Journal of AppliedMath 2024, 2(5), 1700.
https://doi: doi.org/10.59400/jam.v2i5.1700

Article

Differentials of the basis in Clifford Geometric Algebra

Yingqiu Gu

School of Mathematical Science, Fudan University, Shanghai 200433, China; yqgu@fudan.edu.cn

CITATION

Gu Y. Differentials of the basis in Clifford Geometric Algebra. Journal of AppliedMath. 2024; 2(5): 1700. https://doi.org/10.59400/jam.v2i5.1700

ARTICLE INFO

Received: 27 June 2024
Accepted: 6 August 2024
Available online: 18 August 2024

Abstract: In this paper we discuss the dynamic effects of the varying frames. The differential of frame or basis vectors is always equivalent to a linear transformation of the frame, and the linear transformation is not the same in different contexts. In differential geometry, the linear transformation is the connection operator. While in quantum mechanics, the operator algebra corresponds to the differentials of matrices. Corresponding to the variation of the metric, the variation of the frame contains a unusual fourth-order tensor. We also derive the Lie differential of the frame corresponding to the Lorentz transformation group. The definition of differential of the frame is different, so the corresponding linear transformation is also different. In this paper, the unified point of view to deal with the variation of frame or basis vectors will bring great convenience to the research and application of Clifford algebras.

Keywords: Clifford algebra; connection; Dirac-γ; moving frame; variation

MSC(2020): 15A63; 15A67; 15A75; 53A17; 53A45

1. Introduction

Professor W. K. Clifford defined his geometric algebra [1] by combining and extending the Grassmann's exterior algebra [2] and Hamilton's quaternions [3] into a more general algebraic framework, which is a direct and intuitive generalization of vector algebra, with an explicit geometric interpretation [4] and clear relations with linear algebra [5, 6]. Geometric algebra has developed steadily over the past century and has gained popularity by discovering many applications in different scientific fields. It brings new perspectives to multiple mathematical disciplines, and many properties have been derived in new forms [7–9]. An attractive feature of Clifford algebras is that they unify various branches of mathematics. Clifford geometric algebra has gradually become a unified language and effective tool for modern science and is widely used in different fields of mathematics, physics and engineering [10–13]. Geometric algebra is visualized and easily accessible. Some of its recent applications in high-tech are introduced in [14]. The great practical value of standardized geometric algebra in current mathematics and physics courses is evident.

Clifford algebra has many applications in differential geometry [15–17]. In [18] the authors reviewed and discussed a generalization of the Einstein theory of gravity, where the spin of matter and its mass play a dynamical role. The spin of matter in space-time is coupled to a non-Riemannian structure, the Cartan's torsion tensor. Nester made the Clifford algebraic decomposition of the spinor connection [19]. The Cartan's differential forms and Dirac-γ matrices are simultaneously employed to concentrate the relations in differential geometry, resulting in very neat forms [20]. This formalism of "double frames" is used to derive a class of spin curvature identities existing in the Riemann or Riemannian-Cartan geometry in the study of Nester [21]. Each identity involves a quadratic expression of the covariant derivatives of the spinor field, which is a linear combination of the curvature and an exact differential form.

In differential geometry, the basis and coframe of a manifold vary from point to point.

In this paper, we focus on the dynamic effects of the basis vector or generator of the Clifford algebras, which reflects the differentials of basis vector. This problem arises from the discussion on the relations between the variations of basis vector and metric with Professor J. M. Nester, and this issue seems to be neglected by the academic community. A detailed calculation for the case of $1+3$ dimensional space-time was made in paper [22], and some unusual formulas were derived. The following analysis shows that these formulas, such as Equations (21) and (25), may hold for all space-times. There are many different dynamic effects of the basis vector, such as the change of coordinate or coordinate system, moving frames, operator action, etc., which lead to different differential of the basis. Therefore, this paper makes a special survey on this topic, aimed to attract the attention of colleagues in the field.

2. Clifford representation of Riemann Geometry

We consider the n-dimensional pseudo-Riemannian manifold equipped with metric

$$(g_{\mu\nu}) \simeq (\eta_{ab}) = \mathrm{diag}(I_p, -I_q), \qquad (n = p + q). \tag{1}$$

In what follows, unless the dimension is specified, we discuss the manifold $\mathbb{R}^n$ with arbitrary (p, q). The element of the space-time is described by

$$d\mathbf{x} = \gamma_\mu dx^\mu = \gamma^\mu dx_\mu = \gamma_a \delta X^a = \gamma^a \delta X_a, \tag{2}$$

in which $\{\gamma_\mu\}$ is a covariant basis vector or **frame**, and $\{\gamma_a\}$ is a set of orthonormal basis vectors in the tangent space-time at any fixed point, and $\{\gamma^a = \eta^{ab}\gamma_b,\ \gamma^\mu = g^{\mu\nu}\gamma_\nu\}$ are the **coframes**. dx^μ and δX^a are variables that represent the coordinate increments in the tangent space-time, and δX^a can be determined only to a Lorentz transformation. We use the Latin characters $a, b, \cdots$ for the Minkowski indices, and Greek characters (μ, ν) for the curvilinear indices. We have transformation

$$\gamma_\mu = f_\mu^{\ a}\gamma_a, \qquad \gamma^\mu = f^\mu_{\ a}\gamma^a, \tag{3}$$

where $f_\mu^{\ a} \in \mathbb{R}$ and $f^\mu_{\ a} \in \mathbb{R}$ are the **frame coefficients**. The frame and basis satisfy the following Clifford relations

$$\frac{1}{2}(\gamma_a\gamma_b + \gamma_b\gamma_a) = \gamma_a \cdot \gamma_b I = \eta_{ab} I, \qquad \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \gamma_\mu \cdot \gamma_\nu I = g_{\mu\nu} I, \tag{4}$$

where $\gamma_a\gamma_b$ and $\gamma_\mu\gamma_\nu$ are **Clifford products** of vectors, and I is the identity element of Clifford algebra. In the case without confusion, we can directly use 1 to replace I. By Equations (3) and (4) we have the relations between $(f^\mu_{\ a}, f_\mu^{\ a})$ and metric as

$$f_\mu^{\ a} f^\mu_{\ b} = \delta^a_b, \quad f_\mu^{\ a} f^\nu_{\ a} = \delta^\nu_\mu, \quad f^\mu_{\ a} f^\nu_{\ b}\eta^{ab} = g^{\mu\nu}, \quad f_\mu^{\ a} f_\nu^{\ b}\eta_{ab} = g_{\mu\nu}. \tag{5}$$

The space-time $\mathbb{R}^{p+q}$ defined with Clifford product of vectors form a **Clifford algebra** $C\ell(\mathbb{R}^{p+q})$. By Clifford algebra we know that $\{\gamma_a\}$ is isomorphic to a set of special matrices constructed by Pauli matrices [15]. Thus, in the case without confusion, we no longer distinguish between the basis γ_a and its matrix representation.

