Journal of AppliedMath 2024, 2(5), 1829.
https://doi.org/10.59400/jam1829

Article

Positivity results to iterative system of higher order boundary value problems

Kanakayya Namana[1], Sreedhar Namburi[1,*], Rajendra Prasad Kapula[2]

[1] Department of Mathematics, School of Science, GITAM (Deemed to be University), Visakhapatnam 530045, India
[2] Department of Applied Mathematics, College of Science and Technology, Andhra University, Visakhapatnam 530003, India
*** Corresponding author:** Sreedhar Namburi, sreedharnamburi@rediffmail.com

CITATION

Namana K, Namburi S, Kapula RP. Positivity results to iterative system of higher order boundary value problems. Journal of AppliedMath. 2024; 2(5): 1829. https://doi.org/10.59400/jam1829

ARTICLE INFO

Received: 8 October 2024
Accepted: 23 October 2024
Available online: 30 October 2024

Abstract: The present research explores the existence of positive solutions for the iterative system of higher-order differential equations with integral boundary conditions that include a non-homogeneous term. To address the boundary value problem, the solution is expressed as a solution of an equivalent integral equation involving kernels. Subsequently, bounds for these kernels are determined to facilitate further analysis. The primary tool employed in this study is the Guo-Krasnosel'skii fixed-point theorem, which is utilized to establish the existence of positive solutions within a cone of a Banach space. This approach enables a rigorous exploration of the existence of at least one positive solution and provides insights into the behavior of the differential equation under the given boundary conditions.

Keywords: iterative system; non-homogenous conditions; positivity results; kernel

1. Introduction

Differential equations are useful for formulating mathematical models to analyze real-world phenomena. These models are frequently expressed as either initial or boundary value problems. The problems with integral boundary conditions are commonly encountered in various fields of science and engineering, such as thermoelectricity, thermoelasticity, plasma physics, underground water flow, hydrodynamic problems, chemical engineering, and many more; see Cannon [1], Ionkin [2], Chegis [3]. We refer to the study on differential equations of third order, fourth order, and higher order with integral boundary conditions [4–10].

Complex structures with multiple degrees of freedom are frequently described by systems of differential equations under specific conditions. The main challenge is to analyze mathematical models for such structures and to determine the positive solutions using various mathematical techniques. Due to the theoretical and practical significance of this topic, researchers have shown considerable interest in studying positive solutions for iterative systems of nonlinear boundary value problems by determining the parameter intervals. A few papers along these lines include Henderson et al. [11], Henderson et al. [12], Prasad et al. [13], and Oz and Karaca [14] for second-order systems; Bouteraa et al. [15] for fourth-order systems; and Henderson and Ntouyas [16], Prasad et al. [17], and Namburi et al. [18] for nth-order systems. In particular, iterative differential equations provide a novel approach to studying functional differential equations [19,20]. However, only a few studies in the literature address the existence of positive solutions for iterative systems of nonlinear boundary value problems that do not involve parameters [21,22]. In this work, we extend these results to iterative systems of higher-order boundary value problems involving sets of equations.

Based on the above literature, we investigate the presence of positive solutions to the following iterative system of nonlinear boundary value problems,

$$\left.\begin{array}{c} w_i^{(ꞵ)}(æ) + a_i(æ) f_i\big(w_{i+1}(æ)\big) = 0, \qquad 1 \le i \le m,\, æ \in [0,1], \\ w_{m+1}(æ) = w_1(æ),\, æ \in [0,1], \end{array}\right\} \tag{1}$$

fulfilling conditions with non-homogeneous term

$$\left.\begin{array}{c} w_i(0) = 0, w_i'(0) = 0, \dots,\ w_i^{(ꞵ-2)}(0) = 0, \\ w_i^{(ɤ)}(1) - \eta_i \displaystyle\int_0^1 g_i(\xi)\, w_i^{(ɤ)}(\xi) d\xi = ꞵ_i, \qquad 1 \le i \le m, \end{array}\right\} \tag{2}$$

where $ɤ$ is a fixed value ranging from 1 to $ꞵ - 2$, $ꞵ \ge 3$, η_i is a positive real constant, $ꞵ_i \epsilon\, (0, \infty)$ is a parameter for $1 \le i \le m$, by fixed point theorem of Guo-Krasnosel'skii. By giving different values to the constants $ꞵ$, η_i and $ꞵ_i$, we get various order problems, for instance [23,24]. Here, the integral kernel plays a key role in establishing our results. The applications of integral equations are vast, spanning areas such as signal processing, fluid flow, electromagnetics, quantum mechanics, and population dynamics. However, solving integral equations presents challenges due to kernel singularity, high computational costs for high-dimensional systems, and convergence difficulties in nonlinear cases. Advancements in numerical methods continue to expand the applicability of integral equations, providing new insights and solutions to complex real-world problems [25–29].

For $1 \le i \le m$, the below stated assumptions are valid in this paper:

(E1) $g_i \in C([0,1], \mathbb{R}^+)$, $f_i \in C(\mathbb{R}^+, \mathbb{R}^+)$, $a_i(æ) \in C([0,1], \mathbb{R}^+)$ and $a_i(æ)$ fails to vanish identically on any closed subset of [0,1],

(E2) $Ɣ_i = (ꞵ - 1)!\,(1 - \eta_i \theta_i) > 0$, where $\theta_i = \int_0^1 g_i(\xi) \xi^{ꞵ-ɤ-1}\, d\xi$.

Define the nonnegative extended real numbers f_{i0} and $f_{i\infty}$ as $f_{i0} = \lim\limits_{w \to 0^+} \frac{f_i(w)}{w}$ and $f_{i\infty} = \lim\limits_{w \to \infty} \frac{f_i(w)}{w}$, for $1 \le i \le m$.

1.1. Definition

By a positive solution of the problem (1) and (2), we mean that $\big(w_1(æ), w_2(æ), \dots, w_m(æ)\big) \in \big(C^{ꞵ}[0,1]\big)^m$ satisfying the Equations (1) and (2) with $w_i(æ) \ge 0$, $\quad i = 1,2,\dots,m$ for $æ \in [0,1]$ and $\big(w_1(æ), w_2(æ), \dots, w_m(æ)\big) \neq (0,0,\dots,0)$.

The remaining part of the paper is structured as follows: In section 2, the solution for Equations (1) and (2) is represented as a solution of the related integral equation, which includes kernels. Section 3 is devoted to studying the presence of positive solutions to Equations (1) and (2) based on the Guo-Krasnosel'skii theorem. The established results are verified by constructing the examples in section 4. The last section presents the conclusion and future scope of the study.

2. Inequalities for kernels

In this section, we represent the solution of the Equations (1) and (2) as a solution of the related integral equation with kernels. We then derive some inequalities related to these kernels.

