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A logical approach to validate the Goldbach conjecture: Part I

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ABSTRACT: This paper is a first of the series of three papers which provide a general proof to validate the Goldbach conjecture. This conjecture states that every even number can be expressed as a summation of two prime numbers. At the onset, concept of successive-addition of-digits-of-an-integer-number (SADN) and its properties in terms of basic algebraic functions like addition, multiplication and subtraction are discussed. SADN classifies odd numbers into 3 sequences—the S1, S3 and S5 sequences—which comprise of odd numbers having SADN 7 or 4 or 1; SADN 3 or 9 or 6 and SADN 5 or 2 or 8 respectively. The S1 and S5 sequences are of interest in the analysis. Furthermore, composites on the S1 sequence are derived as products of intra-sequence elements of the S1 and S5 sequences while composites on the S5 sequence are derived as products of inter-sequence elements of the S1 and S5 sequences. SADN also shows why such combinations for even numbers of SADN (1, 4, 7) will be found on the S5 sequence while those for even numbers of SADN (2, 5, 8) will lie on the S1 sequence and both the sequences have a role to play in identifying the prime number combinations for even numbers with SADN (3, 6, 9). Thereafter, analysis moves to calculating the total number of combinations for a given even number that would include combinations in the nature of two composites ($c1 + c2$); prime & composite ($p + c$) and two primes ($p1 + p2$). Identifying the total number of $c1 + c2$ and $p + c$ combinations yield the number of $p1 + p2$ combinations. The logic employed in present discussion shows that at least one such $p1 + p2$ combination exists for the even numbers having SADN digit within 1 to 9. This encompasses all even numbers and hence generalizes this method for all even numbers.

KEYWORDS: Goldbach conjecture; prime numbers; number theory; open problems in Mathematics; successive addition of digits of a number

1. Introduction

On 7 June 1742, a Prussian amateur mathematician and historian Christian Goldbach, wrote a letter to Leonard Euler, the content of which has been subsequently interpreted so as to state 'at least it seems that every even number that is greater than 2 is the sum of two primes'[1,2]. I.e., $2k = p1 + p2$; where $2k$ is an even number greater than 2 and $p1$, $p2$ are two prime numbers which may or may not be identical.

In 1938, Pipping showed that the Goldbach conjecture is true for even numbers up to and including 100,000[3]. It has been established, using a computer search, that it is true for even numbers up to and including 4,000,000,000,000,000,000[4]. In words of celebrated mathematician Hardy, 'The unproven

Goldbach conjecture enjoyed the dual importance of being one of such conjectures which were easy enough to be guessed by any fool and yet not proven'[5]. It is generally because of these paradoxical properties attached with Goldbach conjecture that British Mathematicians Hardy and Littlewood attempted to prove it using partition functions and resulted in bringing this problem into contact with the then recognised methods of the Analytic Theory of Numbers[6,7] and the problem was found worthy of being included as selected problems in number theory[8]. Interestingly ternary Goldbach conjecture, also an offshoot of the abovementioned exchange of letters between Euler and Goldbach, has been proved by Helfgott to be true for odd numbers greater than 5[9]. In this context, although the attempts by Pogorzelski have not received any objections, they have not been accepted either, till date[10]. The same reluctance has been observed for the work of Guiasu as well till date[11]. In his attempt to provide a general solution to the Goldbach problem, Watanabe observes that the number of prime-prime combinations for even numbers would be less in case if half of even number is a prime as compared to number of prime-prime combinations in case if half of even number is composite[12]. The present work in due course provides a logical reasoning for this observation made by Watanabe.

The present work which has been divided into three inter-linked papers for analytical convenience provides a definitive general proof for the Goldbach conjecture which covers all even numbers. For this purpose, the approach begins with a brief discussion of the concept of successive-addition-of-digits-of-an-integer-number (SADN) and its properties in terms of basic algebraic functions like addition, multiplication and subtraction. This concept of SADN forms the basis for classifying all odd numbers into 3 sequences—the S1, S3 and S5—which comprise of odd numbers of SADN (7 or 4 or 1), (3 or 9 or 6) and (5 or 2 or 8) respectively and follow a cyclical order. The S1 and S5 sequences are of interest in the analysis since they include both prime and composite numbers while the S3 sequence consists of exclusively composite numbers except the number '3'. The role of SADN is also important in determining the relevant sequence for identifying the combination of primes for a given even number since it shows:

- why such combinations for even numbers of SADN 1, 4 and 7 will be found on the S5 sequence,
- why such combinations for even numbers of SADN 2, 5 and 8 will lie on the S1 sequence and,
- why both the sequences have a role to play in identifying the prime number combinations for even numbers with SADN 3, 6 or 9.

Thereafter, the analysis moves to calculating the total number of acceptable combinations for a given even number that would include combinations of two composites ($c1 + c2$), one prime and one composite ($p + c$) and two primes ($p1 + p2$). A cyclical pattern followed by even numbers is also discussed in this context. Identifying the number of $c1 + c2$ and $p + c$ combinations and thereafter subtracting them from the total number of combinations will yield the number of $p1 + p2$ combinations. For this purpose, the second paper of this series provides a detailed method for deriving the number of combinations of type $c1 + c2$. The third paper of the series thereafter introduces the concept of minimum required number of combinations of type $c1 + c2$ to identify a combination of type $p1 + p2$; for any given even number. The relation between the minimum required number of $c1 + c2$ combinations and actual number of $c1 + c2$ combinations forms the basis for identifying the existence of combinations of type $p1 + p2$. The complete work put together presents this analysis as a proof to validate the Goldbach conjecture.

The present paper is the first among the series of the three papers and is organised in the following manner. The current section, i.e., section 1, is in the form of an introduction to the structure of this manuscript. Section 2 discusses the definition and properties of successive addition of digits of an integer number (SADN). Section 3 discusses the classification of odd numbers into three distinct sequences based on their SADN and extends the analysis for segregation of primes and composites. Section 4 applies the additive property of SADN to draw basic inferences for the identification of prime-prime combinations for a given even number based on its SADN. Section 5 introduces the conceptual framework for a cyclical sequence of even numbers. Based on the discussion in sections 4 and 5, Section 6 discusses the derivation of the total number of acceptable combinations for a given even number as also the nature of these combinations in terms of prime and composite components. Appendix B provides a detailed proof for this derivation. Section 7 highlights the need to identify combinations in the nature of composite-composite ($c1 + c2$) components which is central to the identification of the combinations in the nature of prime-prime ($p1 + p2$) components. The detailed method for calculating the Number of composite-composite combinations will be discussed in the second paper of the series. Section 8 is in the nature of a brief conclusion. Appendix A elaborates on the identification of $p1 + p2$ combinations for any given even number.

2. SADN: Definition and properties

Successive addition of digits of (integer) number (called as SADN) in simple terms refers to the process of adding up all the digits of a multi-digit number till we arrive at a single digit. In order to determine SADN of any given number, its digits are to be successively added until a single digit is obtained which is termed as SADN of the given number.

For example, consider the number 546,289. Addition of its digits = 5 + 4 + 6 + 2 + 8 + 9 = 34 Successive addition of digits = 3 + 4 = 7. Therefore, in our example, SADN of (546,289) = 7

Alternatively, SADN of an integer 'n' may be defined in terms of modular arithmetic as: n is congruent to (SADN of n) modulo 9 or $n \equiv$ (SADN of n) mod 9.

As $[a(10^k) - a]$ is divisible by 9 ⇨ $a(10^k) \equiv a \bmod(9)$. So an integer number

$$abc = a(100) + b(10) + c(1)$$
$$= a(10^2) + b(10^1) + c(10^0)$$
$$\equiv a \bmod(9) + b \bmod(9) + c \bmod(9)$$
$$\equiv (a + b + c) \bmod(9)$$
$$\equiv (\text{SADN of } abc) \bmod 9$$

The SADN exhibits following properties:

i. Idempotence.
ii. Well-defined range.
iii. Distribution over addition.
iv. Distribution over multiplication.
v. Additive Identity for SADN function.
vi. Interchangeability of non-positive SADN and positive SADN.
vii. Distribution over subtraction.
viii. Multiplicative identity for SADN function.

i. Property of idempotence:

In general terms: SADN of (x) = SADN of (SADN of (SADN of (...SADN of (x)))) which says that SADN function is an idempotent function.

ii. Range of SADN function:

The property of idempotence implies that the value of SADN function for any non-zero integer number would be a single digit integer only ranging from 1 to 9.

$1 \leq$ SADN of $(x) \leq 9$ implies that SADN of $(x) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

iii. Property of distribution over addition:

SADN of $(x + y)$ = SADN of (x) + SADN of (y) SADN function is distributive over addition.