There are several definitions of Clifford algebra [13]. However, it is best to treat it as a **hypercomplex system** with addition, subtraction, multiplication and division operations [23–25]. Geometric algebra brings great convenience to study geometry and physics [16, 17]. By Equation (2) we have

$$
\begin{aligned}
d\mathbf{x}^2 &= \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)dx^\mu dx^\nu = g_{\mu\nu}dx^\mu dx^\nu I = \eta_{ab}\delta X^a \delta X^b I, \\
dV_k &= d\mathbf{x}_1 \wedge d\mathbf{x}_2 \wedge \cdots \wedge d\mathbf{x}_k = \gamma_{\mu\nu\cdots\omega}dx_1^\mu dx_2^\nu \cdots dx_k^\omega, \ \ (1 \le k \le n),
\end{aligned}
$$

in which $ds = |d\mathbf{x}|$ is the distance element and dV_k is the oriented volume, $\gamma_{\mu\nu\cdots\omega} = \gamma_\mu \wedge \gamma_\nu \wedge \cdots \wedge \gamma_\omega \in \Lambda^k(\mathbb{R}^{p,q})$ is the unit of oriented volume, and $\wedge$ is the Grassmann's **exterior product**, which is defined by

$$
\gamma_{a_1} \wedge \gamma_{a_2} \cdots \wedge \gamma_{a_k} \equiv \frac{1}{k!}\sum_\sigma \sigma^{b_1 b_2 \cdots b_k}_{a_1 a_2 \cdots a_k} \gamma_{b_1}\gamma_{b_2}\cdots\gamma_{b_k}, \ (1 \le k \le n)
$$

where $a_j \ne a_l$ if $j \ne l$, $\sigma^{b_1 b_2 \cdots b_k}_{a_1 a_2 \cdots a_k}$ is permutation tensor, and if $b_1 b_2 \cdots b_k$ is an even permutation of $a_1 a_2 \cdots a_k$, it is equal to 1, for odd permutation it is equal to -1, otherwise equal to 0. The above formula is a sum over all permutations; that is, it is anti-symmetric for all indices. Then the following Clifford-Grassmann numbers

$$
\mathbf{C} = C_0 I + C_\mu \gamma^\mu + C_{\mu\nu}\gamma^{\mu\nu} + \cdots + C_{12\cdots n}\gamma^{12\cdots n} \tag{6}
$$

form a 2^n-dimensional hypercomplex system over $\mathbb{R}$ according to matrix algebra, in which $C_0, C_\mu, \cdots, C_{12\cdots n} \in \mathbb{R}$. The Calvet's norm is defined by $||\mathbf{C}|| = \sqrt[m]{|\det(\mathbf{C})|}$, where m is the order of matrix $\mathbf{C}$. The Calvet's norm is a scalar under similarity transformations, and satisfies $||\mathbf{AB}|| = ||\mathbf{A}|| \cdot ||\mathbf{B}||$ for any Clifford-Grassmann numbers $\mathbf{A}, \mathbf{B}$. The transformation law of $||\cdot||$ is studied in details in the study of Calvet [26].

For the $1+3$ dimensional realistic space-time, the lowest-order complex matrix representation of the generators of Clifford algebra $C\ell(\mathbb{R}^{1,3})$ is Dirac-γ matrices

$$
\gamma^0 = \gamma_0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \qquad \gamma^a = -\gamma_a = \begin{pmatrix} 0 & -\sigma_a \\ \sigma_a & 0 \end{pmatrix},
$$

which generate the **Grassmann basis elements** of $C\ell(\mathbb{R}^{1,3})$ as

$$
I_4, \quad \gamma^a, \quad \gamma^{ab} = \gamma^a \wedge \gamma^b, \quad \gamma^{abc} = -\epsilon^{abcd}\gamma_d\gamma^{0123}, \quad \gamma^{0123} = -\mathrm{i}\gamma^5, \tag{7}
$$

where σ_a stand for Pauli matrices, $\gamma^5 = \mathrm{diag}(I_2, -I_2)$ and $\epsilon^{0123} = 1$. We have the Clifford-Grassmann number as follows,

$$
\mathbf{K} = sI_4 + A_a\gamma^a + H_{ab}\gamma^{ab} + Q_a\gamma^a\gamma^{0123} + p\gamma^{0123}, \tag{8}
$$

where $(s, p, A_a, \cdots \in \mathbb{R})$. $sI_4 \in \Lambda^0$ is a scalar, $A_a\gamma^a \in \Lambda^1$ is a true vector, $H_{ab}\gamma^{ab} \leftrightarrow (\vec{E}, \vec{B}) \in \Lambda^2$ is a 2-vector, $Q_a\gamma^a\gamma^{0123} \in \Lambda^3$ is a pseudo vector and $p\gamma^{0123} \in \Lambda^4$ is a pseudo scalar. In general, any Clifford algebra $C\ell(\mathbb{R}^{p,q})$ is a system of hypercomplex numbers.

3. Various differentials of basis

3.1. Directional differential of frame

In differential geometry, for a vector field $\mathbf{A} = \gamma_\mu A^\mu$ we define its **absolute differential** as

$$\begin{aligned} d\mathbf{A} &\equiv \lim_{\Delta\mathbf{x}\to d\mathbf{x}}[\mathbf{A}(\mathbf{x}+\Delta\mathbf{x}) - \mathbf{A}(\mathbf{x})] \\ &= (\partial_\alpha A^\mu \gamma_\mu + A^\mu \mathfrak{d}_\alpha \gamma_\mu)dx^\alpha = (\partial_\alpha A_\mu \gamma^\mu + A_\mu \mathfrak{d}_\alpha \gamma^\mu)dx^\alpha, \end{aligned} \tag{9}$$

where $\Delta\mathbf{x} \to d\mathbf{x}$ means the linearization of $\Delta\mathbf{x}$ in the above equation [27, Ch.1]. We call $\mathfrak{d}_\alpha$ the **c**onnection operator. According to its geometric significance, the connection operator should meet the following axioms [15]:

1) It is a real linear transformation in the tangent space $\mathfrak{d}_\alpha : TV \to TV$, so we have

$$\mathfrak{d}_\alpha \gamma_\beta = K^\mu_{\alpha\beta}\gamma_\mu, \qquad (K^\mu_{\alpha\beta} \in \mathbb{R}). \tag{10}$$

2) For any differentiable function $\phi(\mathbf{x})$ we have

$$\mathfrak{d}_\alpha(\phi\gamma_\beta) = (\partial_\alpha \phi)\gamma_\beta + \phi(\mathfrak{d}_\alpha \gamma_\beta). \tag{11}$$

3) For any bilinear product of the vectors or Clifford-Grassmann numbers $\mathbf{A} \circ \mathbf{B}$, it satisfies the Leibniz formula

$$\mathfrak{d}_\alpha(\mathbf{A} \circ \mathbf{B}) = (\mathfrak{d}_\alpha \mathbf{A}) \circ \mathbf{B} + \mathbf{A} \circ (\mathfrak{d}_\alpha \mathbf{B}), \tag{12}$$

or in the form of basis elements

$$\mathfrak{d}_\alpha(\gamma^{\mu\cdots} \circ \gamma^{\nu\cdots}) = (\mathfrak{d}_\alpha \gamma^{\mu\cdots}) \circ \gamma^{\nu\cdots} + \gamma^{\mu\cdots} \circ (\mathfrak{d}_\alpha \gamma^{\nu\cdots}). \tag{13}$$

Here the bilinear product means for arbitrary $a, b \in \mathbb{R}$ we have

$$\begin{aligned} (a\mathbf{A} + b\mathbf{B}) \circ \mathbf{C} &= a\mathbf{A} \circ \mathbf{C} + b\mathbf{B} \circ \mathbf{C}, \\ \mathbf{C} \circ (a\mathbf{A} + b\mathbf{B}) &= a\mathbf{C} \circ \mathbf{A} + b\mathbf{C} \circ \mathbf{B}. \end{aligned}$$

In the study of Cartan [27, Ch.1], the differential $d\mathbf{A}$ is directly defined as

$$d\mathbf{A} = \omega^i \gamma_i, \quad d\gamma_i = \omega^j_i \gamma_j, \quad (\omega^i = \Gamma^i_a dx^a, \quad \omega^j_i = \Gamma^j_{ia} dx^a). \tag{14}$$

Clearly, both Equations (14) and (9) are logically equivalent. The difference between them is that the geometric and physical meanings of Equation (9) is more intuitive and easier for operation. We can define different connection operators for different applications, which will be illustrated by the several application examples. We have the following conclusions [15].

