Article

Gravitational redshift explained as a Doppler Effect in uniformly accelerated frames

A. Sfarti

CS Department, 387 Soda Hall, UC Berkeley, Berkeley, CA 94720, USA; egas@pacbell.net

Abstract: Einstein predicted a change in the energy of photons in the proximity of a gravitational field, the change being directly proportional with the distance from the gravitational source. In the following, we will present an approach to the experiment relying on the Einstein Equivalence Principle and on the recently derived expressions of the Doppler Effect for uniformly accelerated motion of the source and the receiver. We will conclude with a chapter on the numerical limits of applicability of the described method.

Keywords: Pound-Rebka; Gravitational Redshift; Tests of General Relativity; Einstein Equivalence Principle; Doppler Effect for uniformly accelerated motion; Error Analysis for Numerical Methods

1. Standard derivation of Gravitational Redshift

In the early 60's, Pound and Rebka set out to verify Einstein's prediction [1]. The experiment was reprised with even higher precision by Pound and Snider [2]. Later, Vessot [3] reprised the experiment in space with much improved precision. The standard explanation of gravitational redshift falls out straight from the Schwarzschild solution of the Einstein Field Equations (EFE) [4]. We start our derivation with the general form of the Schwarzschild metric for light propagating radially in the gravitational field of the Earth, as observed in the Pound-Rebka tests of General Relativity:

$$c^2 d\tau^2 = (1 - \frac{r_s}{r(t)})c^2 dt^2 - \frac{dr^2}{1 - \frac{r_s}{r(t)}} - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{1}$$

where:

τ is the proper time,

c is the speed of light,

t is the coordinate time,

r is the radial coordinate,

θ is the co-latitude (angle from North) in radians,

φ is the longitude in radians

r_s is the Schwarzschild radius: $r_s = \frac{2GM}{c^2}$.

The rotational terms are negligible and there is no radial motion, so dr = 0:

$$c^2 d\tau^2 = (1 - \frac{r_s}{R})c^2 dt^2 \tag{2}$$

where R is the Earth radius.

For an observer at the top of the tower in the Pound experiment, the following holds:

$$d\tau^2{}_{top} = (1 - \frac{r_s}{R + h})dt^2 \tag{3}$$

where h is the height of the tower in the Pound-Rebka experiment.

For an observer at the bottom of the tower, the following holds:

$$d\tau^2{}_{bottom} = (1 - \frac{r_s}{R})dt^2 \tag{4}$$

From Equations (3) and (4):

$$\frac{d\tau_{top}}{d\tau_{bottom}} = \sqrt{\frac{1 - \frac{r_s}{R+h}}{1 - \frac{r_s}{R}}} \tag{5}$$

This can be rewritten in terms of frequencies:

$$\frac{1/f_{top}}{1/f_{bottom}} = \sqrt{\frac{1 - \frac{r_s}{R+h}}{1 - \frac{r_s}{R}}} \tag{6}$$

bringing us to the famous Pound-Rebka formula whereby the emitter is at the bottom of the tower and the receiver is at the top:

$$f_r = f_e \sqrt{\frac{1 - \frac{r_s}{R}}{1 - \frac{r_s}{R+h}}} \tag{7}$$

Remembering that $r_s << R$ and $\frac{GM}{R^2} = g$, for $h << R$ we can further write:

$$f_r = f_e(1 - \frac{gh}{c^2} + \frac{1}{2}(\frac{gh}{c^2})^2) \tag{8}$$

If we do not neglect the rotational term, we get a higher level of precision:

$$c^2 d\tau^2 = (1 - \frac{r_s}{R})c^2 dt^2 - R^2 d\theta^2 \tag{9}$$

$$d\theta = \omega dt \tag{10}$$

$$c^2 d\tau^2 = [(1 - \frac{r_s}{R})c^2 - R^2\omega^2]dt^2 \tag{11}$$

$$d\tau = dt\sqrt{1 - \frac{r_s}{R} - \frac{R^2\omega^2}{c^2}} \tag{12}$$

Thus, the relationship between frequencies becomes:

$$f_r = f_e \sqrt{\frac{1 - \frac{r_s}{R+h} - \frac{(R+h)^2\omega^2}{c^2}}{1 - \frac{r_s}{R} - \frac{R^2\omega^2}{c^2}}} \tag{13}$$

To put effects in perspective $\frac{r_s}{R} \approx \frac{8*10^{-3}}{6400*10^3} \approx 1.25*10^{-9}$ while $\frac{R^2\omega^2}{c^2} \approx (\frac{400}{3*10^8})^2 \approx 1.7*10^{-12}$. This explains why the rotational effect can be safely neglected.

