\partial y^k}(\partial_jf^l)\sigma^{ij} + \frac{1}{\Delta}\partial_if^k\partial_jf^l\frac{\partial}{\partial y^k}\sigma^{ij}$$

$$\frac{\partial \tau^{ku}}{\partial y^k} = \frac{1}{\Delta}(\partial_{ij} f^u)\sigma^{ij} + \frac{1}{\Delta}\partial_j f^u \partial_i \sigma^{ij}$$

Now, we recall the transformation law of the Christoffel symbols, namely:

$$\partial_r f^u(x)\gamma^r_{ij}(x) = \partial_{ij} f^u(x) + \partial_i f^k(x)\partial_j f^l(x)\bar{\gamma}^u_{kl}(f(x))$$

$$\Rightarrow \frac{1}{\Delta}\partial_r f^u \gamma^r_{ij}\sigma^{ij} = \frac{1}{\Delta}\partial_{ij} f^u \sigma^{ij} + \bar{\gamma}^u_{kl}(y)\tau^{kl}$$

Eliminating the second derivatives of f we finally get:

$$\psi^u = \frac{\partial \tau^{ku}}{\partial y^k} + \bar{\gamma}^u_{kl} = \frac{1}{\Delta}\partial_r f^u(\partial_i \sigma^{ir} + \gamma^r_{ij}\sigma^{ij}) = \frac{1}{\Delta}\partial_r f^u \varphi^r$$

This tricky technical result explains why the additional term we had is just disappearing in fact when σ is a density. We let the reader use a similar procedure for Maxwell equations [28]. □

Let K be a *differential field* with n commuting derivations $(\partial_1, ..., \partial_n)$ and consider the ring $D = K[d_1, ..., d_n] = K[d]$ of differential operators with coefficients in K with n commuting formal derivatives satisfying $d_i a = a d_i + \partial_i a$ in the operator sense. If $P = a^\mu d_\mu \in D = K[d]$, the highest value of $|\mu|$ with $a^\mu \neq 0$ is called the *order* of the *operator* P and the ring D with multiplication $(P, Q) \longrightarrow P \circ Q = PQ$ is filtred by the order q of the operators. We have the *filtration* $0 \subset K = D_0 \subset D_1 \subset ... \subset D_q \subset ... \subset D_\infty = D$. As an algebra, D is generated by $K = D_0$ and $T = D_1/D_0$ with $D_1 = K \oplus T$ if we identify an element $\xi = \xi^i d_i \in T$ with the vector field $\xi = \xi^i(x)\partial_i$ of differential geometry, but with $\xi^i \in K$ now. It follows that $D = {}_D D_D$ is a *bimodule* over itself, being at the same time a left D-module by the composition $P \longrightarrow QP$ and a right D-module by the composition $P \longrightarrow PQ$. We define the *adjoint* functor $ad : D \longrightarrow D^{op} : P = a^\mu d_\mu \longrightarrow ad(P) = (-1)^{|\mu|} d_\mu a^\mu$ and we have $ad(ad(P)) = P$ both with $ad(PQ) = ad(Q)ad(P), \forall P, Q \in D$. Such a definition can be extended to any matrix of operators by using the transposed matrix of adjoint operators (See [29,30] for more details and applications to control theory or mathematical physics).

Accordingly, if $y = (y^1, ..., y^m)$ are differential indeterminates, then D acts on y^k by setting $d_i y^k = y^k_i \longrightarrow d_\mu y^k = y^k_\mu$ with $d_i y^k_\mu = y^k_{\mu+1_i}$ and $y^k_0 = y^k$. We may therefore use the jet coordinates in a formal way as in the previous section. Therefore, if a system of OD/PD equations is written in the form $\Phi^\tau \equiv a^{\tau\mu}_k y^k_\mu = 0$ with coefficients $a \in K$, we may introduce the free differential module $Dy = Dy^1 + ... + Dy^m \simeq D^m$ and consider the differential *module of equations* $I = D\Phi \subset Dy$, both with the residual *differential module* $M = Dy/D\Phi$ or D-module and we may set $M = {}_D M$ if we want to specify the ring of differential operators. We may introduce the formal *prolongation* with respect to d_i by setting $d_i \Phi^\tau \equiv a^{\tau\mu}_k y^k_{\mu+1_i} + (\partial_i a^{\tau\mu}_k) y^k_\mu$ in order to induce maps $d_i : M \longrightarrow M : \bar{y}^k_\mu \longrightarrow \bar{y}^k_{\mu+1_i}$ by residue with respect to I if we use to denote the residue $Dy \longrightarrow M : y^k \longrightarrow \bar{y}^k$ by a bar like in algebraic geometry. However, for simplicity, we shall not write down the bar when the background will indicate clearly if we are in Dy or in M. As a byproduct, the differential modules we shall consider will always be *finitely generated* $(k = 1, ..., m < \infty)$ and *finitely presented* $(\tau = 1, ..., p <$

∞). Equivalently, introducing the *matrix of operators* $\mathcal{D} = (a_k^{\tau\mu} d_\mu)$ with m columns and p rows, we may introduce the morphism $D^p \xrightarrow{\mathcal{D}} D^m : (P_\tau) \longrightarrow (P_\tau \Phi^\tau)$ over D by acting with D *on the left of these row vectors* while acting with $\mathcal{D}$ *on the right of these row vectors* by composition of operators with $im(\mathcal{D}) = I$. The *presentation* of M is defined by the exact cokernel sequence $D^p \xrightarrow{\mathcal{D}} D^m \longrightarrow M \longrightarrow 0$. We notice that the presentation only depends on K, D and Φ or $\mathcal{D}$, that is to say never refers to the concept of (explicit local or formal) solutions. It follows from its definition that M can be endowed with a *quotient filtration* obtained from that of D^m which is defined by the order of the jet coordinates y_q in $D_q y$. We have therefore the *inductive limit* $0 \subseteq M_0 \subseteq M_1 \subseteq ... \subseteq M_q \subseteq ... \subseteq M_\infty = M$ with $d_i M_q \subseteq M_{q+1}$ and $M = DM_q$ for $q \gg 0$ with prolongations $D_r M_q \subseteq M_{q+r}, \forall q, r \geq 0$. It is important to notice that it may be sometimes quite difficult to work out I_q or M_q from a given presentation which is not involutive.