Theorem 1. *For metric $\mathbf{g} = g_{\mu\nu}\gamma^\mu \otimes \gamma^\nu = \gamma_\mu \otimes \gamma^\mu$, where $\otimes$ is the tensor product, we have the metric consistent condition $d\mathbf{g} = 0$, as well as*

$$\mathfrak{d}_\alpha \gamma^\mu = -K^\mu_{\alpha\beta}\gamma^\beta, \qquad \partial_\alpha g_{\mu\nu} = g_{\nu\beta}K^\beta_{\alpha\mu} + g_{\mu\beta}K^\beta_{\alpha\nu}. \tag{15}$$

For the **connection coefficients**

$$K^\alpha_{\mu\nu} = \Pi^\alpha_{\mu\nu} + \mathbf{T}^\alpha_{\mu\nu}, \qquad \Pi^\mu_{\alpha\beta} = \Pi^\mu_{\beta\alpha}, \qquad \mathbf{T}^\mu_{\alpha\beta} = -\mathbf{T}^\mu_{\beta\alpha}$$

we have solutions $\Pi^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + \pi^\alpha_{\mu\nu}$, in which $\Gamma^\alpha_{\mu\nu}$ is the Christoffel symbol. For the **contortion** $\pi^\mu_{\alpha\beta} = \pi^\mu_{\beta\alpha}$ and **torsion** $\mathbf{T}^\mu_{\alpha\beta} = -\mathbf{T}^\mu_{\beta\alpha}$, denoting

$$\pi_{\mu|\nu\alpha} = g_{\mu\beta}\pi^\beta_{\nu\alpha}, \qquad \mathbf{T}_{\mu|\nu\alpha} = g_{\mu\beta}\mathbf{T}^\beta_{\nu\alpha},$$

we have the following relations

$$\begin{aligned} \pi_{\mu|\nu\alpha} &= \mathbf{T}_{\nu|\alpha\mu} + \mathbf{T}_{\alpha|\nu\mu}, \\ \mathbf{T}_{\mu|\nu\alpha} &= \frac{1}{3}(\pi_{\alpha|\mu\nu} - \pi_{\nu|\mu\alpha}) + \widetilde{\mathbf{T}}_{\mu\nu\alpha}, \end{aligned}$$

as well as the consistent condition

$$\pi_{\mu|\nu\alpha} + \pi_{\alpha|\mu\nu} + \pi_{\nu|\alpha\mu} = 0.$$

$\widetilde{\mathbf{T}} = \widetilde{\mathbf{T}}_{\mu\nu\omega}\gamma^{\mu\nu\omega} \in \Lambda^3$ is an arbitrary skew-symmetric tensor.

By the above theorem we obtain the absolute differential (9) of vector **A**. In the case $\pi^\alpha_{\mu\nu} \equiv 0$, the absolute differential of vector **A** is given by

$$d\mathbf{A} = \nabla_\alpha A^\mu \gamma_\mu dx^\alpha = \nabla_\alpha A_\mu \gamma^\mu dx^\alpha, \tag{16}$$

where ∇_α denotes the absolute derivatives of vector defined as follows

$$\begin{aligned} \nabla_\alpha A^\mu = A^\mu_{;\alpha} + \mathbf{T}^\mu_{\alpha\beta}A^\beta, \quad A^\mu_{;\alpha} = \partial_\alpha A^\mu + \Gamma^\mu_{\alpha\nu}A^\nu, \\ \nabla_\alpha A_\mu = A_{\mu;\alpha} - \mathbf{T}^\beta_{\alpha\mu}A_\beta, \quad A_{\mu;\alpha} = \partial_\alpha A_\mu - \Gamma^\nu_{\alpha\mu}A_\nu, \end{aligned}$$

where $A^\mu_{;\alpha}$ and $A_{\mu;\alpha}$ are usual covariant derivatives of vector without torsion. Torsion $\mathbf{T}_{\mu\nu\omega} \in \Lambda^3$ is an antisymmetrical tensor of C^3_n independent components.

By Equation (15) and Equation (11), we have the second order differential of γ^μ as

$$\mathfrak{d}_\omega \mathfrak{d}_\alpha \gamma^\mu = -(\partial_\omega K^\mu_{\alpha\beta} - K^\mu_{\alpha\gamma}K^\gamma_{\omega\beta})\gamma^\beta.$$

Thus we have

$$(\mathfrak{d}_\omega\mathfrak{d}_\alpha - \mathfrak{d}_\alpha\mathfrak{d}_\omega)\gamma^\mu = R^\mu_{\beta\alpha\omega}\gamma^\beta,$$

in which

$$R^\mu_{\beta\alpha\omega} = \partial_\alpha K^\mu_{\omega\beta} - \partial_\omega K^\mu_{\alpha\beta} + K^\mu_{\alpha\gamma}K^\gamma_{\omega\beta} - K^\mu_{\omega\gamma}K^\gamma_{\alpha\beta}.$$

In the case of $K^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta}$, $R^\mu_{\beta\alpha\omega}$ is just Riemann curvature tensor. Similarly, we can calculate the absolute differential for any tensor. It is easy to check the following results.

Theorem 2. *For the basis* (7) *of the Clifford algebra $C\ell(\mathbb{R}^{1,3})$, we have connection calculus*

$$\mathfrak{d}_\alpha\gamma^{0123} = \mathfrak{d}_\alpha\gamma^5 = 0.$$

$$\begin{cases} \mathfrak{d}_\alpha\gamma^{123} = (\mathfrak{d}_\alpha\gamma_0)\gamma^{0123}, \quad \mathfrak{d}_\alpha\gamma^{023} = -(\mathfrak{d}_\alpha\gamma_1)\gamma^{0123}, \\ \mathfrak{d}_\alpha\gamma^{013} = (\mathfrak{d}_\alpha\gamma_2)\gamma^{0123}, \quad \mathfrak{d}_\alpha\gamma^{012} = -(\mathfrak{d}_\alpha\gamma_3)\gamma^{0123}. \end{cases}$$

For the skew-symmetric tensor $\mathbf{S} = \mathbf{S}_{\mu\nu\omega}\gamma^{\mu\nu\omega} = \mathbf{S}_\alpha\gamma^\alpha\gamma^{0123}$, we have

$$\nabla_\alpha\mathbf{S} = (\nabla_\alpha\mathbf{S}_\mu)\gamma^\mu\gamma^{0123}, \quad \nabla_\alpha\mathbf{S}_\mu = \partial_\alpha\mathbf{S}_\beta - (\Gamma^\mu_{\alpha\beta} + \mathbf{T}^\mu_{\alpha\beta})\mathbf{S}_\mu.$$

For the torsion $\mathbf{S} = \mathbf{T}$ we have $\nabla_\alpha \mathbf{T}_\mu = \partial_\alpha \mathbf{T}_\beta - \Gamma^\mu_{\alpha\beta}\mathbf{T}_\mu$.

For k-vector

$$\mathbf{F} = \frac{1}{k!} F_{\mu_1\mu_2\cdots\mu_k}\gamma^{\mu_1\mu_2\cdots\mu_k},$$

the **exterior differential d** and **co-differential** δ are defined as

$$\mathbf{dF} = \frac{1}{k!}\gamma^{\alpha\mu_1\mu_2\cdots\mu_k}\partial_\alpha F_{\mu_1\mu_2\cdots\mu_k}, \qquad \delta\mathbf{F} = \frac{1}{(k-1)!}\gamma^{\nu_1\nu_2\cdots\nu_{k-1}}\partial_\alpha F^{\alpha}_{\ \nu_1\nu_2\cdots\nu_{k-1}}.$$

Then we have the following beautiful results [16, Ch7.1].