Proof. Suppose SADN of $x = x'$ and SADN of $y = y'$ So $x \equiv x'$ mod(9) and $y = y'$ mod(9),

⇨ $(x - x')$ is divisible by 9 and $(y - y')$ is divisible by 9,

⇨ $[(x - x') + (y - y')]$ is divisible by 9,

⇨ $[(x + y) - (x' + y')]$ is divisible by 9,

⇨ $x + y \equiv (x' + y')$ mod(9),

⇨ $x + y \equiv$ (SADN of x + SADN of y) mod(9) Also $x + y \equiv$ (SADN of $(x + y)$) mod(9) Above two statements lead to:

SADN of $(x + y)$ = SADN of (x) + SADN of (y)

⇨ SADN function is distributive over addition. □

For example:

SADN of (52 + 95) = SADN of (147) = 3 (using property of idempotence as mentioned above),

SADN of (52) = 7; SADN of (95) = 5.

Using property of idempotence; SADN of (52) + SADN of (95) = SADN of (SADN of (52) + SADN of (95)) = SADN of (7 + 5) = SADN of (12) = 3 = SADN of (52 + 95).

Hence SADN of (52 + 95) = SADN of (52) + SADN of (95).

iv. Property of distribution over multiplication:

SADN of $(x.y)$ = SADN of (x).SADN of (y) SADN function is distributive over multiplication.

Proof. Suppose SADN of $x = x'$ and SADN of $y = y'$. So $x \equiv x'$ mod(9) and $y = y'$ mod(9),

⇨ $(x - x')$ is divisible by 9 and $(y - y')$ is divisible by 9,

⇨ $(x - x')y$ is divisible by 9 and $x'(y - y')$ is divisible by 9,

⇨ $(xy - x'y)$ is divisible by 9 and $(x'y - x'y')$ is divisible by 9,

⇨ $[(xy - x'y) + (x'y - x'y')]$ is divisible by 9,

⇨ $(xy - x'y')$ is divisible by 9,

⇨ $xy \equiv x'y'$ mod(9),

⇨ $xy \equiv$ [(SADN of x).(SADN of y)] mod(9). Also $xy \equiv$ (SADN of (xy)) mod(9).

Above two statements lead to:

SADN of (xy) = (SADN of x).(SADN of y),

⇨ SADN function is distributive over multiplication. □

For example:

SADN of (12). SADN of (15) = 3.6 = 18.

Using property of idempotence SADN of (12). SADN of (15) = SADN of (SADN of (12). SADN of (15)) = SADN of (18) = 9.

SADN of (12) = 3 and SADN of (15) = 6.

Hence SADN of (12.15) = SADN of (12). SADN of (15).

v. Additive identity for SADN function:

As SADN is primarily a type of addition operator, additive identity zero (0) acts as an additive identity for SADN as well.

Apart from zero, the number nine (9) also acts as an additive identity for SADN function as SADN function is represented in terms of modulo 9.

Proof. Suppose SADN of $x = x'$. So $x \equiv x'$ mod(9),

⇨ $x + 9(10^k) \equiv x'$ mod(9) + 0 mod(9),

$\equiv x'$ mod(9). □

vi. Interchangeability of non-positive SADN and positive SADN:

As $x \equiv$ (SADN of x) mod 9 and $0 \equiv 9$ mod 9,

⇨ $x - 0 \equiv$ [(SADN of x) − 9] mod 9. Or $x \equiv$ [(SADN of x) − 9] mod 9. Above statements lead to:

SADN of $x \equiv$ (SADN of x) − 9.

Thus properties of distribution over addition and identity of SADN function lead to following equivalence between nonpositive and positive values of SADN.

Table 1. Equivalence between nonpositive and positive values of SADN.

Positive SADN digit	1	2	3	4	5	6	7	8	9
Equivalent non-positive SADN digit	−8	−7	−6	−5	−4	−3	−2	−1	0

Both numbers of same columns are considered to be identical and replaceable substitutes of one another if need arises to consider non-positive digit for SADN.

This above-mentioned **Table 1** of interchangeability relates negative and positive SADN as follows:

SADN of $(-a)$ = −SADN of (a).

vii. Distribution over

Subtraction: SADN of $(x - y)$ = SADN of (x) – SADN of (y) SADN function is distributive over subtraction.

Proof. Suppose SADN of $x = x'$ and SADN of $y = y'$. So $x \equiv x'$ mod(9) and $y = y'$ mod(9),

⇨ $(x - x')$ is divisible by 9 and $(y - y')$ is divisible by 9,

⇨ $[(x - x') - (y - y')]$ is divisible by 9,

⇨ $[(x - y) - (x' - y')]$ is divisible by 9,

⇨ $x - y \equiv (x' - y')$ mod(9),

⇨ $x - y \equiv$ (SADN of x − SADN of y) mod(9).

⇨ Also $x - y \equiv$ (SADN of $(x - y)$) mod(9). Above two statements lead to:

SADN of $(x - y)$ = SADN of (x) − SADN of (y),

⇨ SADN function is distributive over subtraction. □

For example:

SADN of (132 − 130) = SADN of (2) = 2.

SADN of (132) − SADN of (130) = 6 − 4 = 2.

Hence SADN of (132 − 130) = SADN of (132) − SADN of (130).

viii. Multiplicative identity for SADN function:

Apart from SADN (1) as multiplicative identity for every SADN function; following are multiplicative identities as special cases:

SADN (4, 7, 1) acts as multiplicative identities for SADN (3, 6) SADN (1, 2, 3, 4, 5, 6, 7, 8, 9) or SADN (n) act as multiplicative identity for SADN (9).

3. Classifying odd numbers into three sequences

Consider the set of natural numbers $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\}$,

$$N = E + D$$

where E is set of even numbers, $E = \{2, 4, 6, 8, 10, \ldots\}$, and D = set of odd numbers = $\{1, 3, 5, 7, 9, 11, \ldots\}$.

All elements of the set of even numbers 'E' are composites (with exception of the number 2) whereas elements of the set of odd numbers 'D' may be prime or composite.

Now consider the following three sets of odd numbers:

$S1 = 1 + 6n$; $n \in \{N\}$; n is a natural number $S1 = \{7, 13, 19, 25, 31, 37, 43, 49, 55, 61, 67, \ldots\}$.

SADN of (element of S1) = {7, 4, 1} in cyclic order,

$S3 = 3 + 6n$, where n is a natural number including zero implying $S3 = \{3, 9, 15, 21, 27, 33, 39, 45, 51, 57, 63, 69, \ldots\}$.

SADN of (element of S3) = {3, 9, 6} in cyclic order

$S5 = 5 + 6n$, where n is a natural number including zero implying $S5 = 6n - 1$; $n \in \{N\}$

$S5 = \{5, 11, 17, 23, 29, 35, 41, 47, 53, 59, 65, 71, \ldots\}$

SADN of (element of $S5$) = {5, 2, 8} in cyclic order.

The above discussion is summarized in the following **Table2**.

Table 2. SADN of odd number determines its Sequence out of S1, S3, S5.

SADN of (n):	**n ϵ:**
7 or 4 or 1	S1
3 or 9 or 6	S3
5 or 2 or 8	S5

Set of natural numbers $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\}$.

Now we say that $N = E + D = E + \{1\} + S1 + S3 + S5$, where E is set of even numbers, $E = \{2, 4, 6, 8, 10, \ldots\}$, and D = set of odd numbers = $\{1\} + S1 + S3 + S5 = \{1, 3, 5, 7, 9, 11, \ldots\}$.

SADN of prime numbers:

Prime numbers are a subset of natural numbers with the unique property that they are divisible by themselves and by the number 1 (one) only. In terms of SADN function, the divisibility test of 3 says that any natural number would be divisible by 3 if its SADN is 3, 6 or 9. It leads to conclude that any natural number whose SADN is 3, 6 or 9 would be a composite as it would be divisible by the number 3, hence SADN of primes can never be 3, 6 or 9 (only exception to this would be the number '3' itself). This discussion in conjunction with the properties of range of SADN allows us to conclude that SADN of primes may be 1, 2, 4, 5, 7 or 8.

If 'p' represents a prime number, then SADN of $(p) = \{1, 2, 4, 5, 7, 8\}$.

The classification of odd numbers in terms of three sequences S1, S3 and S5 leads to segregation of primes and composites. Since sequence $S3$ consists of elements of $6n + 3$ type or $3 \times (2n + 1)$ type, all elements of $S3$ will be multiples of the number 3. This allows us to conclude that sequence $S3$ consists of composites and no prime (number '3' being the only exception). Further, since as discussed above SADN of primes can be 1, 2, 4, 5, 7 or 8, it is possible to conclude that prime numbers will be found on the $S1$ and $S5$ sequence. Specifically, sequence $S1$ comprises of elements of type $6n + 1$ and sequence $S5$ has elements of type $6n - 1$. Therefore, this classification of all odd numbers into the 3sequences based on their SADN segregates the primes of types $6n + 1$ and $6n - 1$ wherein all prime numbers of $6n + 1$ type which will be of SADN 7 or 4 or 1 will lie on the $S1$ sequence while prime numbers of $6n - 1$ type which will be of SADN 2 or 5 or 8 will lie on the $S5$ sequence.