2.1. Lemma [18]

Suppose that (E2) is true. Let $Q(x) \in C([0,1], \mathbb{R}^+)$. Then the solution of the differential equation

$$w_i^{(\beta)}(x) + Q(x) = 0, \qquad 1 \le i \le m, x \in [0,1], \tag{3}$$

fulfilling the Equation (2) is expressed uniquely as

$$w_i(x) = \frac{\beta_i(\beta - \gamma - 1)!\, x^{\beta-1}}{\gamma_i} + \int_0^1 \left[R(x, v) + \frac{\eta_i x^{\beta-1}}{\gamma_i} \int_0^1 S(\xi, v) g_i(\xi)\, d\xi \right] Q(v) dv, \tag{4}$$

$$R(x, v) = \frac{1}{(\beta-1)!} \begin{cases} \left[x^{\beta-1}(1-v)^{\beta-\gamma-1} - (x-v)^{\beta-1}\right], & 0 \le v \le x \le 1, \\ x^{\beta-1}(1-v)^{\beta-\gamma-1}, & 0 \le x \le v \le 1, \end{cases} \tag{5}$$

and

$$S(\xi, v) = \begin{cases} \xi^{\beta-\gamma-1}(1-v)^{\beta-\gamma-1} - (\xi - v)^{\beta-\gamma-1}, & 0 \le v \le \xi \le 1, \\ \xi^{\beta-\gamma-1}(1-v)^{\beta-\gamma-1}, & 0 \le \xi \le v \le 1. \end{cases} \tag{6}$$

2.2. Lemma [18]

The functions $R(x, v)$ and $S(x, v)$ satisfy the below:

(1) $R(x, v)$ and $S(x, v)$ are nonnegative, for every x, $v \in [0,1]$,

(2) $R(x, v) \le R(1, v)$, for every x, $v \in [0,1]$,

(3) $\frac{1}{4^{\beta-1}} R(1, v) \le R(x, v)$, for every $x \in I$ and $v \in [0,1]$, where $I = \left[\frac{1}{4}, \frac{3}{4}\right]$.

We can see that the solution for the Equations (1) and (2) is an m-tuple $(w_1(x), w_2(x), \ldots, w_m(x))$ if and only if $w_i(x)$ fulfills the subsequent equations

$$w_i(x) = \frac{\beta_i(\beta - \gamma - 1)!\, x^{\beta-1}}{\gamma_i} + \int_0^1 \left[R(x, v) + \frac{\eta_i x^{\beta-1}}{\gamma_i} \int_0^1 S(\xi, v) g_i(\xi)\, d\xi \right] a_i(v) f_i\big(w_{i+1}(v)\big) dv,$$

$$1 \le i \le m, x \in [0,1],$$

and

$$w_{m+1}(x) = w_1(x), x \in [0,1].$$

Therefore,

$$w_1(x) = \frac{\beta_1(\beta - \gamma - 1)!\, x^{\beta-1}}{\gamma_1} + \int_0^1 \left[R(x, v_1) + \frac{\eta_1 x^{\beta-1}}{\gamma_1} \int_0^1 S(\xi, v_1) g_1(\xi) d\xi \right]$$

$$a_1(v_1) f_1\left(\frac{\beta_2(\beta - \gamma - 1)!\, v_1^{\beta-1}}{\gamma_2} + \int_0^1 \left[R(v_1, v_2) + \frac{\eta_2 v_1^{\beta-1}}{\gamma_2} \int_0^1 S(\xi, v_2) g_2(\xi) d\xi \right] \right.$$

$$a_2(\mathcal{v}_2)\dots f_{m-1}\left(\frac{\beta_m(\beta-\gamma-1)!\ \mathcal{v}_{m-1}^{\beta-1}}{\gamma_m}\right.$$

$$+\int_0^1\left[\mathrm{R}(\mathcal{v}_{m-1},\mathcal{v}_m)\right.$$

$$\left.+\frac{\eta_m\mathcal{v}_{m-1}^{\beta-1}}{\gamma_m}\int_0^1 \varsigma(\xi,\mathcal{v}_m)g_m(\xi)d\xi\right]a_m(\mathcal{v}_m)f_m\big(w_1(\mathcal{v}_m)\big)\,d\mathcal{v}_m\Big)\dots d\mathcal{v}_2\Big)\,d\mathcal{v}_1.$$

To establish main outcomes, we shall use the Guo-Krasnosel'skii fixed point theorem, which is described below.

2.3. Theorem [30,31]

Let ρ be a cone in a Banach Space B. Let the two open subsets be Λ_1 and Λ_2 of Banach space B such that $0\in\Lambda_1$ and $\overline{\Lambda_1}\subset\Lambda_2$. If the function $\mathrm{T}:\rho\cap(\overline{\Lambda_2}\backslash\Lambda_1)\to\rho$ satisfy the below subsequent inequalities:

(i) $\|\mathrm{T}u\|\le\|u\|$, for $u\in\rho\cap\partial\Lambda_1$ and $\|\mathrm{T}u\|\ge\|u\|$, for $u\in\rho\cap\partial\Lambda_2$, or

(ii) $\|\mathrm{T}u\|\ge\|u\|$, for $u\in\rho\cap\ \partial\Lambda_1$ and $\|\mathrm{T}u\|\le\|u\|$, for $u\in\rho\cap\partial\Lambda_2$,

then there is a fixed point in $\rho\cap(\overline{\Lambda_2}\backslash\Lambda_1)$.

3. Positive solutions

This section contains the presence of positive solutions to the Equations (1) and (2). For construction, let the set $\mathrm{B}=\{w:w\in C([0,1],\mathbb{R})\}$ be a Banach space with

$$\|w\|=\max_{x\in[0,1]}|w(x)|.$$

Let ρ be a cone in a Banach space B and is defined as

$$\rho=\{w\in\mathrm{B}:w(x)\ge 0,\text{for every } x\in[0,1] \text{ with } \min_{x\in I}w(x)\ge\tfrac{1}{4^{\beta-1}}\|w\|\}.$$

For $w_1\in\rho$, define a function $\mathfrak{D}:\rho\to\mathrm{B}$ as

$$\left.\begin{aligned}\mathfrak{D}w_1(x)=\frac{\beta_1(\beta-\gamma-1)!\,x^{\beta-1}}{\gamma_1}+\int_0^1\left[\mathrm{R}(x,\mathcal{v}_1)+\frac{\eta_1x^{\beta-1}}{\gamma_1}\int_0^1\varsigma(\xi,\mathcal{v}_1)g_1(\xi)d\xi\right]\\ a_1(\mathcal{v}_1)f_1\left(\frac{\beta_2(\beta-\gamma-1)!\ \mathcal{v}_1^{\beta-1}}{\gamma_2}+\right.\\ \int_0^1\left[\mathrm{R}(\mathcal{v}_1,\mathcal{v}_2)+\frac{\eta_2\mathcal{v}_1^{\beta-1}}{\gamma_2}\int_0^1\varsigma(\xi,\mathcal{v}_2)g_2(\xi)d\xi\right]\\ a_2(\mathcal{v}_2)\dots f_{m-1}\left(\frac{\beta_m(\beta-\gamma-1)!\ \mathcal{v}_{m-1}^{\beta-1}}{\gamma_m}+\right.\\ \int_0^1\left[\mathrm{R}(\mathcal{v}_{m-1},\mathcal{v}_m)+\frac{\eta_m\mathcal{v}_{m-1}^{\beta\text{-}1}}{\gamma_m}\int_0^1\varsigma(\xi,\mathcal{v}_m)g_m(\xi)d\xi\right]\\ a_m(\mathcal{v}_m)f_m\big(w_1(\mathcal{v}_m)\big)\,d\mathcal{v}_m\Big)\dots d\mathcal{v}_2\Big)\,d\mathcal{v}_1.\end{aligned}\right\}\tag{7}$$

3.1. Lemma

The function $\mathfrak{D}:\rho\to\mathfrak{B}$ stated in Equation (7) is a self-map on ρ.