Definition 7. *An exact sequence of morphisms finishing at M is said to be a resolution of M. If the differential modules involved apart from M are free, that is isomorphic to a certain power of D, we shall say that we have a free resolution of M.*

Having in mind that K is a left D-module with the action $(D, K) \longrightarrow K : (d_i, a) \longrightarrow \partial_i a$ and that D is a bimodule over itself with $PQ \neq QP$, *we have only two possible constructions*:

Definition 8. *We may define the right (care) differential module $hom_D(M, D)$ with $(fP)(m) = (f(m))P \Rightarrow (fPQ)(m) = ((fP)(m))Q = ((f(m))P)Q = (f(m))PQ$.*

Definition 9. *We define the system $R = hom_K(M, K)$ and set $R_q = hom_K(M_q, K)$ as the system of order q. We have the projective limit $R = R_\infty \longrightarrow ... \longrightarrow R_q \longrightarrow ... \longrightarrow R_1 \longrightarrow R_0$. It follows that $f_q \in R_q : y_\mu^k \longrightarrow f_\mu^k \in K$ with $a_k^{\tau\mu} f_\mu^k = 0$ defines a section at order q and we may set $f_\infty = f \in R$ for a section of R. For an arbitrary differential field K, such a definition has nothing to do with the concept of a formal power series solution. One has the following key proposition:*

Proposition 5. *When M is a left D-module, then R is also a left D-module.*

Proof. As D is generated by K and T as we already said, let us define:

$$(af)(m) = af(m) = f(am), \quad \forall a \in K, \forall m \in M$$

$$(\xi f)(m) = \xi f(m) - f(\xi m), \quad \forall \xi = a^i d_i \in T, \forall m \in M$$

In the operator sense, it is easy to check that $d_i a = a d_i + \partial_i a$ and that $\xi\eta - \eta\xi = [\xi, \eta]$ is the standard bracket of vector fields. We finally get $(d_i f)_\mu^k = (d_i f)(y_\mu^k) = \partial_i f_\mu^k - f_{\mu+1_i}^k$ and thus recover *exactly* the Spencer operator of the previous section though *this is not evident at all*. We also get $(d_i d_j f)_\mu^k = \partial_{ij} f_\mu^k - \partial_i f_{\mu+1_j}^k - \partial_j f_{\mu+1_i}^k + f_{\mu+1_i+1_j}^k \Longrightarrow d_i d_j = d_j d_i, \forall i, j = 1, ..., n$ and thus $d_i R_{q+1} \subseteq R_q \Longrightarrow d_i R \subset R$ induces a well defined operator $R \longrightarrow T^* \otimes R : f \longrightarrow dx^i \otimes d_i f$. This operator has been first introduced, up to sign, by F.S. Macaulay as early as in 1916 [25] but this is still not ackowledged [31]. For more details on the Spencer operator and its applications, the reader may look at [3,12,29,32,33]. □

Definition 10. *With any differential module M we shall associate the graded module*

$G = gr(M)$ over the polynomial ring $gr(D) \simeq K[\chi]$ by setting $G = \oplus_{q=0}^{\infty} G_q$ with $G_q = M_q/M_{q-1}$ and we get $g_q = G_q^*$ where the symbol g_q is defined by the short exact sequences:

$$0 \longrightarrow M_{q-1} \longrightarrow M_q \longrightarrow G_q \longrightarrow 0 \quad \Longleftrightarrow \quad 0 \longrightarrow g_q \longrightarrow R_q \longrightarrow R_{q-1} \longrightarrow 0$$

We have the short exact sequences $0 \longrightarrow D_{q-1} \longrightarrow D_q \longrightarrow S_qT \longrightarrow 0$ leading to $gr_q(D) \simeq S_qT$ and we may set as usual $T^* = hom_K(T, K)$ in a coherent way with differential geometry.

The two following definitions, which are well known in commutative algebra, are also valid (with more work) in the case of differential modules [29].

Definition 11. *The set of elements $t(M) = \{m \in M \mid \exists 0 \neq P \in D, Pm = 0\} \subseteq M$ is a differential module called the torsion submodule of M. More generally, a module M is called a torsion module if $t(M) = M$ and a torsion-free module if $t(M) = 0$. In the short exact sequence $0 \to t(M) \to M \to M' \to 0$, the module M' is torsion-free. Its defining module of equations I' is obtained by adding to I a representative basis of $t(M)$ set up to zero and we have thus $I \subseteq I'$.*

Definition 12. *A differential module F is said to be free if $F \simeq D^r$ for some integer $r > 0$ and we shall define $rk_D(F) = r$. If F is the biggest free differential module contained in M, then M/F is a torsion differential module and $hom_D(M/F, D) = 0$. In that case, we shall define the differential rank of M to be $rk_D(M) = rk_D(F) = r$. Accordingly, if M is defined by a linear involutive operator of order q, then $rk_D(M) = \alpha_q^n$.*

Proposition 6. *If $0 \to M' \to M \to M" \to 0$ is a short exact sequence of differential modules and maps or operators, we have $rk_D(M) = rk_D(M') + rk_D(M")$.*

In the general situation, let us consider the sequence $M' \xrightarrow{f} M \xrightarrow{g} M"$ of modules which may not be exact and define $B = im(f) \subseteq Z = ker(g) \Rightarrow H = Z/B$.

In order to conclude this section, we may say that the main difficulty met when passing from the differential framework to the algebraic framework is the " *inversion* " of arrows. Indeed, when an operator is injective, that is when we have the exact sequence $0 \to E \xrightarrow{\mathcal{D}} F$ with $dim(E) = m, dim(F) = p$, like in the case of the operator $0 \to E \xrightarrow{j_q} J_q(E)$, on the contrary, using differential modules, we have the epimorphism $D^p \xrightarrow{\mathcal{D}} D^m \to 0$. The case of a formally surjective operator, like the div operator, described by the exact sequence $E \xrightarrow{\mathcal{D}} F \to 0$ is now providing the exact sequence of differential modules $0 \to D^p \xrightarrow{\mathcal{D}} D^m \to M \to 0$ because $\mathcal{D}$ has no CC.

Theorem 5. *(Double Duality Test) The procedure has 5 steps in the operator language:*

- *STEP 1: Start with the given operator $\mathcal{D}_1$ and the corresponding differential module M_1.*
- *STEP 2: Construct the operator $ad(\mathcal{D}_1)$.*
- *STEP 3: As any operator is the adjoint of an operator, namely its adjoint, denote by $ad(\mathcal{D})$ its generating CC.*
- *STEP 4: Construct $\mathcal{D} = ad(ad(\mathcal{D}))$.*
- *STEP 5: Construct the generating CC $\mathcal{D}_1'$ of $\mathcal{D}$ and compare to $\mathcal{D}_1$.*

If $\mathcal{D}_1$ generates the CC of $\mathcal{D}$, we have obtained a parametrization. Otherwise, M_1 is *not torsion-free and any new CC provides an element of* $t(M_1)$.