Theorem 3. *In the case of torsion-free, we have*

$$\mathbf{d}^2\mathbf{F} = \delta^2\mathbf{F} = 0,$$
$$\nabla\mathbf{F} = (\mathbf{d} + \delta)\mathbf{F}, \qquad \nabla^2\mathbf{F} = (\mathbf{d}\delta + \delta\mathbf{d})\mathbf{F},$$

where $\nabla = \gamma^\alpha\nabla_\alpha$.

3.2. Algebraic derivatives of Basis

In order to find the eigenfunctions of Dirac equation $\hat{H}\psi = E\psi$ in curved space-time, we need to compute the commutative operators [24]. In this case, the γ_a are only regarded as matrices of numbers rather than basis vectors, and the derivatives of the operator-valued Clifford numbers are normal partial derivatives. Here (γ_μ, γ^μ) have no longer geometric meanings, and they are different from the basis vectors (γ_μ, γ^μ) in Equation (3).

We introduce the following Christoffel-like connections $C^\mu_{\alpha\beta} \equiv f^\mu_{\ a}\partial_\alpha f_\beta^{\ a}$, then for the matrices (γ_μ, γ^μ), the **algebraic derivatives** are given by

$$\begin{aligned} \partial_\alpha\gamma_\beta &= \gamma_a\partial_\alpha f_\beta^{\ a} = \gamma_\mu f^\mu_{\ a}\partial_\alpha f_\beta^{\ a} = \gamma_\mu C^\mu_{\alpha\beta}, \\ \partial_\alpha\gamma^\mu &= \gamma^a\partial_\alpha f^\mu_{\ a} = \gamma^\beta f_\beta^{\ a}\partial_\alpha f^\mu_{\ a} = -\gamma^\beta C^\mu_{\alpha\beta}. \end{aligned}$$

In this case, we have $\mathfrak{d}_\alpha\gamma^a = 0$ and $(\partial_\omega\partial_\alpha - \partial_\alpha\partial_\omega)\gamma^\mu = 0$.

Similarly to Equation (16), we can define the covariant algebraic derivatives $\overline{\nabla}_\alpha$ for Clifford numbers as

$$\begin{aligned} \partial_\alpha\mathbf{A} &= \partial_\alpha(A^\mu\gamma_\mu) = \gamma_\mu\overline{\nabla}_\alpha A^\mu = \gamma_\mu\left(\partial_\alpha A^\mu + C^\mu_{\alpha\beta}A^\beta\right) \\ &= \partial_\alpha(A_\mu\gamma^\mu) = \gamma^\mu\overline{\nabla}_\alpha A_\mu = \gamma^\mu\left(\partial_\alpha A_\mu - C^\beta_{\alpha\mu}A_\beta\right), \\ \partial_\alpha\mathbf{N} &= \gamma_{\mu\nu}\overline{\nabla}_\alpha N^{\mu\nu} = \gamma_{\mu\nu}\left(\partial_\alpha N^{\mu\nu} + C^\mu_{\alpha\beta}N^{\beta\nu} + C^\nu_{\alpha\beta}N^{\mu\beta}\right) \\ &= \gamma^{\mu\nu}\overline{\nabla}_\alpha N_{\mu\nu} = \gamma^{\mu\nu}\left(\partial_\alpha N_{\mu\nu} - C^\beta_{\alpha\mu}N_{\beta\nu} - C^\beta_{\alpha\nu}N_{\mu\beta}\right), \end{aligned}$$

and so on. The computing rules of $\bar{\nabla}_\alpha$ is quite similar to that of ∇_α in Equation (16), which also satisfies conditions (10)-(13).

3.3. Variations of frame and metric

In spinor theory in curved space-time, we need the variation of frame $\delta\gamma_\alpha$ instead of $\delta g_{\mu\nu}$ in some cases [22]. By Equation (5) or $g_{\mu\nu} = \gamma_\mu\cdot\gamma_\nu$ we know that map $(\gamma_\mu, \gamma_\nu) \mapsto g_{\mu\nu}$ is a single valued and continuous mapping. However, for $g_{\mu\nu} \mapsto \gamma_\alpha$, equation (5) has multiple roots for γ_α, and γ_α can only be determined to an arbitrary Lorentz transformation $\delta X' = \Lambda\delta X$. For a fixed Lorentz transformation, the map $g_{\mu\nu} \mapsto \gamma_\alpha$ has continuous and bijective branches, and each branch is somewhat similar to the quotient group. Thus the map $g_{\mu\nu} \leftrightarrow \gamma_\alpha$ is a bijection in a connected injective domain D for a fixed Λ, and $\delta g_{\mu\nu} \leftrightarrow \delta\gamma_\alpha$ is a linear transformation. Now we determine one of such linear transformations for a bijective branch. By Sylvester inertial theorem $(g_{\mu\nu}) \simeq (\eta_{ab})$ and Gram-Schmidt orthogonalization process, under some arrangement of the order of coordinates, we have

Theorem 4. *Let us suppose for matrix $(g_{\mu\nu})$ that*

$$(g_{\mu\nu}) = L(\eta_{ab})L^T, \qquad (g^{\mu\nu}) = U(\eta_{ab})U^T, \qquad U = L^{-T}, \tag{17}$$

where L is a real lower triangular matrix and U an upper one

$$L = \begin{pmatrix} L_1^{\ 1} & 0 & \cdots & 0 \\ L_2^{\ 1} & L_2^{\ 2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ L_n^{\ 1} & L_n^{\ 2} & \cdots & L_n^{\ n} \end{pmatrix}, \qquad U = \begin{pmatrix} U_{\ 1}^1 & U_{\ 2}^1 & \cdots & U_{\ n}^1 \\ 0 & U_{\ 2}^2 & \cdots & U_{\ n}^2 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & U_{\ n}^n \end{pmatrix}, \tag{18}$$

and (L, U) have positive diagonal elements $L_a^{\ a} > 0,\ U_{\ a}^a > 0$. The map $g_{\mu\nu} \leftrightarrow L_\alpha^{\ a} \in \mathbb{R}$ is a bijective and continuous map in a connected domain D. We have

$$\delta X = L^T dx, \qquad (\mathbf{e}_a) = (\gamma_\mu)U,$$

where δX and dx are column vectors, $(\mathbf{e}_a)$ and (γ_μ) are raw vectors, namely

$$\delta X = (\delta X^1, \delta X^2, \cdots, \delta X^n)^T, \qquad (\mathbf{e}_a) = (\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n).$$

We take $\mathbf{e}_a = \gamma_a$ to avoid confusion with γ_μ, the corresponding metric is given by (1).

Proof. The decomposition (17) is equivalent to transforming $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ into the sum of squares $ds^2 = \eta_{ab}\delta X^a \delta X^b$ by completing squares. In matrix form, we have

$$\delta X = L^T dx, \qquad d\mathbf{x}^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ab}\delta X^a \delta X^b. \tag{19}$$

Eq(18) is a direct result of Equation (19), but Equation (19) manifestly shows the geometric meanings of the frame coefficients $L_\mu^{\ a}$. By a fixed order of coordinates for completing squares and taking $L_a^{\ a} > 0$, we get a unique solution of L and $U = L^{-T}$. The solution $L_\mu^{\ a} = f(g_{\alpha\beta})$ is analytic in D, so $g_{\mu\nu} \leftrightarrow L_\alpha^{\ a}$ is bijective and continuous. The proof is completed. □

Theorem 5. *For any solution of frame (5) in matrix form $(f_\mu^{\ a})$ and $(f^\mu_{\ a})$, there exists a local Lorentz transformation $\delta X'^a = \Lambda^a_{\ b}\delta X^b$ independent of $g_{\mu\nu}$, such that*

$$(f_\mu^{\ a}) = L\Lambda^T, \qquad (f^\mu_{\ a}) = U\Lambda^{-1}, \qquad \gamma_\mu = f_\mu^{\ a}\gamma_a, \qquad \gamma^\mu = f^\mu_{\ a}\gamma^a, \tag{20}$$

where $\Lambda = (\Lambda^a_{\ b})$ is the matrix of Lorentz transformation.