Proof. For $w_1 \in \rho$, $\mathfrak{D}w_1(æ) \geq 0$ on $æ \in [0,1]$ by the nonnegativity of $\mathrm{R}(æ, \upsilon)$ and $\varsigma(æ, \upsilon)$. Using Lemma 2.2, we see that for $w_1 \in \rho$,

$$\mathfrak{D}w_1(æ) = \frac{\beta_1(\beta-\gamma-1)!æ^{\beta-1}}{\gamma_1} + \int_0^1 \left[\mathrm{R}(æ, \upsilon_1) + \frac{\eta_1 æ^{\beta-1}}{\gamma_1}\int_0^1 \varsigma(\xi, \upsilon_1) g_1(\xi) d\xi\right]$$

$$a_1(\upsilon_1) f_1\left(\frac{\beta_2(\beta-\gamma-1)!\,\upsilon_1^{\beta-1}}{\gamma_2} + \int_0^1 \left[\mathrm{R}(\upsilon_1, \upsilon_2) + \frac{\eta_2 \upsilon_1^{\beta-1}}{\gamma_2}\int_0^1 \varsigma(\xi, \upsilon_2) g_2(\xi) d\xi\right]\right.$$

$$a_2(\upsilon_2) \ldots f_{m-1}\left(\frac{\beta_m(\beta-\gamma-1)!\,\upsilon_{m-1}^{\beta-1}}{\gamma_m}\right.$$

$$+ \int_0^1 \left[\mathrm{R}(\upsilon_{m-1}, \upsilon_m) + \frac{\eta_m \upsilon_{m-1}^{\beta-1}}{\gamma_m}\int_0^1 \varsigma(\xi, \upsilon_m) g_m(\xi) d\xi\right]$$

$$a_m(\upsilon_m) f_m\big(w_1(\upsilon_m)\big)\, d\upsilon_m \vdots\Big) \ldots d\upsilon_2 \vdots\Big)\, d\upsilon_1$$

$$\leq \frac{\beta_1(\beta-\gamma-1)!}{\gamma_1} + \int_0^1 \left[\mathrm{R}(1, \upsilon_1) + \frac{\eta_1}{\gamma_1}\int_0^1 \varsigma(\xi, \upsilon_1) g_1(\xi) d\xi\right]$$

$$a_1(\upsilon_1) f_1\left(\frac{\beta_2(\beta-\gamma-1)!\ \upsilon_1^{\beta-1}}{\gamma_2} + \int_0^1 \left[\mathrm{R}(\upsilon_1, \upsilon_2) + \frac{\eta_2 \upsilon_1^{\beta-1}}{\gamma_2}\int_0^1 \varsigma(\xi, \upsilon_2) g_2(\xi) d\xi\right]\right.$$

$$a_2(\upsilon_2) \ldots f_{m-1}\left(\frac{\beta_m(\beta-\gamma-1)!\,\upsilon_{m-1}^{\beta-1}}{\gamma_m}\right.$$

$$+ \int_0^1 \left[\mathrm{R}(\upsilon_{m-1}, \upsilon_m) + \frac{\eta_m \upsilon_{m-1}^{\beta-1}}{\gamma_m}\int_0^1 \varsigma(\xi, \upsilon_m) g_m(\xi) d\xi\right]$$

$$a_m(\upsilon_m) f_m\big(\mathrm{w}_1(\upsilon_m)\big)\, d\upsilon_m \vdots\Big) \ldots d\upsilon_2 \vdots\Big)\, d\upsilon_1.$$

Then

$$\left.\begin{array}{c} \|\mathfrak{D}w_1(x)\| \leq \dfrac{\beta_1(\beta-\gamma-1)!}{\gamma_1} + \displaystyle\int_0^1 \left[\mathrm{R}(1, \upsilon_1) + \frac{\eta_1}{\gamma_1}\int_0^1 \varsigma(\xi, \upsilon_1) g_1(\xi) d\xi\right] \\ a_1(\upsilon_1) f_1\left(\dfrac{\beta_2(\beta-\gamma-1)!\ \upsilon_1^{\beta-1}}{\gamma_2} + \right. \\ \displaystyle\int_0^1 \left[\mathrm{R}(\upsilon_1, \upsilon_2) + \frac{\eta_2 \upsilon_1^{\beta-1}}{\gamma_2}\int_0^1 \varsigma(\xi, \upsilon_2) g_2(\xi) d\xi\right] \\ a_2(\upsilon_2) \ldots f_{m-1}\left(\dfrac{\beta_m(\beta-\gamma-1)!\ \upsilon_{m-1}^{\beta-1}}{\gamma_m} + \right. \\ \displaystyle\int_0^1 \left[\mathrm{R}(\upsilon_{m-1}, \upsilon_m) + \frac{\eta_m \upsilon_{m-1}^{\beta-1}}{\gamma_m}\int_0^1 \varsigma(\xi, \upsilon_m) g_m(\xi) d\xi\right] \\ a_m(\upsilon_m) f_m\big(w_1(\upsilon_m)\big)\, d\upsilon_m \vdots\Big) \ldots d\upsilon_2) d\upsilon_1. \end{array}\right\} \quad (8)$$