We now return to the basic statement of the Goldbach conjecture which states that all even numbers greater than 2 can be expressed as a summation of 2 prime numbers. The segregation of prime numbers into the $S1$ and $S5$ sequence implies that these are the relevant sequences from the viewpoint of identification of prime-prime combinations for a given even number. In the following section we apply the additive property of SADN to identify the possible SADN of prime components of combinations adding up to a given even number of a given SADN, i.e., the question we are addressing below is that if $2k = p1 + p2$ then considering SADN of $2k$ what could be the SADN of $p1$ and $p2$?

4. Possible combinations of $p1 + p2$ for even number of particular SADN and a particular last-digit

Even numbers can be of SADN1 to 9 and can end in any of the digits 2, 4, 6, 8 or 0. While an even number of a particular SADN will recur after 18 integers, an even number of a particular SADN ending in a specific digit will recur after 90 integers. For example, 12 is an even number of SADN 3 which ends in the digit 2. The next even number with SADN 3 would be 12 + 18 = 30 while the next even number with SADN 3 that ends in the digit 2 will be 12 + 90 = 102. Therefore, if we denote an even number $2k$ as $a//b$, where 'a' is the SADN of the even number and 'b' is the digit in which it ends, then $2k + 18$ will in the form of $a//b - 2$ (or $a//b + 8$ if $b - 2$ is negative); and $2k + 90$ will be in the form of $a//b$. Here $a = 1$ to 9 while $b = 2, 4, 6, 8, 0$.

As mentioned earlier, prime numbers will essentially be odd numbers (with the only exception of the number 2), and can be of SADN 1, 2, 4, 5, 7 or 8 (with the only exception of the number 3) and can end in the digits 1, 3, 7 or 9 (with the only exception of the number 5 which is of SADN 5 and ends in 5). Therefore, prime numbers can be denoted as $ap//bp$ where 'ap' denotes the SADN of the prime number while 'bp' denotes the last digit of the prime. Here ap can be 1, 2, 4, 5, 7 or 8 while bp can be 1, 3, 7 or 9.

If we leave out the two exceptions, the combinations of prime numbers in which even numbers of a given SADN can be summed up has been generalized in the following **Table 3**:

Table 3. SADN of Possible combinations of prime numbers possible for an even number of a given SADN.

SADN of the even number $2k$	SADN of combinations of prime numbers that can add up to the given $2k$
1	2 + 8, 5 + 5
2	1 + 1, 4 + 7
3	2 + 1, 4 + 8, 5 + 7
4	2 + 2, 5 + 8
5	1 + 4, 7 + 7
6	1 + 5, 2 + 4, 7 + 8
7	2 + 5, 8 + 8
8	1 + 7, 4 + 4
9	1 + 8, 2 + 7, 4 + 5

It may be noted here that even numbers of SADN 1, 2, 4, 5, 7 or 8 can be added up in the form of 3 + odd-number-of-a-particular-SADN. For instance, if SADN of $2k$ is 1 then one possible combination of $p + p$ that can add up to $2k$ will be 3 + prime-number-of-SADN 7. Similarly, if $2k$ is of SADN 2 then a possible combination can be of 3 + prime-number-of-SADN 8. However, since odd numbers of SADN 3 are 'generally' composite in nature, in order to consider this combination a limiting condition needs to be placed that it will be applicable only in cases where the corresponding odd number will be a prime number so that when combined with the digit 3, it will qualify to be a $p + p$ combination i.e., such a combination can be considered for numbers where $2k - 3$ will be a prime number. Since a general solution is not conceivable with such specific limiting conditions, the current line of analysis will treat these combinations as non-general and leave them out.

Furthermore, depending on the last digit of $2k$, the last digit of the odd numbers that can add up in $2k$ will also have a role to play. For instance; consider $2k = 20$, it can be denoted as an even number of the form 2//0 i.e., SADN 2 ending in 0. In this case combinations of odd numbers that can add up to $2k$ will be of ap//1 + ap'//9 or ap//3 + ap'//7. If $2k$ ends in 2, then prime numbers ending in 7 cannot be considered in a general solution since in this case $2k$—'the odd number' will end in 5 and all odd numbers ending in 5 (except the number 5) will be composite numbers divisible by 5. The following **Table 4** shows the possible combinations of prime numbers ending in specific digits that can be considered in a general solution:

Table 4. Possible combinations of prime numbers ending in specific digits which can be considered, and others which are prohibited; in a general solution.

Last digit of $2k$	Last digit of combinations of prime numbers that are possible	Last digit that is not possible
2	*ap//1 + ap'//1*, *ap//3 + ap'//9*	*ap//7*
4	*ap//1 + ap'//3*, *ap//7 + ap'//7*	*ap//9*
6	*ap//3 + ap'//3*, *ap//7 + ap'//9*	*ap//1*
8	*ap//1 + ap'//7*, *ap//9 + ap'//9*	*ap//3*
0	*ap//1 + ap'//9*, *ap//3 + ap'//7*	-

On application of additive property of SADN, we obtain a general picture of the combinations of prime numbers that can add up to even numbers of a particular SADN and a particular last digit. These combinations are studied in terms of SADN and the last digit of the prime-prime combination. They can be summarized in the form of 45 matrices as elaborated in Appendix A.

Why $p1 + p2$ of SADN (2, 5, 8) will lie on $S7$ sequence and that of SADN (7, 4, 1) will lie on $S5$ sequence?

The matrices mentioned in Appendix A show that even numbers with SADN (2, 5, 8) can be summed up in prime numbers with SADN (7, 4, 1). Even numbers with SADN (7, 4, 1) can be added up in terms of prime numbers with SADN (2, 5, 8). Even numbers with SADN (3, 6, 9) can be added up in prime numbers of SADN (1, 2, 4, 5, 7, 8). If we consider this in perspective of the 3 sequence of odd numbers discussed in section2, this can be re-stated to suggest that prime combinations ($p1 + p2$) for even numbers with SADN (2, 5, 8) will be found on the $S1$ sequence of odd numbers while prime combinations for even numbers of SADN (7, 4, 1) will be found on the $S5$ sequence of odd numbers. Prime combinations for even numbers with SADN (3, 6, 9) will be in the form of $p1 + p2$ where $p1$ and $p2$ will be found on the $S5$ and $S7$ sequence respectively.

5. Cyclic-Sequence-Element (CSE) of even numbers

There is a cyclic & closed sequence of six numbers viz. 12, 2, 4, 6, 4, 2, 12; where first and last numbers are identical.

Consider any sequence of consecutive even numbers, e.g., 38, 40, 42, 44, 46, 48, 50, 52, 54, ... It may be written as 2×19, 4×10, 6×7, 4×11, 2×23, 12×4, 2×25, 4×13, 6×9, ...

For any length of sequence of consecutive even numbers, the universal sequence of factors will be {2, 4, 6, 4, 2, 12} acting as a unit sequence in cyclic order. We call this sequence of factors as factor-sequence and these elements as factor-elements (*fe*). Hence factor-sequence is given as {2, 4, 6, 4, 2, 12} in this specific order and 2, 4, 6, 12 are factor-elements (*fe*). This cyclic order of the factor-sequence may start from any specific element of the factor-sequence. First element of factor-sequence would depend on the first number of corresponding consecutive-even number-sequence selected for study.

Corresponding factor-elements are said to be '*fe*'. So $fe = \{2, 4, 6, 4, 2, 12\}$ with order preserved. Consider any even number '$2k$'.

Total number of possible combinations resulting in '$2k$' would be 'k'. Suppose factor-element of '$2k$' is 'fe', $2k = [k + n(12/fe)] + [k - n(12/fe)]$; $n = 1, 2, 3, \ldots$ such that $[k - n(12/fe)] > 0$.

The cyclic-relation between factor-element (*fe*) and SADN of even numbers can be understood with the help of following quadrant as mentioned in **Figure 1**:

We will now turn to deriving the number of combinations for a given even number based on their SADN and their position on the cyclical sequence of even numbers. These combinations should exhibit the following properties:

1) All these combinations identified for a given $2k$ would sum up to $2k$.
2) The components of the combinations would be elements of the relevant sequence for the given even number.

This is as discussed below in section 6.