Proof. For any solution (5) we have

$$(g_{\mu\nu}) = L(\eta_{ab})L^T = (f_\mu^{\ a})(\eta_{ab})(f_\mu^{\ a})^T \ \Leftrightarrow\ L^{-1}(f_\mu^{\ a})(\eta_{ab})(L^{-1}(f_\mu^{\ a}))^T = (\eta_{ab}).$$

So we have a Lorentz transformation matrix $\Lambda = (\Lambda^a_{\ b})$, such that

$$L^{-1}(f_\mu^{\ a}) = \Lambda^T \ \Leftrightarrow\ (f_\mu^{\ a}) = L\Lambda^T \qquad \text{or} \qquad f_\mu^{\ a} = L_\mu^{\ b}\Lambda^a_{\ b}.$$

By Equation (5) we have $(f^\mu_{\ a}) = (f_\mu^{\ a})^{-T} = U\Lambda^{-1}$. The proof is finished. □

For any variation of frame $\delta\gamma_\mu = \varepsilon_{\mu\nu}\gamma^\nu$, by Equation (4) we have a variation of metric

$$\delta g_{\mu\nu} \quad = \quad \delta\gamma_\mu \cdot \gamma_\nu + \gamma_\mu \cdot \delta\gamma_\nu = \varepsilon_{\mu\nu} + \varepsilon_{\nu\mu}.$$

Thus in the bijective domain D, we have solution

$$\varepsilon_{\mu\nu} = \frac{1}{2}(\delta g_{\mu\nu} + K_{\mu\nu}^{\alpha\beta}\delta g_{\alpha\beta}), \qquad K_{\mu\nu}^{\alpha\beta} = K_{\mu\nu}^{\beta\alpha} = -K_{\nu\mu}^{\alpha\beta},$$

where $K_{\mu\nu}^{\alpha\beta}$ should be determined by frame coefficients $(f_\mu^{\ a}, f_{\ a}^{\mu})$.

For LU decomposition (18), we define a **spinor coefficient table** by

$$S_{ab}^{\mu\nu} \quad \equiv \quad \begin{pmatrix} 0 & -U_{\ 1}^{\{\mu}U_{\ 2}^{\nu\}} & -U_{\ 1}^{\{\mu}U_{\ 3}^{\nu\}} & \cdots & -U_{\ 1}^{\{\mu}U_{\ n}^{\nu\}} \\ U_{\ 2}^{\{\mu}U_{\ 1}^{\nu\}} & 0 & -U_{\ 2}^{\{\mu}U_{\ 3}^{\nu\}} & \cdots & -U_{\ 2}^{\{\mu}U_{\ n}^{\nu\}} \\ U_{\ 3}^{\{\mu}U_{\ 1}^{\nu\}} & U_{\ 3}^{\{\mu}U_{\ 2}^{\nu\}} & 0 & \cdots & -U_{\ 3}^{\{\mu}U_{\ n}^{\nu\}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ U_{\ n}^{\{\mu}U_{\ 1}^{\nu\}} & U_{\ n}^{\{\mu}U_{\ 2}^{\nu\}} & U_{\ n}^{\{\mu}U_{\ 3}^{\nu\}} & \cdots & 0 \end{pmatrix} = -S_{ba}^{\mu\nu}, \quad (21)$$

in which

$$U_{\ a}^{\{\mu}U_{\ b}^{\nu\}} = \frac{1}{2}(U_{\ a}^{\mu}U_{\ b}^{\nu} + U_{\ a}^{\nu}U_{\ b}^{\mu}) = U_{\ b}^{\{\mu}U_{\ a}^{\nu\}}.$$

$S_{ab}^{\mu\nu} = U_{\ a}^{\{\mu}U_{\ b}^{\nu\}}\mathrm{sign}(a-b) = -S_{ba}^{\mu\nu}$ is symmetrical for Riemann indices (μ,ν) but anti-symmetrical for Minkowski indices (a,b). For any local Lorentz transformation $\delta X' = \Lambda\delta X$, if taking (21) as the proper values and setting Lorentz transformation

$$(S'^{\mu\nu}_{ab}) = \Lambda^{-T}(S_{cd}^{\alpha\beta})\Lambda^{-1}, \tag{22}$$

then $S_{ab}^{\mu\nu}$ becomes a tensor for indices (a,b). Definition (21) fixes the Lorentz transformation.

Theorem 6. *In the $1 \le p+q \le 4$ dimensional space-time $(\mathbb{R}^{p,q}, g_{\mu\nu})$, for frame (20) we have*

$$\delta\gamma_\alpha \quad = \quad \frac{1}{2}\gamma^\beta(\delta g_{\alpha\beta} + K_{\alpha\beta}^{\mu\nu}\delta g_{\mu\nu}), \tag{23}$$

$$\delta\gamma^\lambda \quad = \quad -\frac{1}{2}g^{\lambda\beta}\gamma^\alpha(\delta g_{\alpha\beta} + K_{\alpha\beta}^{\mu\nu}\delta g_{\mu\nu}), \tag{24}$$

in which

$$K_{\alpha\beta}^{\mu\nu} = S_{ab}^{\mu\nu}L_\alpha^{\ a}L_\beta^{\ b} = S'^{\mu\nu}_{ab}f_\alpha^{\ a}f_\beta^{\ b} \tag{25}$$

is independent of any Lorentz transformation.

Proof. By symbolic calculation we can check Equation (25) for the cases of $1 \le p+q \le 4$, so Equation (23) holds. By $\gamma^\lambda = g^{\lambda\alpha}\gamma_\alpha$ and

$$\frac{\partial g^{\lambda\alpha}}{\partial g_{\mu\nu}} = -\frac{1}{2}(g^{\mu\lambda}g^{\nu\alpha} + g^{\nu\lambda}g^{\mu\alpha}),$$

we have

$$\delta\gamma^\lambda = g^{\lambda\alpha}\delta\gamma_\alpha + \gamma_\alpha\frac{\partial g^{\lambda\alpha}}{\partial g_{\mu\nu}}\delta g_{\mu\nu} = g^{\lambda\alpha}\delta\gamma_\alpha - g^{\lambda\alpha}\gamma^\beta\delta g_{\alpha\beta}. \tag{26}$$

Substituting Equation (23) into Equation (26) and using $K_{\alpha\beta}^{\mu\nu} = -K_{\beta\alpha}^{\mu\nu}$ we obtain Equation

(24). The proof is completed. □

In the case of $p+q>4$, Theorem 6 should be also valid, but it seems difficult to generally prove Equations (21) and (25).