If N_1 is the differential module defined by $ad(\mathcal{D}_1)$, it follows from the last step that $t(M_1) = ext^1_D(N_1, D) = ext^1(N_1)$. More generally, we have (See [30] p. 218 for details):

Corollary 3. *If M is the differential module defined by any operator $\mathcal{D}$ and N is the corresponding differential module defined by $ad(\mathcal{D})$, then we have $t(M) = ext^1(N)$ with a slight abuse of language. Moreover, as $ad(ad(\mathcal{D})) = \mathcal{D}$, we have also $t(N) = ext^1(M)$.*

Definition 13. *A parametrization is said to be "minimum" if the differential module defined by $\mathcal{D}$ has a vanishing differential rank and is thus a torsion module.*

Example 6. *If $\mathcal{D} : \xi \to (d_{22}\xi = \eta^2, d_{12}\xi = \eta^1)$ we have $\mathcal{D}_1 = (\eta^1, \eta^2) \to d_1\eta^2 - d_2\eta^1 = \zeta$ and the only first order generating CC of $ad(\mathcal{D}_1) : \lambda \to (d_2\lambda = \mu^1, -d_1\lambda = \mu^2)$ is $d_1\mu^1 + d_2\mu^2 = \nu'$ while $ad(\mathcal{D}) : (\mu^1, \mu^2) \to d_{12}\mu^1 + d_{22}\mu^2 = \nu = d_2\nu'$ is a second order operator like $\mathcal{D}$.*

Example 7. *Many other examples can be found in ordinary differential control theory because it is known that a linear control system is controllable if and only if it is parametrizable (See [30, 33] for more details and examples). In our opinion, the best and simplest one is provided by the so-called double pendulum in which a rigid bar is able to move horizontally with reference position x and we attach two pendulums with respective length l_1 and l_2 making the (small) angles θ_1 and θ_2 with the vertical, the corresponding control system does not depend on the mass of each pendulum and the operator $\mathcal{D}_1$ is defined as follows:*

$$\boxed{d^2x + l_1 d^2\theta^1 + g\theta^1 = 0, \qquad d^2x + l_2 d^2\theta^2 + g\theta^2 = 0}$$

where g is the gravity. The standard way used by any student of the control community, is to prove that this control system is controllable if and only if $l_1 \neq l_2$ through a tedious computation based on the standard Kalman test. We let the reader prove this result as an exercise and apply the previous theorem in order to work out the parametrizing operator $\mathcal{D}$ of order 4, namely:

$$\boxed{\begin{array}{rcl} -l_1 l_2 d^4\phi - g(l_1 + l_2)d^2\phi - g^2\phi & = & x \\ l_2 d^4\phi + g d^2\phi & = & \theta_1 \\ l_1 d^4\phi + g d^2\phi & = & \theta_2 \end{array}}$$

The main problem is that this operator is trivially involutive but that *its adjoint is far from being even FI and the search for a Pommaret basis is quite delicate*. Indeed, multiplying the first OD equation by λ^1, the second by λ^2, adding and integrating by parts, we get:

$$\theta^1 \to l_1 d^2\lambda^1 + g\lambda^1 = \mu^1, \;\; \theta^2 \to l_2 d^2\lambda^2 + g\lambda^2 = \mu^2, \;\; x \to d^2\lambda^1 + d^2\lambda^2 = \mu^3$$

Multiplying the first equation by l_2, the second by l_1 and adding while taking into account the third equation, we get an equation of the form:

$$\boxed{(l_2\lambda^1 + l_1\lambda^2) = \frac{1}{g}(l_2\mu^1 + l_1\mu^2 - l_1 l_2\mu^3) = A(\mu) \in j_0(\mu)}$$

Differentiating twice this equation while using the first and second equations, we also obtain:

$$\boxed{(\frac{l_2}{l_1}\lambda^1 + \frac{l_1}{l_2}\lambda^2) = B(\mu) \in j_2(\mu) \ \Rightarrow \ (l_1 - l_2)\lambda \in j_2(\mu)}$$

When $l_1 - l_2 \neq 0$, it follows that $\lambda \in j_2(\mu)$ and, substituting in the third equation, we find the fourth order CC operator:

$$\boxed{(l_2 d^4 + g d^2)\mu^1 + (l_1 d^4 + g d^2)\mu^2 - (l_1 l_2 d^4 + g(l_1 + l_2)d^2 + g^2)\mu^3 = \nu}$$

Multiplying by a test function ϕ and integrating by parts we obtain the desired fourth order parametrization but the differential module M_1 is torsion-free if and only if $l_1 \neq l_2$. Hence: CONTRARY TO WHAT ENGINEERS STILL BELIEVE, THE CONTROLLABILITY IS A STRUCTURAL PROPERTY OF A CONTROL SYSTEM, NOT DEPENDING ON THE CHOICE OF THE INPUTS AND OUTPUTS AMONG THE SYSTEM VARIABLES.

In the present situation, one usually select a single input (for example x) and the two outputs (for example θ^1 and θ^2) but we invite the reader to spend a few dollars in order to realize this experiment. Of course, if $l_1 = l_2 = l$, setting $\theta = \theta^1 - \theta^2$, we obtain by subtraction $ld^2\theta + g\theta = 0$ and θ is a torsion element as can be seen by any reader doing the experiment. One must finally notice that the control system is controllable if and only if the adjoint of the system operator is injective (See[30] p 204-205 for details when $n = 1$).

For a control system in the Kalman form $-dx + Ax + Bu = 0$, multiplying on the left by a test row vector λ and integrating by parts, the adjoint system becomes $d\lambda + \lambda A = 0, \lambda B = 0 \Rightarrow d\lambda B = 0 \Rightarrow \lambda AB = 0 \Rightarrow \lambda A^2 B = 0$ and so on, a result showing that the Kalman controllability test amounts to the injectivity of the adjoint of the control operator in a purely intrinsic way.

Example 8. *A less academic but much more important example is the problem of parametrizing the Einstein equations. The following diagram is proving that Einstein equations cannot be parametrized [30] and we shall give details in the next section:*

$$\boxed{\begin{array}{ccccccccccc}
 & & & & & & 20 & \overset{Bianchi}{\longrightarrow} & 10 & & \\
 & & & & & \overset{Riemann}{\nearrow} & \downarrow & & \downarrow & & \\
 & & 4 & \overset{Killing}{\longrightarrow} & 10 & \overset{Einstein}{\longrightarrow} & 10 & \overset{div}{\longrightarrow} & 4 & \longrightarrow & 0 \\
 & & & & & & & & & & \\
0 & \longrightarrow & 4 & \overset{Cauchy}{\longleftarrow} & 10 & \overset{Einstein}{\longleftarrow} & 10 & & & &
\end{array}}$$

It is essential to notice that the *Cauchy* and *Killing* operators (*left side*) have *strictly nothing to do* with the *Bianchi* and thus *div* operators (*right side*). According to the last corollary, the $20 - 10 = 10$ new CC are generating the torsion submodule of the differential module defined by the Einstein operator. In the last section we shall explain why such a basis of the torsion module is made by the 10 independent components of the Weyl tensor, *each one killed by the Dalembertian*, a result leading to the so-called *Lichnerowicz waves* (in France) [28].