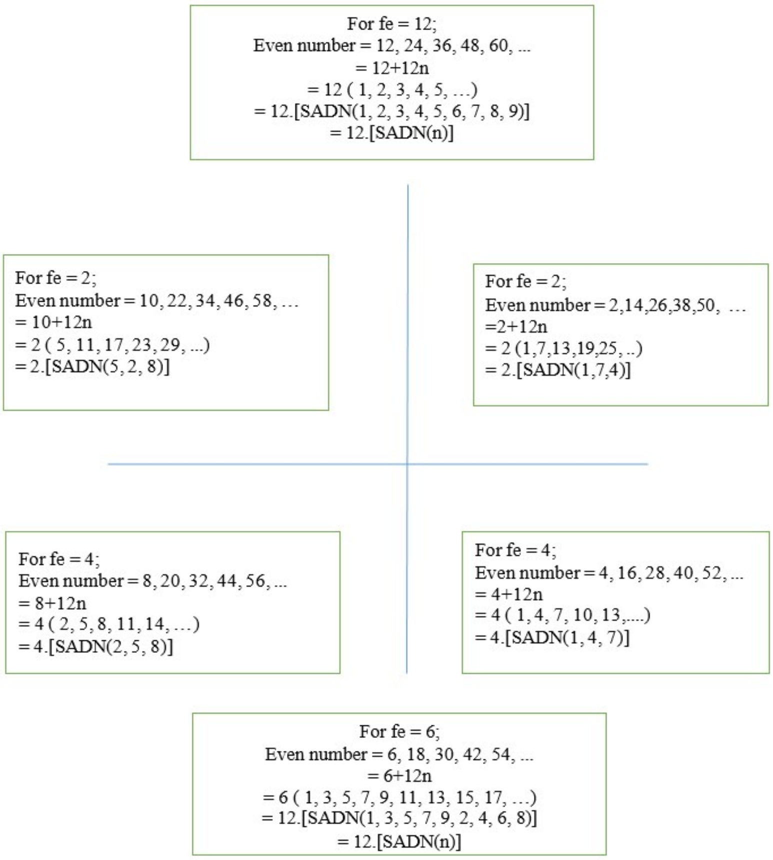

Figure 1. Quadrant figure representing cyclic sequence elements vis-a-vis SADN of even numbers.

6. Identifying *n*TC (i.e., total number of acceptable combinations) for a given even number (2*k*) depending on SADN and CSE

6.1. Even numbers (2*k*) of SADN (2, 5 or 8)

6.1.1. Even numbers (2*k*) of SADN (2, 5 or 8) that are of CSE 2 type

For even numbers of SADN (2, 5 or 8); the relevant sequence of odd numbers will be the $S7$ sequence as mentioned earlier in section 4 and **Table 3**.

As evident from **Figure 2**, $(2k - 2)/6$ gives the total number of elements of $S7$ sequence that exist up to $2k$, including an element given as $2k - 1$. As the number ‘1’ is not considered to be an element in $S7$ sequence, its corresponding element (i.e., $2k - 1$) of $S7$ sequence is also to be not considered. Hence the total number of elements, worth consideration, of the $S7$ sequence up to $2k$ that would include both prime and composite element numbers would be $\{(2k - 2)/6\} - 1$.

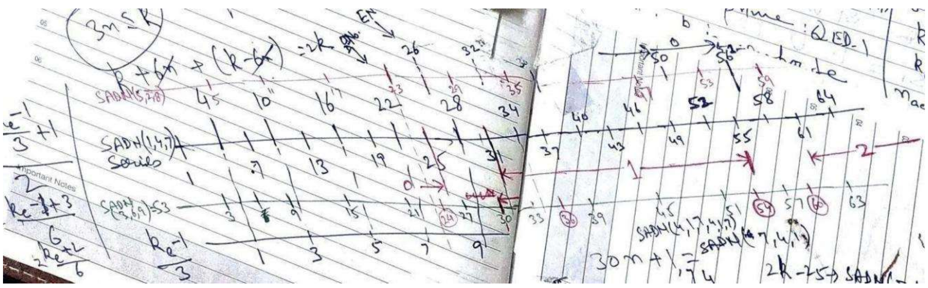

Figure 2. The three sequences; $S1$, $S3$ and $S5$ along with number of unique composites on the relevant sequence for initial terms.

Hence $\{(2k-2)/6\}-1$ will give the total number of elements, worth consideration, of the $S7$ sequence up to $2k$ that would include both prime and composite element numbers.

For instance, if $2k$ is taken as 32 (i.e., SADN 5), then $(32-2)/6=5$. Hence 5 is the number of elements of the $S7$ sequence whose value is less than 32 and the actual numbers are 7, 13, 19, 25 and 31.

If $2k$ is a SADN (2, 5 or 8) number of CSE 2 type, then k will be an odd number (for proof, see Appendix A of supplementary material). $(k-1)/6$ will give the number of combinations (i.e., nTC) of different elements of the $S7$ sequence that will add up to $2k$ (refer **Figure 2**). It is important to note here that all odd numbers of SADN (1, 4, 7) that lie on the $S7$ sequence and which are smaller than $2k$ will find a place in the combinations thus derived irrespective of whether they are prime or composite.

It is important to note here that since there is no consensus on whether the number 1 is a prime or composite, this additional combination of $2k = 1 + (2k-1)$ is only of academic importance and this combination has not been considered to have a role to play, in identifying $p1 + p2$ combinations for the given $2k$, in present paper.

6.1.2. Even numbers ($2k$) of SADN (2, 5 or 8) that are of CSE 4 type

If $2k$ is of SADN (2, 5 or 8) of CSE 4 type, k will be an even number (for proof, see Appendix B of supplementary material). Here, $(k-4)/6$ will give the number of total combinations (i.e., nTC) of elements of the $S7$ sequence that will add up to $2k$ (refer **Figure 2**).

Therefore, for $2k$ of SADN (2, 5, 8); the value of nTC will be: For $2k$ of CSE 2 type: $n\text{TC} = (k-1)/6$

For $2k$ of CSE 4 type: $n\text{TC} = (k-4)/6$

6.2. Even numbers ($2k$) of SADN (1, 4 or 7)

6.2.1. Even numbers ($2k$) of SADN (1, 4 or 7) that are of CSE 2 type $2k$

For even numbers of SADN (1, 4 or 7) the relevant sequence of odd numbers will be the $S5$ sequence. Here the total number of odd numbers lying on the $S5$ sequence up to $2k$ which includes both prime and composite element numbers will be given by $(2k-4)/6$ (for proof, see Appendix B of supplementary material and refer **Figure 2**). For instance, consider $2k$ to be 34 (i.e., of SADN 7). This implies that there are $(34-4)/6$, i.e., 5 elements of the S5 sequence whose value is less than 34 and these actual numbers are 5, 11, 17, 23 and 29.

In case $2k$ is a number of SADN 1, 4 or 7 and of CSE 2 type, then k will be an odd number and $(k+1)/6$ will give the value of nTC.

6.2.2. Even numbers ($2k$) of SADN (1, 4 or 7) that are of CSE 4 type

In this case, k will be an even number and $(k-2)/6$ will give the value of nTC as is proven in Appendix B of supplementary material.

6.3. Even numbers ($2k$) of SADN (3, 6 or 9)

In case of even numbers of SADN 3, 6 or 9; the numbers would be of either CSE 6 or CSE 12 type and the relevant sequence will be both the $S5$ and $S7$ sequence since the possible prime combinations will be such that one term will lie on the $S5$ sequence while the corresponding term will lie on the $S7$ sequence, as discussed in section 3. In this case $(2k/3)-1$ will give the number of elements of the $S5$ and $S7$ sequence of odd numbers whose value is less than $2k$ whereas $(2k/6)-1$ will give the value of nTC as is proven in Appendix B of supplementary material.

It is important to note here that since nTC for all even numbers irrespective of their SADN includes all elements on the relevant sequence irrespective of whether they are prime or composite, these combinations are of following three types:

- Combination 1: $p + c$ where one component is prime and the other is composite.
- Combination 2: $c1 + c2$ where both components being summed up are composites.
- Combination 3: $p1 + p2$ where both components being summed up are primes.

The next step would be to identify the $p1 + p2$ combinations, as discussed in following section.

7. Identifying combinations of type $p1 + p2$ for even number $2k$

Here we need to consider the number of total combinations (nTC) vis-à-vis the number of composites (nC) on the relevant sequence. The relation between these two variables can be in the nature of $n\text{TC} > n\text{C}$ or $n\text{TC} \le n\text{C}$.