3.4. Lie differentials of frame

The most commonly used groups in physics are the continuous transformation groups, such as the rotation transformation group $SO(3)$ on $\mathbb{R}^3$, the proper Lorentz transformation group $SO(1,3)$ on Minkowski space-time, and so on. These groups are Lie groups with infinite orders. We take $SO(1,3)$ as example to show how to define the Lie differentials of fields and frame. Let

$$X=(t,x,y,z)^T, \quad \eta=\text{diag}(1,-1,-1,-1), \quad \delta\varepsilon=(a,b,c,u,v,w)\to 0,$$

then we have infinitesimal Lorentz transformation

$$X'=\Lambda X, \qquad \Lambda^+\eta\Lambda=\eta+\mathrm{O}(|\delta\varepsilon|^2),$$

in which

$$\Lambda=\begin{pmatrix} 1 & a & b & c \\ a & 1 & w & -v \\ b & -w & 1 & u \\ c & v & -u & 1 \end{pmatrix}=I+\mathrm{O}(|\delta\varepsilon|), \qquad \delta X=X'-X=\begin{pmatrix} ax+by+cz \\ at+wy-vz \\ bt+uz-wx \\ ct+vx-uy \end{pmatrix}.$$

The infinitesimal generator of $SO(1,3)$ is defined by

$$\begin{gathered} J=\delta x^k\partial_k=K^x a+K^y b+K^z c+J^x u+J^y v+J^z w, \\ K^x=(x\partial_t-t\partial_x), \quad K^y=(y\partial_t-t\partial_y), \quad K^z=(z\partial_t-t\partial_z), \\ J^x=(z\partial_y-y\partial_z), \quad J^y=(x\partial_z-z\partial_x), \quad J^z=(y\partial_x-x\partial_y). \end{gathered}$$

In the flat Minkowski space-time, we have the corresponding Lie algebra satisfying

$$\begin{gathered} [K^j,K^k]=[J^j,J^k]=\epsilon^{jkl}J_l, \qquad J_l=\delta_{lk}J^k, \\ [K^j,J^k]=\epsilon^{jkl}K_l, \qquad [K^j,J^j]=0. \end{gathered}$$

In which the subalgebra $\{J^k\}$ corresponds to the rotation group $SO(3)$.

For a scalar field $\phi(\mathbf{x})$, its Lie differential is defined as

$$\widetilde{\delta}\phi(\mathbf{x})\equiv\lim_{\delta\varepsilon\to 0}(\phi(\mathbf{x}')-\phi(\mathbf{x}))=(\delta x^k\partial_k)\phi(\mathbf{x})=J\phi(\mathbf{x}).$$

For a vector field $\mathbf{A}(\mathbf{x})\leftrightarrow A=(A^0,A^1,A^2,A^3)^T$, its Lie differential is defined as

$$\begin{aligned} \widetilde{\delta}\mathbf{A}(\mathbf{x}) &\equiv \gamma_a\lim_{\delta\varepsilon\to 0}(A'^a(\mathbf{x}')-A^a(\mathbf{x}))\leftrightarrow\lim_{\delta\varepsilon\to 0}(A'(\mathbf{x}')-A(\mathbf{x})) \\ &= \lim_{\delta\varepsilon\to 0}((A'(\mathbf{x}')-A'(\mathbf{x}))+(A'(\mathbf{x})-A(\mathbf{x}))) \\ &= (\delta x^k\partial_k)A(\mathbf{x})+(\Lambda-I)A=(J+\Lambda-I)A(\mathbf{x}). \end{aligned} \tag{27}$$

On the other hand, by

$$\lim_{\delta\varepsilon\to 0}(A'(\mathbf{x})-A(\mathbf{x})) \quad\leftrightarrow\quad (aA^1+bA^2+cA^3)\gamma_0+(aA^0+wA^2-vA^3)\gamma_1+ (bA^0+uA^3-wA^1)\gamma_2+(cA^0+vA^1-uA^2)\gamma_3,$$

we can take it as $A^a\widetilde{\delta}\gamma_a$, thus we have the **L**ie differentials of frame as

$$(\widetilde{\delta}\gamma_a)=(a\gamma_1+b\gamma_2+c\gamma_3, a\gamma_0+v\gamma_3-w\gamma_2, b\gamma_0+w\gamma_1-u\gamma_3, c\gamma_0+u\gamma_2-v\gamma_1). \tag{28}$$

Substituting (28) into (27) we obtain the universal form

$$\widetilde{\delta}\mathbf{A}(\mathbf{x})=(JA^a)\gamma_a+A^a\widetilde{\delta}\gamma_a. \tag{29}$$

3.5. Differentials of moving frame

The Frenet-Serret frame in an n-dimensional Euclidean space is derived in the study of Snygg [17, Ch7.1] and in the study of Hestenes and Sobczyk [28, pp.27-28]. In pseudo-Euclidean spaces or in spaces embedded in pseudo-Euclidean spaces, there are vectors with length zero. If any such vectors occur in the original basis, then the method outlined will not work. In the next we generalize the results to the pseudo-Euclidean space-time.

In the tangent space with a fixed point $\mathbf{x}_0$, there is a set of orthonormal basis vectors $\mathbf{e}_a$ constructed by Theorem 4. In the neighborhood $U(\mathbf{x}_0)=\{\mathbf{x};|\mathbf{x}-\mathbf{x}_0|<\varepsilon\}$, there is a null hypersurface $\eta_{ab}\delta X^a\delta X^b=0$, Separating U into the time-like region $\{U^t\subset U|\eta_{ab}\delta X^a\delta X^b>0\}$ and space-like region $\{U^s\subset U|\eta_{ab}\delta X^a\delta X^b<0\}$. In the time-like region U^t, C is a smooth curve segment cross $\mathbf{x}_0$. For all points on C, if $g_{\mu\nu}dx^\mu dx^\nu>0$ hold, then the curve segment is called a time-like curve. The length element of the arc C is given by $ds=\sqrt{g_{\mu\nu}dx^\mu dx^\nu}$. For a parameter t with $\mathbf{x}(t=0)=\mathbf{x}_0$, the arc length is calculated by

$$s(t)=\int_0^t\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}dt,\qquad \dot{x}^\mu=\frac{dx^\mu}{dt}.$$

Now we examine the moving frame attached on C. For convenience we take the arc length s as parameter, then we have n vectors constructed by derivatives of C

$$\{\tau_k=\frac{d^k\mathbf{x}}{ds^k};k=1,2,\cdots,n\}.$$

If $\tau_1\wedge\tau_2\wedge\cdots\wedge\tau_n\neq 0$ at $\mathbf{x}_0$, then τ_k's are linearly independent, and they are equivalent to the basis vectors $\{\mathbf{e}_a\}$. Thus we can constructed a natural or intrinsic frame $\{\mathbf{E}_a\}$ from $\{\tau_k\}$ by means of the Gram-Schmidt process.

Theorem 7. *If $pq\neq 0$ and $\tau_1\wedge\tau_2\wedge\cdots\wedge\tau_n\neq 0$, then the following sequence of vectors*

$$\mathbf{E}_1=\frac{\tau_1}{||\tau_1||},\mathbf{E}_2=\frac{\tau_2\wedge\tau_1}{||\tau_2\wedge\tau_1||}\mathbf{E}^1,\cdots,\mathbf{E}_n=\frac{\tau_n\wedge\cdots\wedge\tau_2\wedge\tau_1}{||\tau_n\wedge\cdots\wedge\tau_2\wedge\tau_1||}\mathbf{E}^1\mathbf{E}^2\cdots\mathbf{E}^{n-1}, \tag{30}$$

forms the orthonormal basis vectors of the tangent space-time at $\mathbf{x}_0$. In which the metric is given by

$$h_{ab}\equiv\mathbf{E}_a\cdot\mathbf{E}_b=\mathrm{diag}(1,\pm1,\cdots,\pm1)=h^{ab},\qquad \mathbf{E}^a=h^{ab}\mathbf{E}_b.$$

$||\cdot||$ *is the Calvet's norm of Clifford-Grassmann number.*

Proof. We prove it by induction. For $\mathbf{E}_1$, by the definition (30) we have

$$h_{11} = \mathbf{E}_1 \cdot \mathbf{E}_1 = 1, \qquad h^{11} = 1, \qquad \mathbf{E}^1 = \mathbf{E}_1.$$

For $\mathbf{E}_2$, by

$$\mathbf{E}_2 = \frac{\tau_2 \wedge \tau_1}{||\tau_2 \wedge \tau_1||}\mathbf{E}^1 = \frac{1}{||\tau_2 \wedge \tau_1||}\left(\tau_2||\tau_1|| - (\tau_2 \cdot \mathbf{E}_1)\tau_1\right) \in \Lambda^1,$$

we have $||\mathbf{E}_2|| = 1$ and

$$h_{22} = \mathbf{E}_2 \cdot \mathbf{E}_2 = \pm 1, \qquad h_{12} = h_{21} = \mathbf{E}_2 \cdot \mathbf{E}_1 = 0,$$

where $h_{22} = \pm 1$ means $h_{22} = 1$ or $h_{22} = -1$, which is determined by the values of τ_a.