Example 9. *In continuum mechanics, the Cauchy stress tensor may not be symmetric in the so-called Cosserat media where the Cauchy stress equations are replaced by the Cosserat couple-stress equations which are nothing else than the adjoint of the first Spencer operator, totally different from the third [34–37]. When $n = 2$, we shall see that the single Airy function has strictly nothing to do with any perturbation of the metric having three components.*

Example 10. *A similar comment can be done for electromagnetism through the exterior derivative as the first set of Maxwell equations can be parametrized by the EM potential 1-form while the second set of Maxwell equations, adjoint of this parametrization, can be parametrized by the EM pseudo-potential [28, 29]. These results are even strengthening the comments we shall make in section 4 on the origin and existence of gravitational waves [28, 38], see Appendix.*

As a byproduct of the preceding examples, it is clear that an operator can be FI or involutive but that its adjoint may be neither involutive, nor even FI and the situation is more delicate when using double duality because each step may be as delicate as the previous one, a fact showing out the importance of the intrinsic definition of Pommaret bases that we have given and illustrated. Of course, according to the many examples illustrating Section 2, whenever one is exhibiting a Janet sequence, it may also be possible to exhibit the corresponding Spencer sequence and vice-versa. It follows that only structural properties not depending on the choice of the resolution of a differential module are really important and this is just the reason for introducing differential extension modules as we already said.

4. Einstein equations

Linearizing the *Ricci* tensor ρ_{ij} over the Minkowski metric ω, we obtain the usual second order homogeneous *Ricci* operator $\Omega \to R$ with 4 terms:

$$2R_{ij} = \omega^{rs}(d_{rs}\Omega_{ij} + d_{ij}\Omega_{rs} - d_{ri}\Omega_{sj} - d_{sj}\Omega_{ri}) = 2R_{ji}$$

$$tr(R) = \omega^{ij}R_{ij} = \omega^{ij}d_{ij}tr(\Omega) - \omega^{ru}\omega^{sv}d_{rs}\Omega_{uv}$$

We may define the $Einstein$ operator by setting $E_{ij} = R_{ij} - \frac{1}{2}\omega_{ij}tr(R)$ and obtain the 6 terms [39]:

$$2E_{ij} = \omega^{rs}(d_{rs}\Omega_{ij} + d_{ij}\Omega_{rs} - d_{ri}\Omega_{sj} - d_{sj}\Omega_{ri}) - \omega_{ij}(\omega^{rs}\omega^{uv}d_{rs}\Omega_{uv} - \omega^{ru}\omega^{sv}d_{rs}\Omega_{uv})$$

We have the (locally exact) differential sequence of operators acting on sections of vector bundles where the order of an operator is written under its arrow:

$$\boxed{n \underset{1}{\overset{Killing}{\longrightarrow}} n(n+1)/2 \underset{2}{\overset{Riemann}{\longrightarrow}} n^2(n^2-1)/12 \underset{1}{\overset{Bianchi}{\longrightarrow}} n^2(n^2-1)(n-2)/24}$$

Our purpose is now to study the differential sequence onto which its right part is projecting:

$$\begin{array}{c} S_2T^* \underset{2}{\overset{Einstein}{\longrightarrow}} S_2T^* \underset{1}{\overset{div}{\longrightarrow}} T^* \to 0 \\ n(n+1)/2 \longrightarrow n(n+1)/2 \longrightarrow n \to 0 \end{array}$$

and the following adjoint sequence:

$$\boxed{ad(T) \overset{Cauchy}{\longleftarrow} ad(S_2T^*) \overset{Beltrami}{\longleftarrow} n^2(n^2-1)/12 \overset{Lanczos}{\longleftarrow} n^2(n^2-1)(n-2)/24}$$

In this sequence, if E is a vector bundle over the ground manifold X with dimension n, we may introduce the new vector bundle $ad(E) = \wedge^n T^* \otimes E^*$ where E^* is obtained from E by inverting the transition rules exactly like T^* is obtained from T. We have for example $ad(T) = \wedge^n T^* \otimes T^* \simeq \wedge^n T^* \otimes T \simeq \wedge^{n-1} T^*$ because T^* is isomorphic to T by using the metric ω. The 10×10 $Einstein$ operator matrix is induced from the 10×20 $Riemann$ operator matrix and the 10×4 div operator matrix is induced from the 20×20 $Bianchi$ operator matrix. We advise the reader not familiar with the formal theory of systems or operators to follow the computation in dimension $n = 2$ with the 1×3 $Airy$ operator matrix, which is the formal adjoint of the 3×1 $Riemann$ operator matrix, and $n = 3$ with the 6×6 $Beltrami$ operator matrix which is the formal adjoint of the 6×6 $Riemann$ operator matrix which is easily seen to be self-adjoint up to a change of basis.

With more details for specific dimensions, we have:

- $n = 2$: The stress equations become $d_1\sigma^{11} + d_2\sigma^{12} = 0, d_1\sigma^{21} + d_2\sigma^{22} = 0$. Their second order parametrization $\sigma^{11} = d_{22}\phi, \sigma^{12} = \sigma^{21} = -d_{12}\phi, \sigma^{22} = d_{11}\phi$ has been provided by George Biddell Airy in 1863 [40] and is well known. We get the second order system:

$$\left\{ \begin{array}{rl} \sigma^{11} & \equiv d_{22}\phi = 0 \\ -\sigma^{12} & \equiv d_{12}\phi = 0 \\ \sigma^{22} & \equiv d_{11}\phi = 0 \end{array} \right. \quad \begin{array}{|cc|} 1 & 2 \\ 1 & \bullet \\ 1 & \bullet \\ \hline \end{array}$$

  which is involutive with one equation of class 2, 2 equations of class 1 and it is easy to check that the 2 corresponding first order CC are just the $Cauchy$ equations. Of course, the single $Airy$ function (1 term) has absolutely nothing to do with the perturbation of the metric (3 terms). When ω is the Euclidean metric, we may consider the only component $d_{22}\Omega_{11} + d_{11}\Omega_{22} - 2d_{12}\Omega_{12}$. Multiplying by the Airy function ϕ and integrating by parts, we discover that $Airy = ad(Riemann)$ in the following differential sequences that may be compared to those of the double pendulum:

$$\begin{array}{ccccccccc} & & 2 & \underset{1}{\overset{Killing}{\longrightarrow}} & 3 & \underset{2}{\overset{Riemann}{\longrightarrow}} & 1 & \longrightarrow & 0 \\ & & & & & & & & \\ 0 & \longleftarrow & 2 & \underset{1}{\overset{Cauchy}{\longleftarrow}} & 3 & \underset{2}{\overset{Airy}{\longleftarrow}} & 1 & & \end{array}$$

- $n = 3$: It is quite more delicate to parametrize the 3 PD equations:

$$d_1\sigma^{11} + d_2\sigma^{12} + d_3\sigma^{13} = 0, \quad d_1\sigma^{21} + d_2\sigma^{22} + d_3\sigma^{23} = 0, \quad d_1\sigma^{31} + d_2\sigma^{32} + d_3\sigma^{33} = 0$$