If $n\text{TC} > n\text{C}$ then it directly follows that even if all composites are prime-eaters i.e., are paired with a prime number, there will still be at least one $p1 + p2$ combination. For instance, consider the even number 100. This is a SADN 1//0 type number which implies that S5 is the relevant sequence on which the $p1 + p2$ may be identified. For the number 100 there would be $(2k - 4)/6$, i.e., $(100 - 4)/6$, i.e., 16 elements on the $S5$ sequence, and $(k-2)/6$, i.e., $(50-2)/6$, i.e., 8 number of total acceptable combinations. On the $S5$ sequence, there are 4 composites (viz. 35, 65, 77 and 95) which are smaller than 100; if we consider all these 4 composites to be prime-eaters, they will absorb 4 out of the 8 acceptable combinations. This means that 4 combinations will still be in the nature of $p1 + p2$ combinations. Rather, it would be more appropriate to state that at least 4 of the 8 combinations would be in the nature of $p1 + p2$ combinations. This is because here we have considered all the composites to be prime-eaters and have not explored the possibility that some of these composites could be in the form of $c1 + c2$ combinations. The number of $p1 + p2$ combinations could increase if there are such $c1 + c2$ combinations. In this example, two of the 4 composites (viz. 35 and 65) come together to form a $c1 + c2$ combination. Therefore, in this example, the total combinations can be classified as: 1 out of 8 combinations are of type $c1 + c2$; 2 out of 8 combinations are of type $p + c$; and 5 out of 8 combinations are of type $p1 + p2$.

For even numbers where nTC (i.e., number of total combinations) is greater than nC (i.e., number of composites), finding out the number of $c1 + c2$ type combinations is an exercise for academic purposes only, whereas it becomes mandatory to find them out for numbers where nTC is smaller than nC.

In general terms; the above discussion can be summarised as follows:

For any even number (EN), SADN of EN = {7, 4 or 1} or {5, 2 or 8} or {6, 3 or 9}

Case (1): SADN of EN = {7,4 or 1} EN = $2k$.

Case (1A): EN/2 = k is a prime number

Case (1B): EN/2 = k is a composite number. If 'k' is a composite number:

Number of acceptable combinations is given as n_{acc}.

Case (1BP): If number of composites is less than number of primes (i.e., $n\text{C} < n\text{P}$) then even if all composites are prime-eaters; there exists at least one $p1 + p2$ pair.

Case (1BC): If number of composites is greater than or equal to number of primes (i.e., $n\text{C} \ge n\text{P}$) then we need to find total number of unique $c1 + c2$ pairs.

Case (2): SADN of EN = {5, 2 or 8} EN = $2k$.

Case (2A): EN/2 = k is a prime number.

Case (2B): EN/2 = k is a composite number. If 'k' is a composite number:

Number of acceptable combinations is given as n_{acc}.

Case (2BP): If number of composites is less than number of primes (i.e., nC < nP) then even if all composites are prime-eaters; there exists at least one $p1 + p2$ pair.

Case (2BC): If number of composites is greater than or equal to number of primes (i.e., nC ≥ nP) then we need to find total number of unique $c1 + c2$ pairs.

Case (3): SADN of EN = {6,3 or 9} EN = $2k$.

Case(3A): EN/2 = k is a prime number, which is never possible as midpoint (k) of even numbers ($2k$) of SADN (6, 3 or 9) would themselves be of SADN (3, 6 or 9) respectively and hence a composite number lying on $S3$ sequence.

Case (3B): EN/2 = k is a composite number. If 'k' is a composite number:

Number of acceptable combinations is given as n_{acc}.

Case (3BP): If number of composites is less than number of primes (i.e., nC < nP) then even if all composites are prime-eaters; there exists at least one $p1 + p2$ pair.

Case (3BC): If number of composites is greater than or equal to number of primes (i.e., nC ≥ nP) then we need to find total number of unique $c1 + c2$ pairs.

8. Conclusion

It follows from the above discussion that the Goldbach conjecture stands proven in two cases: first, where k is a prime number and second where nTC > nC for any given even number. In case of even numbers where nTC < nC; identification of $p1 + p2$ combinations requires derivation of the number of $c1 + c2$ combinations. A general method for deriving such combinations has been discussed in detail in the second paper of the series.

Author contributions

Conceptualization, MK and KC; methodology, MK and KC; validation, MK and KC; formal analysis, MK and KC; investigation, MK and KC; resources, MK and KC; data curation, MK and KC; writing—original draft preparation, MK and KC; writing—review and editing, MK and KC; visualization, MK and KC; supervision, MK and KC. All authors have read and agreed to the published version of the manuscript. All authors have contributed equally in preparing this manuscript.

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Conflict of interest

The authors declare no conflict of interest.

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Appendix A

Even number $2k$ of SADN(a/b) indicates that the SADN of $2k$ is ‘a’ and the last digit of $2k$ is ‘b’. **Tables 3** and **4** of paper 1 suggests that all even numbers necessarily belong to one and only one of the following 45 categories, as defined from ‘*a*’ to ‘*ss*’ below.

These 45 matrices represent the combinations of prime numbers that can add up to even numbers of a particular SADN and last digit, in terms of SADN and the last digit of the prime combination.

(with exceptions of rows containing prime numbers 2 and 5 omitted for reasons as discussed in the main–text of the paper):

Table A1. Even number of SADN (1/0).

Last digit of prime	**SADN of *p1* + SADN of *p2* =**					
	2 + 8		**3 + 7**		**5 + 5**	
1	11	71	Xxx	Xxx	41	41
3	83	53	3 as a number	Xxx	23	23
7	47	17	Xxx	97	167	167
9	29	89	Xxx	Xxx	59	59
Possible combination for last digit 0 = last digit 1 + last digit 9	11 + 89, 29 + 71, etc.		…		41 + 59, etc.	
Possible combination for last digit 0 = last digit 3 + last digit 7	83 + 17, 47 + 53, etc.		3 + 97, etc.		23 + 167, etc.	

Table A2. SADN (1/2).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	2 + 8		**3 + 7**		**5 + 5**	
1	11	71	Xxx	Xxx	41	41
3	83	53	3 as a number	Xxx	23	23
7xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
9	29	89	Xxx	79	59	59
Possible combination for last digit 2 = last digit 1 + last digit 1	11 + 71, etc.		…		41 + 41, etc.	
Possible combination for last digit 2 = last digit 3 + last digit 9	83 + 89, 29 + 53, etc.		3 + 79, etc.		23 + 59, etc.	

Table A3. SADN (1/4).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	2 + 8		**3 + 7**		**5 + 5**	
1	11	71	Xxx	61	41	41
3	83	53	3 as a number	xxx	23	23
7	47	17	Xxx	xxx	167	167
9xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
Possible combination for last digit 4 = last digit 1 + last digit 3	11 + 53, 83 + 71, etc.		3 + 61, etc.		41 + 23, etc.	
Possible combination for last digit 4 = last digit 7 + last digit 7	47 + 17, etc.		…		167 + 167, etc.	

Table A4. SADN (1/6).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	2 + 8		**3 + 7**		**5 + 5**	
1xxx	Xxx	Xxxx	Xxxx	Xxxx	Xxxx	Xxx
3	83	53	3 as a number	43	23	23
7	47	17	Xxx	xxx	167	167
9	29	89	Xxx	Xxx	59	59
Possible combination for last digit 6 = last digit 3 + last digit 3	83 + 53, etc.		3 + 43, etc.		23 + 23, etc.	
Possible combination for last digit 6 = last digit 7 + last digit 9	47 + 89, 29 + 17, etc.		…		167 + 59, etc.	

Table A5. SADN (1/8).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	2 + 8		**3 + 7**		**5 + 5**	
1	11	71	Xxx	Xxx	41	41
3xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
7	47	17	Xxx	xxx	167	167
9	29	89	Xxx	Xxx	59	59
Possible combination for last digit 8 = last digit 1 + last digit 7	11 + 17, 47 + 71, etc.		…		41 + 167, etc.	
Possible combination for last digit 8 = last digit 9 + last digit 9	29 + 89, etc.		…		59 + 59, etc.	

Table A6. SADN (2/0).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 1		**3 + 8**		**4 + 7**	
1	181	181	Xxx	Xxx	31	61
3	73	73	3 as a number	Xxx	13	43
7	37	37	Xxx	17	67	97
9	19	19	Xxx	xxx	139	79
Possible combination for last digit 0 = last digit 1 + last digit 9	181 + 19, etc.		…		31 + 79, 61 + 139, etc.	
Possible combination for last digit 0 = last digit 3 + last digit 7	73 + 37, etc.		3 + 17, etc.		13 + 97, 43 + 67, etc.	

Table A7. SADN (2/2).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 1		**3 + 8**		**4 + 7**	
1	181	181	Xxx	Xxx	31	61
3	73	73	3 as a number	Xxx	13	43
7xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
9	19	19	Xxx	89	139	79
Possible combination for last digit 2 = last digit 1 + last digit 1	181 + 181, etc.		…		31 + 61, etc.	
Possible combination for last digit 2 = last digit 3 + last digit 9	73 + 19, etc.		3 + 89, etc.		13 + 79, 43 + 139, etc.	