2. The explanation of the Gravitational Redshift as a Doppler Effect for observer and source in uniformly accelerated motion

When tidal effects (the variation of g with distance) are negligible, then physics in the presence of a gravitational field is approximately equivalent to physics in a uniformly accelerated coordinate system. This equivalence allows the solving of certain problems involving gravity by converting them to an equivalent problem that does not involve gravity, and then solving "that problem" using special relativity. So,

for example, if one wants to know how clocks are affected by elevation, one converts the problem to the corresponding case of clocks aboard an accelerating rocket in the absence of gravity. Then one solves that problem using Special Relativity. Viewed this way, the equivalence principle is a "tool" for solving problems involving gravity without bringing in the full machinery of general relativity. Will [5] points out: "If the frequency of a given type of atomic clock is the same when measured in a local, momentarily comoving freely falling frame (Lorentz frame), independent of the location or velocity of that frame, then the comparison of frequencies of two clocks at rest at different locations boils down to a comparison of the velocities of two local Lorentz frames, one at rest with respect to one clock at the moment of emission of its signal, the other at rest with respect to the other clock at the moment of reception of the signal. The frequency shift is then a consequence of the first-order Doppler shift between the frames."

In the following, we will show that the principle of equivalence could be tested to a second order effect by comparing the red-shift produced by uniform acceleration with the one produced by a uniform gravitational field. Imagine that we could transport the Harvard Tower to a location very far from any gravitational field. At the time an electromagnetic pulse is emitted from the top of the tower, the tower floor is accelerated away from the direction of the light front with the acceleration g in order to generate the equivalent red-shift (see **Figure 1**).
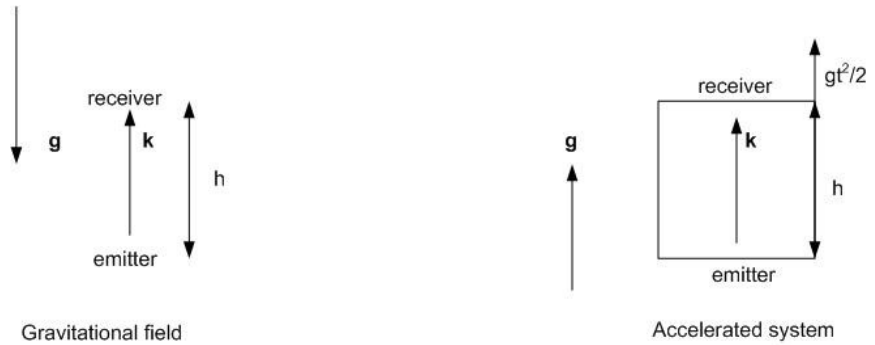
Figure 1. Left: Light emitted against a uniform gravitational field. Right: Light emitted in the direction of an accelerated motion of emitter.

The light front encounters the tower ceiling after the time t given by the equations:

$$ct = h + \frac{1}{2}gt^2$$
$$t = \frac{c - \sqrt{c^2 - 2gh}}{g} \tag{14}$$

The speed of the ceiling at the time when the light strikes it is therefore:

$$V(t) = gt = c - \sqrt{c^2 - 2gh}$$
$$\frac{V(t)}{c} = 1 - \sqrt{1 - \frac{2gh}{c^2}} \tag{15}$$