Assuming for given $k < n$ the conclusions hold. For expression of $\mathbf{E}_k$, by $\mathbf{E}_a \cdot \mathbf{E}^b = \delta_a^b$ and Clifford calculus we have

$$\tau_k \wedge \cdots \wedge \tau_2 \wedge \tau_1 \propto \mathbf{E}_k \mathbf{E}_{k-1} \cdots \mathbf{E}_1 = \mathbf{E}_k \wedge \cdots \wedge \mathbf{E}_2 \wedge \mathbf{E}_1. \tag{31}$$

For the case $\mathbf{E}_{k+1}$, according to Gram-Schmidt process, let

$$\mathbf{X} = \tau_{k+1} - \sum_{a=1}^{k} h^{aa}(\tau_{k+1} \cdot \mathbf{E}_a)\mathbf{E}_a \in \Lambda^1,$$

then for $a \leq k$ we have $\mathbf{X} \cdot \mathbf{E}_a = 0$. By using Equation (31) we have

$$\tau_{k+1} \wedge (\tau_k \wedge \cdots \wedge \tau_2 \wedge \tau_1) \propto \tau_{k+1} \wedge (\mathbf{E}_k \wedge \cdots \wedge \mathbf{E}_2 \wedge \mathbf{E}_1) = \mathbf{X}\mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1. \tag{32}$$

Solving Equation (32) for $\mathbf{X}$, we obtain

$$\begin{aligned} \mathbf{X} &\propto (\tau_{k+1} \wedge \tau_k \wedge \cdots \wedge \tau_2 \wedge \tau_1)\mathbf{E}^1\mathbf{E}^2 \cdots \mathbf{E}^k, \\ \mathbf{E}_{k+1} &\equiv \frac{\mathbf{X}}{||\mathbf{X}||} = \frac{\tau_{k+1} \wedge \tau_k \wedge \cdots \wedge \tau_2 \wedge \tau_1}{||\tau_{k+1} \wedge \tau_k \wedge \cdots \wedge \tau_2 \wedge \tau_1||}\mathbf{E}^1\mathbf{E}^2 \cdots \mathbf{E}^k, \end{aligned}$$

and $h_{k+1,k+1} = \pm 1$, $h_{a,k+1} = \mathbf{E}_{k+1} \cdot \mathbf{E}_a = 0$, $(a \leq k)$. The proof is completed. □

Theorem 8. *The Frenet-Serret frame satisfies*

$$\frac{d}{ds}\begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \vdots \\ \mathbf{E}_n \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0 & \cdots & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & \cdots & 0 & 0 \\ 0 & -\kappa_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \kappa_{n-1} \\ 0 & 0 & 0 & \cdots & -\kappa_{n-1} & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{E}^1 \\ \mathbf{E}^2 \\ \vdots \\ \mathbf{E}^n \end{pmatrix}, \tag{33}$$

where $\kappa_a(s) \in \mathbb{R}$ are the characteristic quantities of the curve C. By selecting the sign of $\pm\mathbf{E}_k$, we can set all $\kappa_a > 0$. The hypercomplex formalism of Equation (33) *becomes*

$$\frac{d\boldsymbol{E}^k}{ds} = \boldsymbol{E}^k\boldsymbol{M} - \boldsymbol{M}\boldsymbol{E}^k, \qquad \boldsymbol{M} \equiv \frac{1}{2}\sum_{a=1}^{n-1}\kappa_a\boldsymbol{E}^a\boldsymbol{E}^{a+1}. \tag{34}$$

Proof. Since $\{\mathbf{E}_a\}$ are orthonormal basis vectors, we have

$$\frac{d\mathbf{E}_a}{ds} = \omega_{ab}\mathbf{E}^b, \qquad \omega_{ab} = -\omega_{ba}.$$

For $1 \le k \le n$ denoting

$$V_k \equiv \text{span}(\mathbf{E}_1, \mathbf{E}_2, \cdots, \mathbf{E}_k) = \text{span}\left(\frac{d\mathbf{x}}{ds}, \frac{d^2\mathbf{x}}{ds^2}, \cdots, \frac{d^k\mathbf{x}}{ds^k}\right).$$

By the definition of $\mathbf{E}_a$ we find $\frac{d}{ds}\mathbf{E}_a \in V_{a+1}$ for $a < n$, thus we have

$$\frac{d\mathbf{E}_a}{ds} = \sum_{b=1}^{a+1} \omega_{ab}\mathbf{E}^b, \qquad \frac{d\mathbf{E}_n}{ds} = \sum_{b=1}^{n} \omega_{ab}\mathbf{E}^b.$$

Noticing $\omega_{ab} = -\omega_{ba}$ we obtain Equation (33) by taking $\omega_{k,k+1} = \kappa_k$.

For $1 < k < n$, by Clifford calculus we have

$$\begin{cases} \mathbf{E}^k\mathbf{E}^a\mathbf{E}^{a+1} = \mathbf{E}^k(\mathbf{E}^a \wedge \mathbf{E}^{a+1}) = h^{ka}\mathbf{E}^{a+1} - h^{k,a+1}\mathbf{E}^a + \mathbf{E}^k \wedge \mathbf{E}^a \wedge \mathbf{E}^{a+1}, \\ \mathbf{E}^a\mathbf{E}^{a+1}\mathbf{E}^k = (\mathbf{E}^a \wedge \mathbf{E}^{a+1})\mathbf{E}^k = h^{k,a+1}\mathbf{E}^a - h^{ka}\mathbf{E}^{a+1} + \mathbf{E}^k \wedge \mathbf{E}^a \wedge \mathbf{E}^{a+1}. \end{cases} \tag{35}$$

Substituting Equation (35) into Equation (34), we find Equation (34) holds for $1 < k < n$. In the case $k = 1$, we should have $a > 0$ in Equation (35), thus Equation (34) holds for $k = 1$. In the case $k = n$, we should have $a < n$ in Equation (35), thus Equation (34) holds for $k = n$. The proof is completed. □

If the space-time is flat, then the moving frame $\{\mathbf{E}_a\}$ and the fixed frame $\{\mathbf{e}_a\}$ can be transformed each other. At this time, the change of the moving frame can be regarded as the change of the Lorentz transformation matrix Λ with the parameter s. Thus, the evolution equations of $\Lambda(s)$ can be established from Equation (33), so that the equations of motion (33) can be simplified. This method can be extended to the cases of high-dimensional surfaces, associated with the equivariant moving frame, computing the symmetry groups of partial differential equations and solving the group classification problem [29]. The new equivariant formulation of moving frames has led to a wide variety of novel and unexpected applications in pure and applied mathematics [30,31].