A direct computational approach has been provided by Eugenio Beltrami in 1892 [41] through the 6 *stress functions* $\phi_{ij} = \phi_{ji}$ in the $Beltrami$ parametrization. The corresponding system:

$$\left\{\begin{array}{rl} \sigma^{11} \equiv & d_{33}\phi_{22} + d_{22}\phi_{33} - 2d_{23}\phi_{23} = 0 \\ -\sigma^{12} \equiv & d_{33}\phi_{12} + d_{12}\phi_{33} - d_{13}\phi_{23} - d_{23}\phi_{13} = 0 \\ \sigma^{22} \equiv & d_{33}\phi_{11} + d_{11}\phi_{33} - 2d_{13}\phi_{13} = 0 \\ \sigma^{13} \equiv & d_{23}\phi_{12} + d_{12}\phi_{23} - d_{22}\phi_{13} - d_{13}\phi_{22} = 0 \\ -\sigma^{23} \equiv & d_{23}\phi_{11} + d_{11}\phi_{23} - d_{12}\phi_{13} - d_{13}\phi_{12} = 0 \\ \sigma^{33} \equiv & d_{22}\phi_{11} + d_{11}\phi_{22} - 2d_{12}\phi_{12} = 0 \end{array}\right. \quad \begin{array}{|ccc|} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \end{array}$$

is involutive with 3 equations of class 3, 3 equations of class 2 and no equation of class 1. The three characters are thus $\alpha_2^3 = 1 \times 6 - 3 = 3 < \alpha_2^2 = 2 \times 6 - 3 = 9 < \alpha_2^1 = 3 \times 6 - 0 = 18$ and we have $dim(g_2) = \alpha_2^1 + \alpha_2^2 + \alpha_2^3 = 18 + 9 + 3 = 30 = dim(S_2T^* \otimes S_2T^*) - dim(S_2T^*) == 6 \times 6 - 6$. The 3 CC are describing the stress equations which admit therefore a parametrization ... but without any geometric framework, in particular without any possibility to imagine that the above second order operator is *nothing else but* the *formal adjoint* of the *Riemann operator*, namely the (linearized) Riemann tensor with $n^2(n^2-1)/2 = 6$ independent components when $n = 3$ by duality.

Breaking the canonical form of the six equations which is associated with the Janet tabular, we may rewrite the Beltrami parametrization of the Cauchy stress equations as follows, after exchanging the third row with the fourth row, keeping the ordering $\{(11) < (12) < (13) < (22) < (23) < (33)\}$:

$$\begin{pmatrix} d_1 & d_2 & d_3 & 0 & 0 & 0 \\ 0 & d_1 & 0 & d_2 & d_3 & 0 \\ 0 & 0 & d_1 & 0 & d_2 & d_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & d_{33} & -2d_{23} & d_{22} \\ 0 & -d_{33} & d_{23} & 0 & d_{13} & -d_{12} \\ 0 & d_{23} & -d_{22} & -d_{13} & d_{12} & 0 \\ d_{33} & 0 & -2d_{13} & 0 & 0 & d_{11} \\ -d_{23} & d_{13} & d_{12} & 0 & -d_{11} & 0 \\ d_{22} & -2d_{12} & 0 & d_{11} & 0 & 0 \end{pmatrix} \equiv 0$$

as an identity where 0 on the right denotes the zero operator. However, if Ω is a perturbation of the metric ω, the standard implicit summation used in continuum mechanics is, when $n = 3$:

$$\begin{array}{rcl}\sigma^{ij}\Omega_{ij} & = & \sigma^{11}\Omega_{11}+2\sigma^{12}\Omega_{12}+2\sigma^{13}\Omega_{13}+\sigma^{22}\Omega_{22}+2\sigma^{23}\Omega_{23}+\sigma^{33}\Omega_{33}\\ & = & \Omega_{22}d_{33}\phi_{11}+\Omega_{33}d_{22}\phi_{11}-2\Omega_{23}d_{23}\phi_{11}+...\\ & & +\Omega_{23}d_{13}\phi_{12}+\Omega_{13}d_{23}\phi_{12}-\Omega_{12}d_{33}\phi_{12}-\Omega_{33}d_{12}\phi_{12}+...\end{array}$$

because *the stress tensor density σ is supposed to be symmetric*. Integrating by parts in order to construct the adjoint operator, we get:

$$\begin{array}{rcl}\phi_{11} & \longrightarrow & d_{33}\Omega_{22}+d_{22}\Omega_{33}-2d_{23}\Omega_{23}\\ \phi_{12} & \longrightarrow & d_{13}\Omega_{23}+d_{23}\Omega_{13}-d_{33}\Omega_{12}-d_{12}\Omega_{33}\end{array}$$

and so on. The identifications $Beltrami = ad(Riemann)$, $Lanczos = ad(Bianchi)$ in the diagram:

$$\boxed{\begin{array}{ccccccccc} & 3 & \underset{1}{\overset{Killing}{\longrightarrow}} & 6 & \underset{2}{\overset{Riemann}{\longrightarrow}} & 6 & \underset{1}{\overset{Bianchi}{\longrightarrow}} & 3 & \longrightarrow 0\\ 0\longleftarrow & 3 & \underset{1}{\overset{Cauchy}{\longleftarrow}} & 6 & \underset{2}{\overset{Beltrami}{\longleftarrow}} & 6 & \underset{1}{\overset{Lanczos}{\longleftarrow}} & 3 & \end{array}}$$

prove that the $Cauchy$ operator has nothing to do with the $Bianchi$ operator [28]. When ω is the Euclidean metric, the link between the two sequences is established by means of the elastic constitutive relations $2\sigma_{ij}=\lambda tr(\Omega)\omega_{ij}+2\mu\Omega_{ij}$ with the Lamé elastic constants (λ,μ) but mechanicians use to set $\Omega_{ij}=2\epsilon_{ij}$. Substituting in the dynamical equation $d_i\sigma^{ij}=\rho d^2/dt^2\xi^j$ where ρ is the mass per unit volume, we get the longitudinal and transverse wave equations, responsible for earthquakes! Multiplying the second, third and fifth row by 2, we get the new 6×6 operator matrix with rank 3 which is clearly self-adjoint, namely:

$$\left(\begin{array}{cccccc}0 & 0 & 0 & d_{33} & -2d_{23} & d_{22}\\ 0 & -2d_{33} & 2d_{23} & 0 & 2d_{13} & -2d_{12}\\ 0 & 2d_{23} & -2d_{22} & -2d_{13} & 2d_{12} & 0\\ d_{33} & 0 & -2d_{13} & 0 & 0 & d_{11}\\ -2d_{23} & 2d_{13} & 2d_{12} & 0 & -2d_{11} & 0\\ d_{22} & -2d_{12} & 0 & d_{11} & 0 & 0\end{array}\right)$$

When $n=4$, Taking the adjoint of the second order PD equations defining the so-called gravitational waves, we have proved in many books [28]) or papers [38] the following crucial theorem which is showing that the Einstein operator is useless contrary to the classical GR literature [42].

Theorem 6. *The GW equations are defined by the adjoint of the Ricci operator which is not self-adjoint contrary to the Einstein operator which is indeed self-adjoint.*

We shall finally prove that this result only depends on the second order jets of the conformal group of transformations of space-time, *a result highly not evident at first sight for sure*.