Next, if $w_1 \in \rho$, we have from Lemma 2.2 and Equation (8) that

$$\min_{æ \in \mathrm{I}} \mathfrak{D}w_1(æ) = \min_{æ \in \mathrm{I}} \left\{\frac{\beta_1(\beta-\gamma-1)!æ^{\beta-1}}{\gamma_1} + \int_0^1 \left[\mathrm{R}(æ, \upsilon_1) + \frac{\eta_1 æ^{\beta-1}}{\gamma_1}\int_0^1 \varsigma(\xi, \upsilon_1) g_1(\xi) d\xi\right]\right.$$

$$a_1(\upsilon_1)f_1\left(\frac{\beta_2(\beta-\gamma-1)!\,\upsilon_1^{\beta-1}}{\gamma_2}+\int_0^1\left[R(\upsilon_1,\upsilon_2)+\frac{\eta_2\upsilon_1^{\beta-1}}{\gamma_2}\int_0^1 \varsigma(\xi,\upsilon_2)g_2(\xi)d\xi\right]\right.$$

$$a_2(\upsilon_2)\dots f_{m-1}\left(\frac{\beta_m(\beta-\gamma-1)!\ \upsilon_{m-1}^{\beta-1}}{\gamma_m}\right.$$

$$+\int_0^1\left[R(\upsilon_{m-1},\upsilon_m)+\frac{\eta_m\upsilon_{m-1}^{\beta-1}}{\gamma_m}\int_0^1 \varsigma(\xi,\upsilon_m)g_m(\xi)d\xi\right]$$

$$\left.\left.a_m(\upsilon_m)f_m\big(w_1(\upsilon_m)\big)\,d\upsilon_m \vdots\right)\dots d\upsilon_2 \vdots\right)d\upsilon_1\Big\}$$

$$\geq\frac{1}{4^{\beta-1}}\left\{\frac{\beta_1(\beta-\gamma-1)!}{\gamma_1}+\int_0^1\left[R(1,\upsilon_1)+\frac{\eta_1}{\gamma_1}\int_0^1 \varsigma(\xi,\upsilon_1)g_1(\xi)d\xi\right]\right.$$

$$a_1(\upsilon_1)f_1\left(\frac{\beta_2(\beta-\gamma-1)!\ \upsilon_1^{\beta-1}}{\gamma_2}+\int_0^1\left[R(\upsilon_1,\upsilon_2)+\frac{\eta_2\upsilon_1^{\beta-1}}{\gamma_2}\int_0^1 \varsigma(\xi,\upsilon_2)g_2(\xi)d\xi\right]\right.$$

$$a_2(\upsilon_2)\dots f_{m-1}\left(\frac{\beta_m(\beta-\gamma-1)!\ \upsilon_{m-1}^{\beta-1}}{\gamma_m}\right.$$

$$+\int_0^1\left[R(\upsilon_{m-1},\upsilon_m)+\frac{\eta_m\upsilon_{m-1}^{\beta-1}}{\gamma_m}\int_0^1 \varsigma(\xi,\upsilon_m)g_m(\xi)d\xi\right]$$

$$\left.a_m(\upsilon_m)f_m\big(w_1(\upsilon_m)\big)\,d\upsilon_m \vdots\right)\dots d\upsilon_2)d\upsilon_1\Big\}$$

$$\geq\frac{1}{4^{\beta-1}}\|\mathfrak{D}w_1(x)\|.$$

Hence, $\mathfrak{D}:\rho\to\rho$. This completes the proof.

In addition, we can see that the function $\mathfrak{D}$ is completely continuous by applying the Arzela-Ascoli theorem [32].

3.2. Theorem

Suppose that (E1) and (E2) are met. If $f_{i0}=0$ and $f_{i\infty}=\infty$ are true, then the Equations (1) and (2) has at least one positive solution and $\beta_i\in(0,\infty)$ small enough for $1\leq i\leq m$.

Proof. For $1\leq i\leq m$, by using the definition of $f_{i0}=0$, there exist $\varrho_i>0$ and $H_1>0$ such that

$$f_i(w)\leq\varrho_i w,\ \text{ for } 0<w\leq H_1,$$

where ϱ_i satisfies

$$\varrho_i\int_0^1 2\left[R(1,\upsilon_i)+\frac{\eta_i}{\gamma_i}\int_0^1 \varsigma(\xi,\upsilon_i)g_i(\xi)\,d\xi\right]a_i(\upsilon_i)d\upsilon_i\leq 1. \tag{9}$$

For $1\leq i\leq m$, let β_i be chosen

$$0<\beta_i\leq\frac{\gamma_i H_1}{(\beta-\gamma-1)!\,2}.$$

Let $w_1\in\rho$ with $\|w_1\|=H_1$. Then using Lemma 2.2 and for $0\leq\upsilon_{m-1}\leq 1$, we have

$$\frac{\beta_m(\beta-\gamma-1)!\,v_{m-1}^{\beta-1}}{\Upsilon_m}$$

$$+\int_0^1\left[\mathrm{R}(v_{m-1},v_m)+\frac{\eta_m v_{m-1}^{\beta-1}}{\Upsilon_m}\int_0^1 \mathrm{S}(\xi,v_m)g_m(\xi)\,d\xi\right]a_m(v_m)f_m\big(w_1(v_m)\big)dv_m$$

$$\leq \frac{\beta_m(\beta-\gamma-1)!}{\Upsilon_m}$$

$$+\int_0^1\left[\mathrm{R}(1,v_m)+\frac{\eta_m}{\Upsilon_m}\int_0^1 \mathrm{S}(\xi,v_m)g_m(\xi)\,d\xi\right]a_m(v_m)\varrho_m w_1(v_m)dv_m$$

$$\leq \frac{H_1}{2}+\varrho_m\int_0^1\left[\mathrm{R}(1,v_m)+\frac{\eta_m}{\Upsilon_m}\int_0^1 \mathrm{S}(\xi,v_m)g_m(\xi)\,d\xi\right]a_m(v_m)dv_m\,\|w_1\|$$

$$\leq \tfrac{H_1}{2}+\tfrac{H_1}{2}=H_1.$$

It comes in the same way as Lemma 2.2 and for $0\leq \vartheta_{m-2}\leq 1$,

$$\frac{\beta_{m-1}(\beta-\gamma-1)!\,v_{m-2}^{\beta-1}}{\Upsilon_{m-1}}+\int_0^1\left[\mathrm{R}(v_{m-2},v_{m-1})+\frac{\eta_{m-1}v_{m-2}^{\beta-1}}{\Upsilon_{m-1}}\int_0^1 \mathrm{S}(\xi,v_{m-1})g_{m-1}(\xi)d\xi\right]$$

$$a_{m-1}(v_{m-1})f_{m-1}\left(\tfrac{\beta_m(\beta-\gamma-1)!\,v_{m-1}^{\beta-1}}{\Upsilon_m}+\int_0^1\left[\mathrm{R}(v_{m-1},v_m)+\tfrac{\eta_m v_{m-1}^{\beta-1}}{\Upsilon_m}\int_0^1 \mathrm{S}(\xi,v_m)g_m(\xi)d\xi\right]\right.$$

$$\left.a_m(v_m)f_m\big(w_1(v_m)\big)\,dv_m\right)dv_{\mathrm{m-1}}$$

$$\leq \frac{\beta_{m-1}(\beta-\gamma-1)!}{\Upsilon_{m-1}}+\int_0^1\left[\mathrm{R}(1,v_{m-1})+\frac{\eta_{m-1}}{\Upsilon_{m-1}}\int_0^1 \mathrm{S}(\xi,v_{m-1})g_{m-1}(\xi)d\xi\right]a_{m-1}(v_{m-1})dv_{m-1}\varrho_{m-1}H_1$$