Table A8. SADN (2/4).

Last digit of prime	SADN of *p*1 + SADN of *p*2 =					
	1 + 1		**3 + 8**		**4 + 7**	
1	181	181	Xxx	71	31	61
3	73	73	3 as a number	Xxx	13	43
7	37	37	Xxx	Xxx	67	97
9	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
Possible combination for last digit 4 = last digit 1 + last digit 3	181 + 73, etc.		71 + 3, etc.		31 + 43, 61 + 13, etc.	
Possible combination for last digit 4 = last digit 7 + last digit 7	37 + 37, etc.		…		67 + 97, etc.	

Table A9. SADN (2/6).

Last digit of prime	SADN of *p*1 + SADN of *p*2 =					
	1 + 1		**3 + 8**		**4 + 7**	
1xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
3	73	73	3 as a number	53	13	43
7	37	37	Xxx	Xxx	67	97
9	19	19	Xxx	Xxx	139	79
Possible combination for last digit 6 = last digit 3 + last digit 3	73 + 73, etc.		3 + 53, etc.		13 + 43, etc.	
Possible combination for last digit 6 = last digit 7 + last digit 9	37 + 19, etc.		…		67 + 79, 139 + 97, etc.	

Table A10. SADN (2/8).

Last digit of prime	SADN of *p*1 + SADN of *p*2 =					
	1 + 1		**3 + 8**		**4 + 7**	
1	181	181	Xxx	Xxx	31	61
3xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
7	37	37	Xxx	Xxx	67	97
9	19	19	Xxx	Xxx	139	79
Possible combination for last digit 8 = last digit 1 + last digit 7	181 + 37, etc.		…		31 + 97, 61 + 67, etc.	
Possible combination for last digit 8 = last digit 9 + last digit 9	19 + 19, etc.		…		139 + 79, etc.	

Table A11. SADN (3/0).

Last digit of prime	SADN of *p*1 + SADN of *p*2 =					
	1 + 2		**4 + 8**		**5 + 7**	
1	181	11	31	71	41	61
3	73	83	13	53	23	43
7	37	47	67	17	167	97
9	19	29	139	89	59	79
Possible combination for last digit 0 = last digit 1 + last digit 9	181+29, 19+11, etc.		31+89, 139+71, etc.		41+79, 59+61, etc.	
Possible combination for last digit 0 = last digit 3 + last digit 7	73+47, 37+83, etc.		13+17, 67+53, etc.		23+97, 167+43, etc.	

Table A12. SADN (3/2).

Last digit of prime	**SADN of *p1* + SADN of *p2* =**					
	1 + 2		**4 + 8**		**5 + 7**	
1	181	11	31	71	41	61
3	73	83	13	53	23	43
7xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
9	19	29	139	89	59	79
Possible combination for last digit 2 = last digit 1 + last digit 1	181 + 11, etc.		31 + 71, etc.		41 + 61, etc.	
Possible combination for last digit 2 = last digit 3 + last digit 9	73 + 29, 19 + 83, etc.		13 + 89, 139 + 53, etc.		23 + 79, 59 + 43, etc.	

Table A13. SADN (3/4).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 2		**4 + 8**		**5 + 7**	
1	181	11	31	71	41	61
3	73	83	13	53	23	43
7	37	47	67	17	167	97
9xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
Possible combination for last digit 4 = last digit 1 + last digit 3	181 + 83, 73 + 11, etc.		31 + 53, 13 + 71, etc.		41 + 43, 23 + 61, etc.	
Possible combination for last digit 4 = last digit 7 + last digit 7	37 + 47, etc.		67 + 17, etc.		167 + 97, etc.	

Table A14. SADN (3/6).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 2		**4 + 8**		**5 + 7**	
1xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
3	73	83	13	53	23	43
7	37	47	67	17	167	97
9	19	29	139	89	59	79
Possible combination for last digit 6 = last digit 3 + last digit 3	73 + 83, etc.		13 + 53, etc.		23 + 43, etc.	
Possible combinatio n for lastdigit 6 = last digit 7 + last digit 9	37 + 29, 19 + 47, etc.		67 + 89, 139 + 17, etc.		167 + 79, 59 + 97, etc.	

Table A15. SADN (3/8).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 2		**4 + 8**		**5 + 7**	
1	181	11	31	71	41	61
3xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
7	37	47	67	17	167	97
9	19	29	139	89	59	79
Possible combination for last digit 8 = last digit 1 + last digit 7	181 + 47, 11 + 37, etc.		31 + 17, 71 + 67, etc.		41 + 97, 61 + 167, etc.	
Possible combination for last digit 8 = last digit 9 + last digit 9	19 + 29, etc.		139 + 89, etc.		59 + 79, etc.	

Table A16. SADN (4/0).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 3		**2 + 2**		**5 + 8**	
1	Xxx	xxx	11	11	41	71
3	Xxx	3 as a number	83	83	23	53
7	37	Xxx	47	47	167	17
9	Xxx	xxx	29	29	59	89
Possible combination for last digit 0 = last digit 1 + last digit 9	…		11 + 29, etc.		41 + 89, 71 + 59, etc.	
Possible combination for last digit 0 = last digit 3 + last digit 7	3 + 37, etc.		83 + 47, etc.		23 + 17, 53 + 167, etc.	

Table A17. SADN (4/2).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 3		**2 + 2**		**5 + 8**	
1	Xxx	xxx	11	11	41	71
3	Xxx	3 as a number	83	83	23	53
7xxx	Xxx	Xxx	Xxxx	Xxx	Xxx	Xxx
9	19	Xxx	29	29	59	89
Possible combination for last digit 2 = last digit 1 + last digit 1	…		11 + 11, etc.		41 + 71, etc.	
Possible combination for last digit 2 = last digit 3 + last digit 9	3 + 19, etc.		83 + 29, etc.		23 + 89, 53 + 59, etc.	

Table A18. SADN (4/4).

Last digit of prime	**SADN of *p1* + SADN of *p2* =**					
	1 + 3		**2 + 2**		**5 + 8**	
1	181	Xxx	11	11	41	71
3	Xxx	3 as a number	83	83	23	53
7	Xxx	Xxx	47	47	167	17
9xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
Possible combination for last digit 4 = last digit 1 + last digit 3	181 + 3, etc.		11 + 83, etc.		41 + 53, 71 + 23, etc.	
Possible combination for last digit 4 = last digit 7 + last digit 7	…		47 + 47, etc.		167 + 17, etc.	

Table A19. SADN (4/6).

Last digitof prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 3		**2 + 2**		**5 + 8**	
1xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
3	73	3 as a number	83	83	23	53
7	Xxx	Xxx	47	47	167	17
9	Xxx	Xxx	29	29	59	89
Possible combination for lastdigit 6 = last digit 3 + last digit 3	73 + 3, etc.		83 + 83, etc.		23 + 53, etc.	
Possible combination for last digit 6 = last digit 7 + last digit 9	…		47 + 29, etc.		167 + 89, 17 + 59, etc.	

Table A20. SADN (4/8).

Last digit of prime	SADN of *p*1 + SADN of *p*2 =					
	1 + 3		**2 + 2**		**5 + 8**	
1	Xxx	Xxx	11	11	41	71
3xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
7	Xxx	Xxx	47	47	167	17
9	Xxx	Xxx	29	29	59	89
Possible combination for last digit 8 = last digit 1 + last digit 7	…		11 + 47, etc.		41 + 17, 71 + 167, etc.	
Possible combination for last digit 8 = last digit 9 + last digit 9	…		29 + 29, etc.		59 + 89, etc.	

Table A21. SADN (5/0).

Last digit of prime	SADN of *p*1 + SADN of *p*2 =					
	1 + 4		**2 + 3**		**7 + 7**	
1	181	31	Xxx	xxx	61	61
3	73	13	Xxx	3 as a number	43	43
7	37	67	47	xxx	97	97
9	19	139	Xxx	xxx	79	79
Possible combination for last digit 0 = last digit 1 + last digit 9	181 + 139, 31 + 19, etc.				61 + 79, etc.	
Possible combination for last digit 0 = last digit 3 + last digit 7	73 + 67, 13 + 37, etc.		3 + 47, etc.		43 + 97, etc.	

Table A22. SADN (5/2).

Last digit of prime	SADN of *p*1 + SADN of *p*2 =					
	1 + 4		**2 + 3**		**7 + 7**	
1	181	31	Xxx	xxx	61	61
3	73	13	Xxx	3 as a number	43	43
7xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
9	19	139	29	xxx	79	79
Possible combination for last digit 2 = last digit 1 + last digit 1	181 + 31, etc.		…		61 + 61, etc.	
Possible combination for last digit 2 = last digit 3 + last digit 9	73 + 139, 13 + 19, etc.		3 + 29, etc.		43 + 79, etc.	