5. Conformal group

We start proving that the structure of the conformal group with $(n+1)(n+2)/2$ parameters may not be related to a classification of Lie algebras [23]. For this, introducing the Christoffel symbols γ for the metric ω and the standard Lie derivative $\mathcal{L}$ of geometric objects, let us consider the strict inclusions of second order infinitesimal Lie equations $R_2 \subset \tilde{R}_2 \subset \hat{R}_2 \subset J_2(T)$:

$$
\begin{array}{lll}
R_2 & \mathcal{L}(\xi)\omega = 0, & \mathcal{L}(\xi)\gamma = 0 \\
\tilde{R}_2 & \mathcal{L}(\xi)\omega = 2\,A(x)\omega, & \mathcal{L}(\xi)\gamma = 0 \\
\hat{R}_2 & \mathcal{L}(\xi)\omega = 2\,A(x)\omega, & \mathcal{L}(\xi)\gamma = \delta_i^k A_j(x) + \delta_j^k A_i(x) - \omega_{ij}\omega^{kr}A_r(x)
\end{array}
$$

For $n = 1$, the projective group of the real line is defined by the Schwarzian OD equation:

$$\Phi(y, y_x, y_{xx}, y_{xxx}) \equiv \frac{y_{xxx}}{y_x} - \frac{3}{2}(\frac{y_{xx}}{y_x})^2 = \nu(x)$$

Setting $\mathcal{L}(\xi) = L(j_3(\xi))$, the linearization with symbol $g_3 = 0$ is the Medolaghi OD equation:

$$L(\xi_3)\nu \equiv \xi_{xxx} + 2\nu(x)\xi_x + \xi\partial_x\nu(x) = 0$$

When $\nu = 0$, the general solution is simply $\xi = ax^2 + bx + c$ with 3 parameters and there is no CC.

The 3 infinitesimal generators of the Lie group action are $\{\theta_1 = \partial_x, \theta_2 = x\,\partial_x,\ \theta_3 = \frac{1}{2}x^2\partial_x\}$, namely *1 translation + 1 dilatation + 1 elation*

For $n = 2$, eliminating the conformal factor in the case of the Euclidean metric of the plane provides the two Cauchy-Riemann equations defining the infinitesimal complex transformations of the plane. The *only possibility* coherent with homogeneity is thus to consider the following system and to prove that it is defining a system of infinitesimal Lie equations, leading to 6 infinitesimal generators, namely: *2 translations + 1 rotation + 1 dilatation + 2 elations*:

$$
\left\{
\begin{array}{c}
\xi^k_{ijr} = 0 \\
\xi^2_{22} - \xi^1_{12} = 0, \xi^1_{22} + \xi^2_{12} = 0, \xi^2_{12} - \xi^1_{11} = 0, \xi^1_{12} + \xi^2_{11} = 0 \\
\xi^2_2 - \xi^1_1 = 0, \xi^1_2 + \xi^2_1 = 0
\end{array}
\right.
$$

$$\{\theta_1 = \partial_1, \theta_2 = \partial_2, \theta_3 = x^1\partial_2 - x^2\partial_1, \theta_4 = x^1\partial_1 + x^2\partial_2, \theta_5 = -\frac{1}{2}((x^1)^2 + (x^2)^2)\partial_1 + x^1(x^1\partial_1 + x^2\partial_2), \theta_6\}$$

with the elation θ_6 obtained from θ_5 by exchanging x^1 with x^2. We have $\hat{g}_3 = 0$ when $n = 1, 2$.

Lemma 3. *We have [43]:*

- $\hat{g}_1$ *is finite type with* $\hat{g}_3 = 0, \forall n \geq 3$.
- $\hat{g}_2$ *is* 2*-acyclic when* $n \geq 4$.
- $\hat{g}_2$ *is* 3*-acyclic when* $n \geq 5$.

In order to prove that conformal differential geometry must be *almost entirely* revisited, let us prove that the analogue of the Weyl tensor is made by a third order operator when $n = 3$, a result which is neither known nor acknowledged today. As before, we shall proceed by diagram chasing as the local computation can only be done by using computer algebra and does not thus provide any geometric insight (See Appendix by A. Quadrat in arXiv:1603.05030 and [24] for all the details).

We have $E = T$ and $dim(\hat{F}_0) = 5$ in the diagram:

$$\boxed{\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & \hat{g}_4 & \rightarrow & S_4T^*\otimes T & \rightarrow & S_3T^*\otimes \hat{F}_0 & \rightarrow & \hat{F}_1 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \parallel & & & & \\
0 & \rightarrow & T^*\otimes \hat{g}_3 & \rightarrow & T^*\otimes S_3T^*\otimes T & \rightarrow & T^*\otimes S_2T^*\otimes \hat{F}_0 & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & \wedge^2T^*\otimes \hat{g}_2 & \rightarrow & \wedge^2T^*\otimes S_2T^*\otimes T & \rightarrow & \wedge^2T^*\otimes T^*\otimes \hat{F}_0 & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & \wedge^3T^*\otimes \hat{g}_1 & \rightarrow & \wedge^3T^*\otimes T^*\otimes T & \rightarrow & \wedge^3T^*\otimes \hat{F}_0 & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}}$$

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & 0 & \rightarrow & 45 & \rightarrow & 50 & \rightarrow & 5 & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & 0 & \rightarrow & 90 & \rightarrow & 90 & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & 9 & \rightarrow & 54 & \rightarrow & 45 & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & 4 & \rightarrow & 9 & \rightarrow & 5 & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}$$

We have 10 parameters: *3 translations + 3 rotations + 1 dilatation + 3 elations* and the totally unexpected fact that *second order CC do not exist, contrary to the Riemannian case*:

$$0 \rightarrow \hat{R}_3 \rightarrow J_3(T) \rightarrow J_2(\hat{F}_0) \rightarrow 0 \ \Rightarrow \ 0 \rightarrow 10 \rightarrow 60 \rightarrow 50 \rightarrow 0$$

$$0 \rightarrow \hat{R}_4 \rightarrow J_4(T) \rightarrow J_3(\hat{F}_0) \rightarrow \hat{F}_1 \rightarrow 0 \ \Rightarrow \ 0 \rightarrow 10 \rightarrow 105 \rightarrow 100 \rightarrow 5 \rightarrow 0$$

$$0 \rightarrow \hat{R}_5 \rightarrow J_5(T) \rightarrow J_4(\hat{F}_0) \rightarrow J_1(\hat{F}_1) \rightarrow \hat{F}_2 \rightarrow 0 \ \Rightarrow \ 0 \rightarrow 10 \rightarrow 168 \rightarrow 175 \rightarrow 20 \rightarrow 3 \rightarrow 0$$

We obtain the minimum differential sequence, *which is nervertheless not a Janet sequence*:

$$0 \rightarrow \hat{\Theta} \rightarrow T \underset{1}{\overset{\hat{\mathcal{D}}}{\rightarrow}} \hat{F}_0 \underset{3}{\overset{\hat{\mathcal{D}}_1}{\rightarrow}} \hat{F}_1 \underset{1}{\overset{\hat{\mathcal{D}}_2}{\rightarrow}} \hat{F}_2 \rightarrow 0 \ \Rightarrow \ 0 \rightarrow \hat{\Theta} \rightarrow 3 \underset{1}{\overset{\hat{\mathcal{D}}}{\rightarrow}} 5 \underset{3}{\rightarrow} 5 \underset{1}{\rightarrow} 3 \rightarrow 0$$

with $\hat{\mathcal{D}}$ the conformal Killing operator and vanishing Euler-Poincaré characteristic $3 - 5 + 5 - 3 = 0$. We have even proved recently in [57] that this 5×5 third order operator is indeed self-adjoint.