$$\leq \tfrac{H_1}{2}+\tfrac{H_1}{2}=H_1.$$

Applying the similar argument, one can obtain, for $0\leq x\leq 1$,

$$\frac{\beta_1(\beta-\gamma-1)!\,x^{\beta-1}}{\Upsilon_1}+\int_0^1\left[\mathrm{R}(x,v_1)+\frac{\eta_1 x^{\beta-1}}{\Upsilon_1}\int_0^1 \mathrm{S}(\xi,v_1)g_1(\xi)d\xi\right]$$

$$a_1(v_1)f_1\left(\frac{\beta_2(\beta-\gamma-1)!\,v_1^{\beta-1}}{\Upsilon_2}+\int_0^1\left[\mathrm{R}(v_1,v_2)+\frac{\eta_2 v_1^{\beta-1}}{\Upsilon_2}\int_0^1 \mathrm{S}(\xi,v_2)g_2(\xi)d\xi\right]\right.$$

$$a_2(v_2)\ldots f_{m-1}\left(\frac{\beta_m(\beta-\gamma-1)!\,v_{m-1}^{\beta-1}}{\Upsilon_m}+\int_0^1\left[\mathrm{R}(v_{m-1},v_m)+\frac{\eta_m v_{m-1}^{\beta-1}}{\Upsilon_m}\int_0^1 \mathrm{S}(\xi,v_m)g_m(\xi)d\xi\right]\right.$$

$$\left.\left.a_m(v_m)f_m\big(w_1(v_m)\big)\,dv_m\right)\ldots dv_2\right)dv_1\leq H_1,$$

so that for $0\leq x\leq 1$,

$$\mathfrak{D}w_1(x)\leq H_1.$$

Hence, $\|\mathfrak{D}w_1\|\leq H_1=\|w_1\|$. Take

$$\Lambda_1=\{w\in \mathrm{B}:\|w\|<H_1\}.$$

Then

$$\|\mathfrak{D}w_1\| \le \|w_1\|, \qquad \text{for } w_1 \in \rho \cap \partial\Lambda_1. \tag{10}$$

Since $f_{i\infty} = \infty$, $1 \le i \le m$, there exist $\sigma_i > 0$ and $\overline{H_2} \ge 0$, such that

$$f_i(w) \ge \sigma_i w, \text{ for } w \ge \overline{H_2},$$

where σ_i satisfies

$$\frac{\sigma_i}{4^{2\beta-2}} \int_{\upsilon_i \in I} \left[R(1, \upsilon_i) + \frac{\eta_i}{\gamma_i} \int_{\xi \in I} \varsigma(\xi, \upsilon_i) g_i(\xi) d\xi \right] a_i(\upsilon_i) d\upsilon_i \ge 1. \tag{11}$$

Let

$$H_2 = \max\{2H_1, 4^{\beta-1}\overline{H_2}\}.$$

Choose $w_1 \in \rho$ and $\|w_1\| = H_2$. Then

$$\min_{\text{æ} \in I} w_1(\text{æ}) \ge \frac{1}{4^{\beta-1}} \|w_1\| \ge \overline{H_2}.$$

By Lemma 2.2 and for $\frac{1}{4} \le \upsilon_{m-1} \le \frac{3}{4}$, we have

$$\frac{\beta_m(\beta - \gamma - 1)!\, \upsilon_{m-1}^{\beta-1}}{\gamma_m}$$

$$+ \int_0^1 \left[R(\upsilon_{m-1}, \upsilon_m) + \frac{\eta_m \upsilon_{m-1}^{\beta-1}}{\gamma_m} \int_0^1 \varsigma(\xi, \upsilon_m) g_m(\xi)\, d\xi \right] a_m(\upsilon_m) f_m\big(w_1(\upsilon_m)\big) d\upsilon_m$$

$$\ge \int_0^1 \left[R(\upsilon_{m-1}, \upsilon_m) + \frac{\eta_m \upsilon_{m-1}^{\beta-1}}{\gamma_m} \int_0^1 \varsigma(\xi, \upsilon_m) g_m(\xi)\, d\xi \right] a_m(\upsilon_m) f_m\big(w_1(\upsilon_m)\big) d\upsilon_m$$

$$\ge \frac{1}{4^{\beta-1}} \int_{\upsilon_m \in I} \left[R(1, \upsilon_m) + \frac{\eta_m}{\gamma_m} \int_{\xi \in I} \varsigma(\xi, \upsilon_m) g_m(\xi)\, d\xi \right] a_m(\upsilon_m) \sigma_m w_1(\upsilon_m) d\upsilon_m$$

$$\ge \frac{1}{4^{2\beta-2}} \int_{\upsilon_m \in I} \left[R(1, \upsilon_m) + \frac{\eta_m}{\gamma_m} \int_{\xi \in I} \varsigma(\xi, \upsilon_m) g_m(\xi)\, d\xi \right] a_m(\upsilon_m) d\upsilon_m \sigma_m ||w_1||$$

$$\ge \|w_1\| = H_2.$$

It comes in the same way as Lemma 2.2 and for $\frac{1}{4} \le \upsilon_{m-1} \le \frac{3}{4}$,

$$\frac{\beta_{m-1}(\beta-\gamma-1)!\upsilon_{m-2}^{\beta-1}}{\gamma_{m-1}} + \int_0^1 \left[R(\upsilon_{m-2}, \upsilon_{m-1}) + \frac{\eta_{m-1}\upsilon_{m-2}^{\beta-1}}{\gamma_{m-1}} \int_0^1 \varsigma(\xi, \upsilon_{m-1}) g_{m-1}(\xi) d\xi \right]$$

$$a_{m-1}(\upsilon_{m-1}) f_{m-1}\Bigg(\frac{\beta_m(\beta-\gamma-1)!\, \upsilon_{m-1}^{\beta-1}}{\gamma_m} + \int_0^1 \Big[R(\upsilon_{m-1}, \upsilon_m) +$$

$$\frac{\eta_m \upsilon_{m-1}^{\beta-1}}{\gamma_m} \int_0^1 \varsigma(\xi, \upsilon_m) g_m(\xi) d\xi \Big] a_m(\upsilon_m) f_m\big(w_1(\upsilon_m)\big)\, d\upsilon_m \vdots \Bigg) d\upsilon_{m-1}$$

$$\ge \frac{1}{4^{\beta-1}} \int_{\upsilon_{m-1} \in I} \Bigg[R(1, \upsilon_{m-1})$$

$$+ \frac{\eta_{m-1}}{\gamma_{m-1}} \int_{\xi \in I} \varsigma(\xi, \upsilon_{m-1}) g_{m-1}(\xi)\, d\xi \Bigg] a_{m-1}(\upsilon_{m-1}) d\upsilon_{m-1} \sigma_{m-1} H_2$$