Table A23. SADN (5/4).

Last digit of prime	SADN of *p*1 + SADN of *p*2 =					
	1 + 4		**2 + 3**		**7 + 7**	
1	181	31	11	xxx	61	61
3	73	13	Xxx	3 as a number	43	43
7	37	67	Xxx	Xxx	97	97
9xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
Possible combination for last digit 4 = last digit 1 + last digit 3	181 + 13, 31 + 73, etc.		11 + 3, etc.		61 + 43, etc.	
Possible combination for last digit 4 = last digit 7 + last digit 7	37 + 67, etc.		….		97 + 97, etc.	

Table A24. SADN (5/6).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 4		**2 + 3**		**7 + 7**	
1xxx	Xxx	Xxx	Xxx	xxx	Xxx	Xxx
3	73	13	83	3 as a number	43	43
7	37	67	Xxx	Xxx	97	97
9	19	139	Xxx	xxx	79	79
Possible combination for last digit 6 = last digit 3 + last digit 3	73 + 13, etc.		83 + 3, etc.		43 + 43, etc.	
Possible combination for last digit 6 = last digit 7 + last digit 9	37 + 139, 67 + 19, etc.		….		97 + 79, etc.	

Table A25. SADN (5/8).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 4		**2 + 3**		**7 + 7**	
1	181	31	Xxx	xxx	61	61
3xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
7	37	67	Xxx	Xxx	97	97
9	19	139	Xxx	xxx	79	79
Possible combination for last digit 8 = last digit 1 + last digit 7	181 + 67, 31 + 37, etc.		…		61 + 97, etc.	
Possible combination for lastdigit 8 = last digit 9 + last digit 9	19 + 139, etc.		…		79 + 79, etc.	

Table A26. SADN (6/0).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 5		**2 + 4**		**7 + 8**	
1	181	41	11	31	61	71
3	73	23	83	13	43	53
7	37	167	47	67	97	17
9	19	59	29	139	79	89
Possible combination for last digit 0 = last digit 1 + last digit 9	181 + 59, 41 + 19, etc.		11 + 139, 31 + 29, etc.		61 + 89, 71 + 79, etc.	
Possible combination for last digit 0 = last digit 3 + last digit 7	73 + 167, 23 + 37, etc.		83 + 67, 13 + 47, etc.		43 + 17, 53 + 97, etc.	

Table A27. SADN (6/2).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 5		**2 + 4**		**7 + 8**	
1	181	41	11	31	61	71
3	73	23	83	13	43	53
7xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
9	19	59	29	139	79	89
Possible combination for last digit 2 = last digit 1 + last digit 1	181 + 41, etc.		11 + 31, etc.		61 + 71, etc.	
Possible combination for last digit 2 = last digit 3 + last digit 9	73 + 59, 23 + 19, etc.		83 + 139, 13 + 29, etc.		43 + 89, 53 + 79, etc.	

Table A28. SADN (6/4).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 5		**2 + 4**		**7 + 8**	
1	181	41	11	31	61	71
3	73	23	83	13	43	53
7	37	167	47	67	97	17
9xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
Possible combination for last digit 4 = last digit 1 + last digit 3	181 + 23, 41 + 73, etc.		11 + 13, 31 + 83, etc.		61 + 53, 71 + 43, etc.	
Possible combination for last digit 4 = last digit 7 + last digit 7	37 + 167, etc.		47 + 67, etc.		97 + 17, etc.	

Table A29. SADN (6/6).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 5		**2 + 4**		**7 + 8**	
1xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
3	73	23	83	13	43	53
7	37	167	47	67	97	17
9	19	59	29	139	79	89
Possible combination for last digit 6 = last digit 3 + last digit 3	73 + 23, etc.		83 + 13, etc.		43 + 53, etc.	
Possible combination for last digit 6 = last digit 7 + last digit 9	37 + 59, 167 + 19, etc.		47 + 139, 67 + 29, etc.		97 + 89, 17 + 79, etc.	

Table A30. SADN (6/8).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 5		**2 + 4**		**7 + 8**	
1	181	41	11	31	61	71
3xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
7	37	167	47	67	97	17
9	19	59	29	139	79	89
Possible combination for lastdigit 8 = last digit 1 + last digit 7	181 + 167, 41 + 37, etc.		11 + 67, 31 + 47, etc.		61 + 17, 71 + 97, etc.	
Possible combination for last digit 8 = last digit 9 + last digit 9	19 + 59, etc.		29 + 139, etc.		79 + 89, etc.	

Table A31. SADN (7/0).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	2 + 5		**3 + 4**		**8 + 8**	
1	11	41	Xxx	Xxx	71	71
3	83	23	3 as a number	Xxx	53	53
7	47	167	Xxx	67	17	17
9	29	59	Xxx	Xxx	89	89
Possible combination for last digit 0 = last digit 1 + last digit 9	11 + 59, 41 + 29, etc.		…		71 + 89, etc.	
Possible combination for last digit 0 = last digit 3 + last digit 7	83 + 167, 23 + 47, etc.		3 + 67, etc.		53 + 17, etc.	

Table A32. SADN (7/2).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	2 + 5		**3 + 4**		**8 + 8**	
1	11	41	Xxx	Xxx	71	71
3	83	23	3 as a number	Xxx	53	53
7xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
9	29	59	Xxx	139	89	89
Possible combination for last digit 2 = last digit 1 + last digit 1	11 + 41, etc.		…		71 + 71, etc.	
Possible combination for last digit 2 = last digit 3 + last digit 9	83 + 59, 23 + 29, etc.		3 + 139, etc.		53 + 89, etc.	

Table A33. SADN (7/4).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	2 + 5		**3 + 4**		**8 + 8**	
1	11	41	Xxx	31	71	71
3	83	23	3 as a number	Xxx	53	53
7	47	167	Xxx	Xxx	17	17
9xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
Possible combination for lastdigit 4 = last digit 1 + last digit 3	11 + 23, 41 + 83, etc.		3 + 31, etc.		71 + 53, etc.	
Possible combination for last digit 4 = last digit 7 + last digit 7	47 + 167, etc.		…		17 + 17, etc.	

Table A34. SADN (7/6).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	2 + 5		**3 + 4**		**8 + 8**	
1xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
3	83	23	3 as a number	13	53	53
7	47	167	Xxx	Xxx	17	17
9	29	59	Xxx	Xxx	89	89
Possible combination for last digit 6 = last digit 3 + last digit 3	83 + 23, etc.		3 + 13, etc.		53 + 53, etc.	
Possible combination for last digit 6 = last digit 7 + last digit 9	47 + 59, 167 + 29, etc.		...		17 + 89, etc.	

Table A35. SADN (7/8).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	2 + 5		**3 + 4**		**8 + 8**	
1	11	41	Xxx	Xxx	71	71
3xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
7	47	167	Xxx	Xxx	17	17
9	29	59	Xxx	Xxx	89	89
Possible combination for last digit 8 = last digit 1 + last digit 7	11 + 167, 41 + 47, etc.		...		71 + 17, etc.	
Possible combination for last digit 8 = last digit 9 + last digit 9	29 + 59, etc.		…		89 + 89, etc.	

Table A36. SADN (8/0).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 7		**3 + 5**		**4 + 4**	
1	181	61	Xxx	Xxx	31	31
3	73	43	3 as a number	Xxx	13	13
7	37	97	Xxx	167	67	67
9	19	79	Xxx	Xxx	139	139
Possible combination for last digit 0 = last digit 1 + last digit 9	181 + 79, 61 + 19, etc.		…		31 + 139, etc.	
Possible combination for lastdigit 0 = last digit 3 + last digit 7	73 + 97, 43 + 37, etc.		3 + 167, etc.		13 + 67, etc.	

Table A37. SADN (8/2).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 7		**3 + 5**		**4 + 4**	
1	181	61	Xxx	Xxx	31	31
3	73	43	3 as a number	Xxx	13	13
7xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
9	19	79	Xxx	59	139	139
Possible combination for last digit 2 = last digit 1 + last digit1	181 + 61, etc.		…		31 + 31, etc.	
Possible combination for last digit 2 = last digit 3 + last digit9	73 + 79, 43 + 19, etc.		3 + 59, etc.		13 + 139, etc.	

Table A38. SADN (8/4).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 7		**3 + 5**		**4 + 4**	
1	181	61	Xxx	41	31	31
3	73	43	3 as a number	Xxx	13	13
7	37	97	Xxx	Xxx	67	67
9xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
Possible combination for last digit 4 = last digit 1 + last digit 3	181 + 43, 61 + 73, etc.		3 + 41, etc.		31 + 13, etc.	
Possible combination for last digit 4 = last digit 7 + last digit 7	37 + 97, etc.		…		67 + 67, etc.	