When $n = 4$, we have 15 parameters: *4 translations + 6 rotations + 1 dilatation + 4 elations* and $\hat{g}_3 = 0 \Rightarrow \hat{g}_4 = 0 \Rightarrow \hat{g}_5 = 0$ in the commutative diagram with exact vertical long δ-sequences *but the left one* and where the second row proves that *there* *cannot exist first order Bianchi-like identities for the Weyl tensor*, contrary to what is still believed today:

$$
\boxed{
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \hat{g}_3 & \rightarrow & S_3T^*\otimes T & \rightarrow & S_2T^*\otimes \hat{F}_0 & \rightarrow & \hat{F}_1 & \rightarrow & 0 \\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \\
0 & \rightarrow & T^*\otimes\hat{g}_2 & \rightarrow & T^*\otimes S_2T^*\otimes T & \rightarrow & T^*\otimes T^*\otimes\hat{F}_0 & \rightarrow & 0 \\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \\
0 & \rightarrow & \wedge^2T^*\otimes\hat{g}_1 & \rightarrow & \wedge^2T^*\otimes T^*\otimes T & \rightarrow & \wedge^2T^*\otimes\hat{F}_0 & \rightarrow & 0 \\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow & & \\
0 & \rightarrow & \wedge^3T^*\otimes T & = & \wedge^3T^*\otimes T & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
}
$$

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & 0 & \rightarrow & 80 & \rightarrow & 90 & \rightarrow & 10 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow\delta & & \downarrow\delta & & & & \\
0 & \rightarrow & 16 & \rightarrow & 160 & \rightarrow & 144 & \rightarrow & 0 & & \\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & & & \\
0 & \rightarrow & 42 & \rightarrow & 96 & \rightarrow & 54 & \rightarrow & 0 & & \\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow & & & & \\
0 & \rightarrow & 16 & = & 16 & \rightarrow & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & 0 & & 0 & & & & & &
\end{array}
$$

A diagonal snake chase proves that $\hat{F}_1 \simeq H^2(\hat{g}_1)$. However, we obtain at once $dim(B^2(\hat{g}_1)) = 16$ but, in order to prove that the number of components of the Weyl tensor is $42 - 32 = 10$ or, equivalently, to prove that $dim(Z^2(\hat{g}_1)) = 42 - 16 = 26$, we have to prove that the last map δ in the left Weyl δ-sequence is surjective, a result that it is almost impossible to prove in local coordinates. Let us prove it by means of circular diagram chasing in the preceding commutative diagram as follows. Lift any $a \in \wedge^3T^*\otimes T$ to $b \in \wedge^2T^*\otimes T^*\otimes T$ because the vertical δ-sequence for S_3T^* is exact. Project it by the symbol map $\sigma_1(\hat{\Phi})$ to $c \in \wedge^2T^*\otimes\hat{F}_0$. Then, lift c to $d \in T^*\otimes T\otimes\hat{F}_0$ that we may lift *backwards horizontally* to $e \in T^*\otimes S_2T^*\otimes T$ to which we may apply δ to obtain $f \in \wedge^2T^*\otimes T^*\otimes T$. By commutativity, both f *and* b map to c and the difference $f - b$ maps thus to zero. Finally, we may find $g \in \wedge^2T^*\otimes\hat{g}_1$ such that $b = g + \delta(e)$ and we obtain thus $a = \delta(g) + \delta^2(e) = \delta(g)$, proving therefore the desired surjectivity.

As a byproduct, we end this paper with the following *fundamental diagram* II first presented in 1983 (See [12], p. 446) but still not yet acknowledged as it only depends on the Spencer δ-maps, explaining both the splitting vertical sequence on the right and the link existing between the Ricci vector bundle and the symbol bundle $\hat{g}_2 \simeq T^*$ of second order jets of conformal elations. Needless to say that the diagonal chase providing the isomorphism $Ricci \simeq S_2T^*$ could not be even imagined by using classical methods because its involves Spencer δ-cohomology with the standard notations $B = im(\delta), Z = ker(\delta), H = Z/B$ for *coboundary, cocycle, cohomology* at $\wedge^s T^* \otimes g_{q+r}$ when g_{q+r} is the r-prolongation of a symbol g_q. It is important to notice that all the bundles appearing in this diagram only depend on the metric ω but *not* on any conformal factor.

$$
\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
 & & & & & & 0 & & Ricci & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & \longrightarrow & Z^2_1(g_1) & \longrightarrow & Riemann & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & T^*\otimes \hat{g}_2 & \stackrel{\delta}{\longrightarrow} & Z^2_1(\hat{g}_1) & \longrightarrow & Weyl & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_2T^* & \stackrel{\delta}{\longrightarrow} & T^*\otimes T^* & \stackrel{\delta}{\longrightarrow} & \wedge^2 T^* & \longrightarrow & 0 & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
 & & & & & & 0 & & 10 & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & \longrightarrow & 20 & \longrightarrow & 20 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & 16 & \stackrel{\delta}{\longrightarrow} & 26 & \longrightarrow & 10 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 10 & \stackrel{\delta}{\longrightarrow} & 16 & \stackrel{\delta}{\longrightarrow} & 6 & \longrightarrow & 0 & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
$$

When $n = 4$, we have explained in [12] that the splitting horizontal lower sequence provides an isomorphism $T^* \otimes \hat{g}_2 \simeq T^* \otimes T^* \simeq S_2T^* \oplus \wedge^2 T^*$ which can be locally described by (R_{ij}, F_{ij}) in which (R_{ij}) is the GR part and (F_{ij}) the EM part as a unification of gravitation and electromagnetism, only depending thus on the second order jets of conformal transformations, *contrary to the philosophy of GR today*. We finally notice that $T^* \otimes \hat{g}_2 = T^* \otimes (\hat{R}_2/\tilde{R}_2) = (T^* \otimes \hat{R}_2)/(T^* \otimes \tilde{R}_2) = \hat{C}_1/\tilde{C}_1$, a result contradicting the mathematical foundations of classical gauge theory while allowing to understand the confusion done by E. Cartan and followers between "curvature alone" (F_1) and "curvature + torsion" (C_2) (See [44] for more details).