$$\geq \frac{1}{4^{2\beta-2}} \int_{\upsilon_{m-1} \in I} \left[\mathrm{R}(1, \upsilon_{m-1}) + \frac{\eta_{m-1}}{\gamma_{m-1}} \int_{\xi \in \mathrm{I}} \varsigma(\xi, \upsilon_{m-1}) g_{m-1}(\xi)\, d\xi \right] a_{m-1}(\upsilon_{m-1}) d\upsilon_{m-1} \sigma_{m-1} H_2$$

$$\geq H_2.$$

Proceeding in a similar argument, we have

$$\frac{\beta_1(\beta-\gamma-1)!\,x^{\beta-1}}{\gamma_1} + \int_0^1 \left[\mathrm{R}(x, \upsilon_1) + \frac{\eta_1 x^{\beta-1}}{\gamma_1} \int_0^1 \varsigma(\xi, \upsilon_1) g_1(\xi) d\xi \right]$$

$$a_1(v_1) f_1 \left(\frac{\beta_2(\beta-\gamma-1)!\, \upsilon_1^{\beta-1}}{\gamma_2} + \int_0^1 \left[\mathrm{R}(\upsilon_1, \upsilon_2) + \frac{\eta_2 \upsilon_1^{\beta-1}}{\gamma_2} \int_0^1 \varsigma(\xi, \upsilon_2) g_2(\xi) d\xi \right] \right.$$

$$a_2(\upsilon_2) \dots f_{m-1} \left(\frac{\beta_m(\beta-\gamma-1)!\ \upsilon_{m-1}^{\beta-1}}{\gamma_m} + \int_0^1 \left[\mathrm{R}(\upsilon_{m-1}, \upsilon_m) + \frac{\eta_m \upsilon_{m-1}^{\beta-1}}{\gamma_m} \int_0^1 \varsigma(\xi, \upsilon_m) g_m(\xi) d\xi \right] \right.$$

$$\left. \left. a_m(\upsilon_m) f_m\big(w_1(\upsilon_m)\big)\, d\upsilon_m \right) \dots d\upsilon_2 \right) d\upsilon_1 \geq H_2,$$

so that, for $0 \leq x \leq 1$,

$$\mathfrak{D} w_1(x) \geq H_2 = \|w_1\|.$$

Hence, $\|\mathfrak{D} w_1\| \geq \|w_1\|$.

Take $\Lambda_2 = \{w \in \mathrm{B} : \|w_1\| < \mathrm{H}_2\}$, then

$$\|\mathfrak{D} w_1\| \geq \|w_1\|, \qquad \text{for } w_1 \in \rho \cap \partial\Lambda_2. \tag{12}$$

Using Theorem 2.3 to the Equations (10) and (12), the function $\mathfrak{D}$ has a fixed point $w_1 \in \rho \cap (\overline{\Lambda_2} \backslash \Lambda_1)$ and that point is the positive solution $(w_1, w_2, \dots, w_m)$ of the Equations (1) and (2) by taking $w_{m+1} = w_1$. Therefore, the solution can be iteratively expressed as

$$w_i(x) = \frac{\beta_i(\beta-\gamma-1)!\,x^{\beta-1}}{\gamma_i} + \int_0^1 \left[\mathrm{R}(x, \upsilon) + \frac{\eta_i x^{\beta-1}}{\gamma_i} \int_0^1 \varsigma(\xi, \upsilon) g_i(\xi)\, d\xi \right] a_i(\upsilon) f_i\big(w_{i+1}(\upsilon)\big) d\upsilon,$$

$$i = m, m-1, \dots, 1.$$

3.3. Theorem

Suppose that (E1) and (E2) are met. If $f_{i0} = \infty$ and $f_{i\infty} = 0$ are true, then the Equations (1) and (2) has at least one positive solution and $\beta_i \in (0, \infty)$ small enough for $1 \leq i \leq m$.

Proof. For $1 \leq i \leq m$, by the definition of f_{i0}, there exist $\varepsilon_i > 0$ and $H_3 > 0$ such that

$$f_i(w) \geq \varepsilon_i w, \qquad \text{for } 0 < w \leq H_3,$$

where $\varepsilon_i \geq \sigma_i$ and σ_i is given in Equation (11).

Choose $w_1 \in \rho$ and $\|w_1\| = H_3$. Then, we have from Lemma 2.2 and for $\frac{1}{4} \leq \upsilon_{m-1} \leq \frac{3}{4}$, we have

$$\frac{\beta_m(\beta-\gamma-1)!\ \upsilon_{m-1}^{\beta-1}}{\Upsilon_m} + \int_0^1 \left[\mathrm{R}(\upsilon_{m-1},\upsilon_m) + \frac{\eta_m \upsilon_{m-1}^{\beta-1}}{\Upsilon_m}\int_0^1 \varsigma(\xi,\upsilon_m) g_m(\xi)\, d\xi\right] a_m(\upsilon_m) f_m\big(w_1(\upsilon_m)\big) d\upsilon_m$$

$$\geq \int_0^1 \left[\mathrm{R}(\upsilon_{m-1},\upsilon_m) + \frac{\eta_m \upsilon_{m-1}^{\beta-1}}{\Upsilon_m}\int_0^1 \varsigma(\xi,\upsilon_m) g_m(\xi)\, d\xi\right] a_m(\upsilon_m) f_m\big(w_1(\upsilon_m)\big) d\upsilon_m$$

$$\geq \frac{1}{4^{\beta-1}} \int_{\upsilon_m \in I} \left[\mathrm{R}(1,\upsilon_m) + \frac{\eta_m}{\Upsilon_m}\int_{\xi\,\in\,\mathrm{I}} \varsigma(\xi,\upsilon_m) g_m(\xi)\, d\xi\right] a_m(\upsilon_m)\varepsilon_m w_1(\upsilon_m) d\upsilon_m$$

$$\geq \frac{1}{4^{2\beta-2}} \int_{\upsilon_m \in I} \left[\mathrm{R}(1,\upsilon_m) + \frac{\eta_m}{\Upsilon_m}\int_{\xi\,\in\,\mathrm{I}} \varsigma(\xi,\upsilon_m) g_m(\xi) d\xi\right] a_m(\upsilon_m) d\upsilon_m \varepsilon_m \|w_1\|$$

$$\geq \|w_1\| = H_3.$$

It comes in the same way as Lemma 2.2 and for $\frac{1}{4} \leq \upsilon_{m-1} \leq \frac{3}{4}$,