Table A39. SADN (8/6).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 7		**3 + 5**		**4 + 4**	
1xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
3	73	43	3 as a number	23	13	13
7	37	97	Xxx	Xxx	67	67
9	19	79	Xxx	Xxx	139	139
Possible combination for last digit 6 = last digit 3 + last digit 3	73 + 43, etc.		3 + 23, etc.		13 + 13, etc.	
Possible combination for last digit 6 = last digit 7 + last digit 9	37 + 79, 97 + 19, etc.		…		67 + 139, etc.	

Table A40. SADN (8/8).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 7		**3 + 5**		**4 + 4**	
1	181	61	Xxx	Xxx	31	31
3xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
7	37	97	Xxx	Xxx	67	67
9	19	79	Xxx	Xxx	139	139
Possible combination for last digit 8 = last digit 1 + last digit 7	181 + 97, 61 + 37, etc.		…		31 + 67, etc.	
Possible combination for last digit 8 = last digit 9 + last digit 9	19 + 79, etc.		…		139 + 139, etc.	

Table A41. SADN (9/0).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 8		**2 + 7**		**4 + 5**	
1	181	71	11	61	31	41
3	73	53	83	43	13	23
7	37	17	47	97	67	167
9	19	89	29	79	139	59
Possible combination for last digit 0 = last digit 1 + last digit 9	181 + 89, 71 + 19, etc.		11 + 79, 61 + 29, etc.		31 + 59, 41 + 139, etc.	
Possible combination for last digit 0 = last digit 3 + last digit 7	73 + 17, 53 + 37, etc.		83 + 97, 43 + 47, etc.		13 + 167, 23 + 67, etc.	

Table A42. SADN (9/2).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 8		**2 + 7**		**4 + 5**	
1	181	71	11	61	31	41
3	73	53	83	43	13	23
7xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
9	19	89	29	79	139	59
Possible combination for last digit 2 = last digit 1 + last digit 1	181 + 71, etc.		11 + 61, etc.		31 + 41, etc.	
Possible combination for last digit 2 = last digit 3 + last digit 9	73 + 89, 53 + 19, etc.		83 + 79, 43 + 29, etc.		13 + 59, 23 + 139, etc.	

Table A43. SADN (9/4).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 8		**2 + 7**		**4 + 5**	
1	181	71	11	61	31	41
3	73	53	83	43	13	23
7	37	17	47	97	67	167
9xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
Possible combination for last digit 4 = last digit 1 + last digit 3	181 + 53, 71 + 73, etc.		11 + 43, 61 + 83, etc.		31 + 23, 41 + 13, etc.	
Possible combination for last digit 4 = last digit 7 + last digit 7	37 + 17, etc.		47 + 97, etc.		67 + 167, etc.	

Table A44. SADN (9/6).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 8		**2 + 7**		**4 + 5**	
1xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
3	73	53	83	43	13	23
7	37	17	47	97	67	167
9	19	89	29	79	139	59
Possible combination for last digit 6 = last digit 3 + last digit 3	73 + 53, etc.		83 + 43, etc.		13 + 23, etc.	
Possible combination for last digit 6 = last digit 7 + last digit 9	37 + 89, 17 + 19, etc.		47 + 79, 97 + 29, etc.		67 + 59, 167 + 139, etc.	

Table A45. SADN (9/8).

Last digit of prime	**SADN of *p*1 + SADN of *p*2 =**					
	1 + 8		**2 + 7**		**4 + 5**	
1	181	71	11	61	31	41
3xxx	Xxx	Xxx	Xxx	Xxx	Xxx	Xxx
7	37	17	47	97	67	167
9	19	89	29	79	139	59
Possible combination for last digit 8 = last digit 1 + last digit 7	181 + 17, 71 + 37, etc.		11 + 97, 61 + 47, etc.		31 + 167, 41 + 67, etc.	
Possible combination for last digit 8 = last digit 9 + last digit 9	19 + 89, etc.		29 + 79, etc.		139 + 59, etc.	

Appendix B

Derivation of total number of acceptable combination (to be referred along with section 6 and **Figure 2** of the manuscript)

Case-1: For even number 2*k* of SADN (5, 2 or 8).

SADN of (2*k*) = {5, 2, 8}.

Case-1P: '*k*' is prime implies that Goldbach conjecture holds true for this case.

Case-1C: '*k*' is composite.

Case-1C-odd: composite '*k*' is an odd number and is represented as kodd or ko.

As kodd or ko will not be divisible by 2 ⇨ CSE can not be 4 or 6 or 12; hence for such evennumbers CSE = 2.

Suppose components of acceptable combinations are ko + 6*n* and ko − 6*n* [Refer diagram 5.1 of the paper].

As even numbers will not be primes hence components of acceptable combinations need tobe odd only; hence ko ± 6*n* must be an odd number.

⇨ 6*n* must be an even number

⇨ *n* may be odd or even number. As ko − 6n ≥ 1

⇨ *n* ≤ (ko − 1)/6

Since odd as well as even values of 'n' are acceptable in present case hence number of acceptable combinations i.e., nacc ≤ floor function of (ko − 1)/6

Now floor function of (ko − 1)/6 = (ko − 1)/6 [because ko = SADN (7, 4, 1)].

Therefore, nacc = (ko − 1)/6

Therefore, number of acceptable combinations = nacc = (ko − 1)/6

Case-1C-even: composite '*k*' is an even number and is represented as keven or ke.

As keven or ke will be divisible by 2 but not divisible by 3 (as ke is of SADN (7, 4, 1)) ⇨ CSE cannot be 6 or 12 or 2; hence for such even numbers CSE = 4.

Suppose components of acceptable combinations are ke + 3*n* and ke − 3*n*. [Refer diagram 5.1 of the paper].

As even numbers will not be primes hence components of acceptable combinations need tobe odd only; hence ke ± 3*n* must be an odd number.

⇨ 3*n* must be an odd number

⇨ *n* must be odd number As ke − 3n ≥ 1

⇨ *n* ≤ (ke − 1)/3

⇨ *n* = floor function of [(ke − 1)/3] = [(ke − 1)/3] [because ke = SADN (7, 4, 1)].

As only odd values of 'n' are acceptable in present case hence number of acceptablecombinations nacc = (*n* − 1)/2.

Therefore, number of acceptable combinations = nacc = (*n* − 1)/2 = [{((ke − 1)/3) − 1}/2]or

number of acceptable combinations = nacc = (ke − 4)/6.

Case-2: For even number $2k$ of SADN (7, 4 or 1). Even number is defined as $2k$, SADN of ($2k$) = {7, 4, 1}.

Case-2P: 'k' is prime implies that Goldbach conjecture holds true for this case.

Case-2C: 'k' is composite.

Case-2C-odd: composite 'k' is an odd number and is represented as kodd or ko.

As kodd or ko will not be divisible by 2 ⇨ CSE can not be 4 or 6 or 12; hence for such evennumbers CSE = 2.

Suppose components of acceptable combinations are ko+ $6n$ and ko − $6n$. [Refer diagram 5.1 of the paper].

As even numbers will not be primes hence components of acceptable combinations need tobe odd only; hence ko ± $6n$ must be an odd number.

⇨ $6n$ must be an even number

⇨ n may be odd or even number as ko − $6n \geq 1$

⇨ $n \leq$ (ko − 1)/6

As odd as well as even values of 'n' are acceptable in present case hence number ofacceptable combinations i.e., nacc ≤ floor function of (ko – 1)/6

Now floor function of (ko − 1)/6 = (ko − 1)/6 [because ko = SADN (5, 2, 8)].

Therefore, nacc = (ko − 1)/6.

Therefore, number of acceptable combinations = nacc = (ko − 1)/6.

Case-2C-even: composite 'k' is an even number and is represented as keven or ke.

As keven or ke will be divisible by 2 but not divisible by 3 (as ke is of SADN (7, 4, 1)) ⇨ CSEcan not be 6 or 12 or 2; hence for such even numbers CSE = 4

Suppose components of acceptable combinations are ke + $3n$ and ke − $3n$ [Refer diagram 5.1 of the paper].

As even numbers will not be primes hence components of acceptable combinations need tobe odd only; hence ke ± $3n$ must be an odd number.

⇨ $3n$ must be an odd number

⇨ n must be odd number as ke − $3n \geq 1$

⇨ $n \leq$ (ke − 1)/3

⇨ n = floor function of [(ke − 1)/3] = [(ke − 1)/3] [because ke = SADN (5, 2, 8)].

As only odd values of 'n' are acceptable in present case hence number of acceptable combinations nacc= (n – 1)/2.

Therefore, number of acceptable combinations = nacc = (n – 1)/2 = [{((ke – 1)/3) – 1}/2]or **number of acceptable combinations = nacc = (ke − 4)/6.**