6. Conclusion

It is not even known that a classical OD control system defined by a surjective operator $\mathcal{D}$ is controllable if and only if $ad(\mathcal{D})$ is injective or, *equivalently*, if the operator $\mathcal{D}$ can be parametrized. The simplest example is the Kalman system $y_x = Ay + Bu$ with input u and output y leading to the Kalman test because the adjoint system $\lambda_x + \lambda A = 0, \lambda B = 0$ is *quite far* from being FI as we have also $\lambda AB = 0$ and so on. This result cannot be extended to an arbitrary PD control system with many independent variables [29]. In this case, one needs the double differential duality test for checking if the corresponding differential module M is torsion-free or, *equivalently*, if $\mathcal{D}$ can be parametrized. Then, one has in general to use twice the PP procedure which is already delicate for the OD case (See the double pendulum) but may become quite difficult for the PD case, like in the study of the Killing operator for the Kerr metric [18]. However, it is a fact that both the control, computer algebra and physics communities largely refused to use the Spencer operator and we don't speak about the mechanical community still not accepting that the Cosserat couple-stress equations are nothing else than the adjoint of the first Spencer operator in the Spencer sequence for the group of rigid motions in space or that the second set of Maxwell equations in electromagnetism are similarly induced by the adjoint of the first Spencer operator for the group of conformal transformations in space-time along the dream of Weyl [55]).

In the case of Special Relativity (1905), it is now known that, contrary to his claim, Einstein was aware of the Michelson and Morley experiment (1887) but only a footnote in his main paper "Electrodynamics of Moving Bodies" provides reference to the *conformal group of space-time* for the Minkowski metric ω. However, proving the local invariance of Maxwell equations by such a group of transformations was not possible at that time because of the non-linear elations introduced by E. Cartan quite later on (1922). The situation of General Relativity is even more delicate and we have proved that Einstein (1915), following Beltrami (1892), made a *terrible confusion* between the Cauchy operator and the div operator induced from the Bianchi operator and thus between the stress functions and the components of the metric, exactly like confusing the single Airy stress function with the 3 components of the metric in plane deformation. Also, the Killing system, which is FI for the Minkowski metric, is no longer FI for the Schwarzschild and Kerr metrics, the groups of invariance being reduced to 4 or 2 transformations respectively, contrary to the Poincaré group with 10 transformations, a result bringing doubts about the existence of black holes.

As a byproduct, GRAVITATIONAL WAVES CANNOT EXIST because the main problem is not a technical question of DETECTION but a foundational question of EQUATION. We can only hope that such a poor effective situation will indeed be recognized and improved in the future as we have only outlined the solution !

On the contrary, the idea of Weyl (1918) for unifying electromagnetism and gravitation had been to *enlarge* the group to the conformal group with 15 transformations on space-time [45]. As homological algebra and Spencer δ-cohomology are *absolutely needed in this case*, CONFORMAL GEOMETRY MUST BE ENTIRELY REVISITED along the *fundamental diagrams* I and II [46].

All these results are now available in a constructive way through the OreModules symbolic package initiated by my former PhD student A. Quadrat (INRIA) (See [5] for other computer references). The author finally thanks an anonymous referee for his many critical comments that have been taken into account.

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Appendix

Example 10: *A similar comment can be done for electromagnetism through the exterior derivative as the first set of Maxwell equations can be parametrized by the EM potential 1-form while the second set of Maxwell equations, adjoint of this parametrization, can be parametrized by the EM pseudo-potential [28, 29]. These results are even strengthening the comments we shall make in section* 4 *on the origin and existence of gravitational waves [28, 38]. With more details, let us only consider the beginning of the Poincaré sequence introduced in the Introduction:*

$$\boxed{\wedge^0 T^* \xrightarrow{d} \wedge^1 T^* \xrightarrow{d} \wedge^2 T^* \xrightarrow{d} \wedge^3 T^*}$$

Using standard notations, we denote by $A \in \wedge^1 T^* = T^*$ the EM potential, by $F = (F_{ij}) \in \wedge^2 T^*$ the EM field and the first set of Maxwel equations, namely $dF = 0$, is parametrized by $dA = F$. Denoting by $\mathcal{F} = (\mathcal{F}^{ijj}) \in \wedge^4 T^* \otimes \wedge^2 T$ the EM induction, *a tensorial density*, the second set of Maxwell equations is usually written as $\partial_i \mathcal{F}^{ij} = \mathcal{J}^j$ and thus $ad(d)\mathcal{F} = \mathcal{J} \in \wedge^4 T^* \otimes \wedge^0 T^* = \wedge^4 T^*$. with the 4 CC $\partial_j \mathcal{J}^j = 0$ describing the so-called conservation of current. The problem that we faced while teaching EM during twenty years, is that *only tensors are used in most textbooks* and the above formulas, if they are used by physicists, are not correct *at all* from a mathematical point of view. When E is any vector bundle over a manifold X of dimension n, the idea, as we shall see in section 4, is to introduced the adjoint vector bundle $ad(E) = \wedge^n T^* \otimes E^*$ with E^* defined by patching the inverse transition matrices, exactly like T^* is obtained from T. Such a formal approach, lacking in the literature, allows to describe both the second set of Maxwell equations and the conservation of current in the following dual sequence existing when $n = 4$:

$$\boxed{0 \leftarrow ad(\wedge^0 T^*) \xleftarrow{ad(d)} ad(\wedge^1 T^*) \xleftarrow{ad(d)} ad(\wedge^2 T^*) \xleftarrow{ad(d)} ad(\wedge^3 T^*)}$$

in which $ad(d)$ is going "*backwards* ", that is from right to left. For the reader knowing more mathematics, such a procedure may be simplified by using *Hodge duality* with the volume form $dx = dx^1 \wedge ... \wedge dx^n$ as a natural way to obtain the dual sequence when $n = 4$ in the form:

$$\boxed{0 \leftarrow \wedge^4 T^* \xleftarrow{d} \wedge^3 T^* \xleftarrow{d} \wedge^2 T^* \xleftarrow{d} \wedge^1 T^*}$$

Such a *confusing procedure* has in fact to do with the so-called *side changing functor* in differential homological algebra but is far out of the purpose of this paper. Of course, in the actual practice of computer algebra and electromagnetism, the two dual sequences can be written, up to sign, as:

$$\boxed{\begin{array}{ccccccccc} & & 1 & \xrightarrow{d} & 4 & \xrightarrow{d} & 6 & \xrightarrow{d} & 4 \\ & & & & & & & & \\ 0 & \leftarrow & 1 & \xleftarrow{d} & 4 & \xleftarrow{d} & 6 & \xleftarrow{d} & 4 \end{array}} \quad \begin{array}{c} Maxwell \quad I \\ \\ Maxwell \quad II \end{array}$$

Let us finally simply say that it is a way to transform a left differential module into a right differential module and vice-versa, one of the most difficult concepts that must be used when studying differential extension modules and the reason for which an adjoint operator must always be written “backwards ” as we saw (See [29] for more details and examples).