$$\frac{\beta_{m-1}(\beta-\gamma-1)!\upsilon_{m-2}^{\beta-1}}{\Upsilon_{m-1}} + \int_0^1 \left[\mathrm{R}(\upsilon_{m-2},\upsilon_{m-1}) + \frac{\eta_{m-1}\upsilon_{m-2}^{\beta-1}}{\Upsilon_{m-1}}\int_0^1 \varsigma(\xi,\upsilon_{m-1}) g_{m-1}(\xi) d\xi\right]$$
$$a_{m-1}(\upsilon_{m-1}) f_{m-1}\left(\frac{\beta_m(\beta-\gamma-1)!\ \upsilon_{m-1}^{\beta-1}}{\Upsilon_m} + \int_0^1 \left[\mathrm{R}(\upsilon_{m-1},\upsilon_m) + \frac{\eta_m \upsilon_{m-1}^{\beta-1}}{\Upsilon_m}\int_0^1 \varsigma(\xi,\upsilon_m) g_m(\xi) d\xi\right] a_m(\upsilon_m) f_m\big(w_1(\upsilon_m)\big)\, d\upsilon_m \vdots\right) d\upsilon_{m-1}$$

$$\geq \frac{1}{4^{\beta-1}} \int_{\upsilon_{m-1} \in I} \left[\mathrm{R}(1,\upsilon_{m-1}) + \frac{\eta_{m-1}}{\Upsilon_{m-1}}\int_{\xi\,\in\,\mathrm{I}} \varsigma(\xi,\upsilon_{m-1}) g_{m-1}(\xi)\, d\xi\right] a_{m-1}(\upsilon_{m-1}) d\upsilon_{m-1}\varepsilon_{m-1} H_3$$

$$\geq \frac{1}{4^{2\beta-2}} \int_{\upsilon_{m-1} \in I} \left[\mathrm{R}(1,\upsilon_{m-1}) + \frac{\eta_{m-1}}{\Upsilon_{m-1}}\int_{\xi\,\in\,\mathrm{I}} \varsigma(\xi,\upsilon_{m-1}) g_{m-1}(\xi)\, d\xi\right] a_{m-1}(\upsilon_{m-1}) d\upsilon_{m-1}\varepsilon_{m-1} H_3$$

$$\geq H_3.$$

Using a similar argument, we have

$$\frac{\beta_1(\beta-\gamma-1)!\, æ^{\beta-1}}{\Upsilon_1} + \int_0^1 \left[\mathrm{R}(æ,\upsilon_1) + \frac{\eta_1 æ^{\beta-1}}{\Upsilon_1}\int_0^1 \varsigma(\xi,\upsilon_1) g_1(\xi) d\xi\right]$$

$$a_1(\upsilon_1) f_1\left(\frac{\beta_2(\beta-\gamma-1)!\ \upsilon_1^{\beta-1}}{\Upsilon_2} + \int_0^1 \left[\mathrm{R}(\upsilon_1,\upsilon_2) + \frac{\eta_2 \upsilon_1^{\beta-1}}{\Upsilon_2}\int_0^1 \varsigma(\xi,\upsilon_2) g_2(\xi) d\xi\right]\right.$$

$$a_2(\mathcal{v}_2)\ldots f_{m-1}\left(\frac{\beta_m(\beta-\gamma-1)!\ \mathcal{v}_{m-1}^{\beta-1}}{\Upsilon_m}\right.$$
$$+\int_0^1\left[R(\mathcal{v}_{m-1},\mathcal{v}_m)+\frac{\eta_m\mathcal{v}_{m-1}^{\beta-1}}{\Upsilon_m}\int_0^1 \mathfrak{S}(\xi,\mathcal{v}_m)g_m(\xi)d\xi\right]$$
$$a_m(\mathcal{v}_m)f_m\big(w_1(\mathcal{v}_m)\big)\,d\mathcal{v}_m\vdots\Big)\ldots d\mathcal{v}_2\vdots\Big)\,d\mathcal{v}_1\geq H_3\,,$$

so that

$$\mathfrak{D}w_1\geq H_3=\|w_1\|.$$

Hence, $\|\mathfrak{D}w_1\|\geq\|w_1\|$. If we put

$$\Lambda_3=\{w\in\mathcal{B}:\|w_1\|<H_3\},$$

then

$$\|\mathfrak{D}w_1\|\geq\|w_1\|,\qquad \text{for } w_1\in\ \rho\cap\partial\Lambda_3. \tag{13}$$

Since $f_{i\infty}=0$, for $1\leq i\leq m$, there exist $\zeta_i>0$ and $\overline{H}_4>0$ such that

$$f_i\,(w)\leq\ \zeta_i w,\ \text{for } w\geq\overline{H}_4,$$

where $\zeta_i\ \leq\varrho_i$ and ϱ_i is given in Equation (9).

For $1\leq i\leq m$, set

$$f_i^*(w)=\sup_{0\leq s\leq w} f_i(s).$$

It is evident from the fact that for $1\leq i\leq m$, the real-valued function $f_i^*(w)$ is non-decreasing, $f_i\leq f_i^*$ and

$$\lim_{w\to\infty}\frac{f_i^*(w)}{w}=0.$$

As a result, for $1\leq i\leq m$, there exists $H_4>\max\{2H_3,\overline{H}_4\}$ such that

$$f_i^*(w)\leq f_i^*(H_4),\ \ 0<w\leq H_4.$$

Let β_i, $1\leq i\leq m$, fulfills

$$0<\beta_i\leq\frac{\Upsilon_i H_4}{(\beta-\gamma-1)!2}\,.$$

Choose $w_1\in\rho$ and $\|w_1\|=H_4$. Apply same argument repeatedly, we have

$$\mathfrak{D}w_1(\mathfrak{x})=\frac{\beta_1(\beta-\gamma-1)!\,\mathfrak{x}^{\beta-1}}{\Upsilon_1}+\int_0^1\left[R(\mathfrak{x},\mathcal{v}_1)+\frac{\eta_1\mathfrak{x}^{\beta-1}}{\Upsilon_1}\int_0^1\mathfrak{S}(\xi,\mathcal{v}_1)g_1(\xi)d\xi\right]$$
$$a_1(\mathcal{v}_1)f_1\left(\frac{\beta_2(\beta-\gamma-1)!\ \mathcal{v}_1^{\beta-1}}{\Upsilon_2}+\int_0^1\left[R(\mathcal{v}_1,\mathcal{v}_2)+\frac{\eta_2\mathcal{v}_1^{\beta-1}}{\Upsilon_2}\int_0^1\mathfrak{S}(\xi,\mathcal{v}_2)g_2(\xi)d\xi\right]\right.$$
$$a_2(\mathcal{v}_2)\ldots f_m\big(w_1(\mathcal{v}_m)\big)\,d\mathcal{v}_m\vdots\Big)\ldots d\mathcal{v}_2\vdots\Big)\,d\mathcal{v}_1$$
$$\leq\frac{\beta_1(\beta-\gamma-1)!}{\Upsilon_1}+\int_0^1\left[R(1,\mathcal{v}_1)+\frac{\eta_1}{\Upsilon_1}\int_0^1\mathfrak{S}(\xi,\mathcal{v}_1)g_1(\xi)d\xi\right]$$