The relativistic Doppler Effect for the case of accelerated receiver and emitter is:

$$f_r = f_e \sqrt{\frac{1 - V/c}{1 + V/c}} \tag{16}$$

Exactly like Equation (7) is the exact formula of gravitational redshift, expression Equation (16) is the exact symbolic formula for relativistic Doppler red-shift. On the other hand, from Equation (15) we obtain:

$$\frac{V}{c} = 1 - \sqrt{1 - \frac{2gh}{c^2}} \approx \frac{gh}{c^2} + \frac{1}{2}(\frac{gh}{c^2})^2 \tag{17}$$

$$f_r = f_e \sqrt{\frac{1 - \frac{V}{c}}{1 + \frac{V}{c}}} = \frac{1 - \frac{V}{c}}{\sqrt{1 - \frac{V^2}{c^2}}} \approx 1 - \frac{gh}{c^2} + \frac{1}{2}(\frac{gh}{c^2})^2 \tag{18}$$

Approximate expressions Equations (8) and (18) agree in the first and second order. The value for h needs to be kept under the length of existent rockets for practical reasons as well as for theoretical reasons since tidal forces can become a source of error for larger laboratories in space. Given that $\frac{1}{2}(\frac{gh}{c^2})^2 \sim 10^{-13}$ such effects are just becoming measurable with existent technologies [6,7].

3. High order effect numerical analysis

We have seen in the previous paragraph that LPI can be confirmed symbolically and experimentally to a second order Taylor expansion in terms of $\frac{gh}{c^2}$ a truly amazing precision. In reality, we are limited by the practicalities of measuring g and h. In the following, we will restate the problem in terms of the precision in the measurement of g and h. Let ε and δ be the measurement errors for g and h respectively. The overall measurement error is

$$\Delta = (1 - \frac{(g + \varepsilon)(h + \delta)}{c^2} + \frac{1}{2}(\frac{(g + \varepsilon)(h + \delta)}{c^2})^2) - (1 - \frac{gh}{c^2} + \frac{1}{2}(\frac{gh}{c^2})^2) \tag{19}$$

Ignoring the higher order terms in $\delta\varepsilon$ we obtain:

$$\Delta \approx (\frac{gh}{c^2} - 1)\frac{g\delta + h\varepsilon}{c^2} \tag{20}$$

Current precision measurements of the Earth gravitational constant constrain ε to less than 10^{-9}. Since Equation (20) is linear in ε there is nothing to be gained from measuring h with a precision higher than 10^{-9}.

4. Conclusions

We have established the theoretical limits of testing the equivalence principle by comparing the uniform gravitational field results with the results of a hypothetical time dilation experiment executed in a uniformly accelerated rocket in free space. We have derived the exact symbolic formulas, and we have shown that their series approximations agree to the second order term. Prior results have shown agreement up to only the first order term.

Conflict of interest: The author declares no conflict of interest.

References

1. Pound RV, Rebka GA Jr. Resonant absorption of the 14.4-kev gamma Ray from 0.10-μsec Fe^{57}. Physical Review Letters. 1959; 3(12): 554. doi: 10.1103/PhysRevLett.3.554

2. Pound RV, Snider JL. Effect of gravity on nuclear resonance. Physical Review Letters. 1964; 13(18): 539-540. doi: 10.1103/physrevlett.13.539

3. Vessot RFC, Levine MW, Mattison EM, et al. Test of relativistic gravitation with a space-borne hydrogen maser. Physical Review Letters. 1980; 45(26): 2081-2084. doi: 10.1103/physrevlett.45.2081

4. Einstein A. On the electrodynamics of moving bodies. Annalen der Physik. 1905; 17: 891-921.

5. Will CM. The confrontation between general relativity and experiment. Living Reviews in Relativity. 2001; 4(1): 4. doi: 10.12942/lrr-2001-4

6. .Niebauer TM, Sasagawa GS, Faller JE, et al. A new generation of absolute gravimeters. Metrologia. 1995; 32(3): 159-180. doi: 10.1088/0026-1394/32/3/004

7. Sfarti A. QED-based derivation of the general forms of the relativistic Doppler effect and of the relativistic aberration for uniformly rotating frames and for uniformly accelerated frames. Modern Physics Letters A. 2022; 37: 37n38. doi: 10.1142/s0217732322502388