$$a_1(\mathcal{v}_1)f_1^*\left(\frac{\beta_2(\text{ß}-\gamma-1)!\ \mathcal{v}_1^{\beta-1}}{\text{Ɣ}_2}+\int_0^1\left[\mathrm{R}(\mathcal{v}_1,\mathcal{v}_2)+\frac{\eta_2\mathcal{v}_1^{\beta-1}}{\text{Ɣ}_2}\int_0^1 \text{Ş}(\xi,\mathcal{v}_2)g_2(\xi)d\xi\right]\right.$$

$$\left.a_2(\mathcal{v}_2)\ldots f_m\big(w_1(\mathcal{v}_m)\big)\,d\mathcal{v}_m \vdots\right)\ldots d\mathcal{v}_2 \vdots\Big)\,d\mathcal{v}_1$$

$$\leq \frac{H_4}{2}+\int_0^1\left[\mathrm{R}(1,\mathcal{v}_1)+\frac{\eta_1}{\text{Ɣ}_1}\int_0^1 \text{Ş}(\xi,\mathcal{v}_1)g_1(\xi)\,d\xi\right]a_1(\mathcal{v}_1)f_1^*(H_4)d\mathcal{v}$$

$$\leq \frac{H_4}{2}+\int_0^1\left[\mathrm{R}(1,\mathcal{v}_1)+\frac{\eta_1}{\text{Ɣ}_1}\int_0^1 \text{Ş}(\xi,\mathcal{v}_1)g_1(\xi)\,d\xi\right]a_1(\mathcal{v}_1)d\mathcal{v}_1\zeta_1 H_4$$

$$\leq \frac{H_4}{2}+\frac{H_4}{2}=H_4.$$

Hence, $\|\mathfrak{D}w_1\|\leq\|w_1\|$. Thus, we choose

$$\Lambda_4=\{w\in\text{Ɓ}:\|w_1\|<H_4\},$$

then

$$\|\mathfrak{D}w_1\|\leq\|w_1\|,\qquad \text{for } w_1\in\rho\cap\partial\Lambda_4. \tag{14}$$

Using Theorem 2.3 to the Equations (13) and (14), it can see that $\mathfrak{D}$ has a fixed point $w_1\in\rho\cap(\overline{\Lambda_4}\backslash\Lambda_3)$, that gives an m-tuple $(w_1,w_2,\ldots,w_m)$ fulfilling the Equations (1) and (2) with $w_{m+1}=w_1$.

4. Examples

Let us present the examples to support our conclusions.

4.1. Example

Consider the third order problem with $r=1$,

$$\left.\begin{aligned} w_1'''(\text{æ})+a_1(\text{æ})f_1(w_2)=0,\quad \text{æ}\in[0,1],\\ w_2'''(\text{æ})+a_2(\text{æ})f_2(w_3)=0,\quad \text{æ}\in[0,1],\\ w_3'''(\text{æ})+a_3(\text{æ})f_3(w_1)=0,\quad \text{æ}\in[0,1],\end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} w_1(0)=0,\ w_1'(0)=0,\ w_1'(1)-1\int_0^1 w_1'(\xi)d\xi=\beta_1,\\ w_2(0)=0,\ w_2'(0)=0,\ w_2'(1)-\frac{1}{2}\int_0^1 \xi w_2'(\xi)d\xi=\beta_2,\\ w_3(0)=0,\ w_3'(0)=0,\ w_3'(1)-\frac{1}{3}\int_0^1 \xi^2 w_3'(\xi)d\xi=\beta_3,\end{aligned}\right\} \tag{16}$$

where

$$a_1(\text{æ})=\frac{1}{4},a_2(\text{æ})=\frac{1}{2},a_3(\text{æ})=\frac{3}{4},g_1(\xi)=1,\qquad g_2(\xi)=\xi,\qquad g_3(\xi)=\xi^2,$$

$$f_1(w_2)=w_2^2(1+e^{-3\,w_2}),\ f_2(w_3)=w_3^2(1-3e^{-2\,w_3}),\ f_3(w_1)=w_1^2(1+4e^{-w_1}).$$

Then $f_{i0}=0$ and $f_{i\infty}=\infty$ for $i=1,2,3$. So, all the assumptions of Theorem 3.2 are met, and hence, the problem (15)-(16) has at least one positive solution by selecting β_1,β_2 and β_3 that are sufficiently small.

4.2. Example

Consider the third order problem with $r = 1$,

$$\left.\begin{aligned} w_1'''(\text{æ}) + a_1(\text{æ})f_1(w_2) = 0, \text{æ} \in [0,1], \\ w_2'''(\text{æ}) + a_2(\text{æ})f_2(w_3) = 0, \text{æ} \in [0,1], \\ w_3'''(\text{æ}) + a_3(\text{æ})f_3(w_1) = 0, \text{æ} \in [0,1], \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} w_1(0) = 0,\ w_1'(0) = 0,\ w_1'(1) - 1\int_0^1 w_1'(\xi)d\xi = \beta_1, \\ w_2(0) = 0,\ w_2'(0) = 0,\ w_2'(1) - 2\int_0^1 \xi w_2'(\xi)d\xi = \beta_2, \\ w_3(0) = 0,\ w_3'(0) = 0,\ w_3'(1) - 3\int_0^1 \xi^2 w_3'(\xi)d\xi = \beta_3, \end{aligned}\right\} \tag{18}$$

where

$$a_1(\text{æ}) = \text{æ}, a_2(\text{æ}) = \text{æ}^2, a_3(\text{æ}) = \text{æ}^3,\ g_1(\xi) = 1,\ g_2(\xi) = \xi,\ g_3(\xi) = \xi^2,$$

$$f_1(w_2) = w_2^{\frac{2}{3}}, f_2(w_3) = w_3^{\frac{3}{4}},\ f_3(w_1) = w_1^{\frac{1}{2}}.$$

Then $f_{i0} = \infty$ and $f_{i\infty} = 0$ for $i = 1, 2, 3$. So, all the assumptions of Theorem 3.3 are met, and hence, the problem (17) and (18) has at least one positive solution by selecting β_1, β_2 and β_3 that are small enough.

5. Conclusion

This study employs the Guo-Krasnosel'skii fixed point theorem as a central tool to investigate the existence of positive solutions for an iterative system of higher order boundary value problems associated with non-homogeneous integral boundary conditions. This theorem, foundational in nonlinear analysis, provides conditions under which a compact operator has at least one fixed point within a cone in a Banach space. By applying this theorem, the study aims to rigorously establish criteria that guarantee the existence of positive solutions for the boundary value problems under consideration. It may be interesting that the researchers extend the results to multipoint boundary value problems and obtain multiple positive solutions by using various new fixed-point theorems.

Author contributions: Conceptualization, RPK and SN; methodology, RPK, SN and KN; validation, RPK, SN and KN; investigation, RPK, SN and KN; resources, RPK, SN and KN; writing—original draft preparation, RPK, SN and KN; writing—review and editing, RPK, SN and KN; supervision, SN. All authors have read and agreed to the published version of the manuscript.

Conflict of interest: The authors declare no conflict of interest.

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