4. Covariant differentials of quaternion

The connection operators can be also defined for general hypercomplex numbers. In this section we take quaternion as example to show the covariant differentials. If taking the quaternions $\mathbb{H}$ as Clifford algebra $C\ell(\mathbb{R}^{0,2})$ and $(\mathbf{i}, \mathbf{j})$ as the generators, by the above procedure we can get a 2-dimensional differential geometry. But this treatment is obviously unnatural, because the intrinsically symmetric coordinates will be artificially graded. Therefore, we should introduce the coordinate transformation and the connection coefficients in a new way [23].

Let $(\mathbf{e}_a = I_2, \mathbf{i}, \mathbf{j}, \mathbf{k})$, we should have transformation rules of basis and coordinate as

$$d\mathbf{x} = dx^\mu\mathbf{e}_\mu = \delta x^a\mathbf{e}_a, \quad \mathbf{e}_\mu = f_\mu^{\ a}\mathbf{e}_a, \quad \mathbf{e}_a = f^\mu_{\ a}\mathbf{e}_\mu, \quad x^\mu = f^\mu_{\ a}\delta x^a.$$

Denoting the multiplication rules of basis vectors as

$$\mathbf{e}_a\mathbf{e}_b = C^c_{ab}\mathbf{e}_c, \qquad \mathbf{e}_\mu\mathbf{e}_\nu = C^\omega_{\mu\nu}\mathbf{e}_\omega, \qquad C^\omega_{\mu\nu} = f_\mu^{\ a}f_\nu^{\ b}f^\omega_{\ c}C^c_{ab}, \tag{36}$$

we get the multiplication matrix and the matrix form of the structure coefficients as

$$\mathbf{M} = (\mathbf{e}_j\mathbf{e}_k) = \begin{pmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \mathbf{e}_1 & -\mathbf{e}_0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ \mathbf{e}_2 & -\mathbf{e}_3 & -\mathbf{e}_0 & \mathbf{e}_1 \\ \mathbf{e}_3 & \mathbf{e}_2 & -\mathbf{e}_1 & -\mathbf{e}_0 \end{pmatrix}, \qquad \mathbf{C}^m = (C^m_{jk}) = \frac{\partial \mathbf{M}}{\partial \mathbf{e}_m}.$$

The structure coefficients matrices read $\mathbf{C}^0 = \text{diag}(1, -1, -1, -1)$ and

$$\mathbf{C}^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \ \mathbf{C}^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ \mathbf{C}^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

For given m we have $(C^m_{jk} = 0, \pm 1)$ and $|\det(\mathbf{C}^m)| = 1$. The matrix form of Equation (36) is given by

$$\mathbf{C}^\omega = F\left(f^\omega_{\ a}\mathbf{C}^a\right)F^T, \quad F = (f_\mu^{\ a}), \quad (f^\omega_{\ a}) = (F^T)^{-1}.$$

The determinant of the quaternion is a scalar, so we have

$$||d\mathbf{x}||^2 = \det(d\mathbf{x}) = \delta_{ab}\delta x^a \delta x^b = g_{\mu\nu}dx^\mu dx^\nu, \qquad g_{\mu\nu} = \delta_{ab} f_\mu^{\ a} f_\nu^{\ b}.$$

The above equations clarify the geometric meaning and the computing method of $f_\mu^{\ a}$. For an arbitrary quaternionic function $\mathbf{q} = q^\mu(\mathbf{x})\mathbf{e}_\mu$, the absolute derivative is defined as

$$d\mathbf{q} = (\partial_\alpha q^\beta + q^\mu K^\beta_{\alpha\mu})\mathbf{e}_\beta dx^\alpha, \qquad \mathfrak{d}_\alpha \mathbf{e}_\mu = K^\beta_{\alpha\mu}\mathbf{e}_\beta.$$

Substituting $\mathfrak{d}_\alpha \mathbf{e}_\mu = K^\beta_{\alpha\mu}\mathbf{e}_\beta$ into Equation (36), We obtain the consistent condition for the connection coefficients $K^\beta_{\alpha\mu}$ as

$$C^\gamma_{\mu\beta}K^\beta_{\alpha\nu} + C^\gamma_{\beta\nu}K^\beta_{\alpha\mu} - C^\beta_{\mu\nu}K^\gamma_{\alpha\beta} = \partial_\alpha C^\gamma_{\mu\nu}.$$

We have the following solution.

Theorem 9. *Suppose $a = 0, 1, 2, 3$ and $k = 1, 2, 3$, $p_a^{\ k} \in \mathbb{R}$ are any given smooth functions. Let*

$$\mathbf{P}_a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & p_a^{\ 3} & -p_a^{\ 2} \\ 0 & -p_a^{\ 3} & 0 & p_a^{\ 1} \\ 0 & p_a^{\ 2} & -p_a^{\ 1} & 0 \end{pmatrix}, \tag{37}$$

then we have

$$K^\mu_{\alpha\beta} = f^\mu_{\ c}\partial_\alpha f_\beta^{\ c} + f^\mu_{\ c} f_\alpha^{\ a} f_\beta^{\ b}(\mathbf{P}_a)_b^{\ c}. \tag{38}$$

Proof. By the properties of connection operator (10)-(13), we have

$$K^\mu_{\alpha\beta}\mathbf{e}_\mu = \mathfrak{d}_\alpha \mathbf{e}_\beta = \mathfrak{d}_\alpha(f_\beta^{\ b}\mathbf{e}_b) = (\partial_\alpha f_\beta^{\ b})\mathbf{e}_b + f_\alpha^{\ a} f_\beta^{\ b}\mathfrak{d}_a \mathbf{e}_b. \tag{39}$$

Denoting $\mathfrak{d}_a \mathbf{e}_b = (\mathbf{P}_a)_b{}^c \mathbf{e}_c$, by multiplication relation (36) we have

$$(\mathbf{P}_d)_a{}^e C^c_{eb} + (\mathbf{P}_d)_b{}^e C^c_{ae} = C^e_{ab} (\mathbf{P}_d)_e{}^c,$$

or in the form of matrix

$$\mathbf{P}_d \mathbf{C}^c + \mathbf{C}^c \mathbf{P}_d^T = \mathbf{C}^e (\mathbf{P}_d)_e{}^c. \tag{40}$$

The solution of Equation (40) can be easily found. By straightforward calculation we obtain (37). Substituting the solution $\mathbf{P}_a$ into $\mathfrak{d}_a \mathbf{e}_b = (\mathbf{P}_a)_b{}^c \mathbf{e}_c$, and then into Equation (39), we obtain Equation (38). $\mathbf{P}_a$ is similar to the torsion in a space-time. The proof is completed. □

5. Discussion and conclusion

In recent years, it has been strongly suggested that theoretical physicists should be all familiar with the differential forms. However, in the context of the Clifford algebra, the differential forms and the co-forms can be greatly simplified. In differential geometry, dx^μ and ∂_μ are presented as coordinate basis vectors of dual spaces in abstract significance. Indeed, dx^μ and γ^μ have the same coordinate transformation laws, because (2) is independent of coordinate system. Thus taking dx^μ as a basis vector usually does not lead to contradictory conclusions. But the true geometric meaning of dx^μ is the coordinate increment, which is just a real variable rather than a vector. The double roles of dx^μ in differential geometry leads to unnecessary complexity and confusion. A manifold is essentially a generalization of the vector space in curved space-time, so the direct introduction of the basis $\{\gamma^\mu\}$ at each point will greatly simplify the description.

The differentials of frame are always equivalent to a linear transformation of the frame, and the linear transformation is distinct in different contexts. In Riemannian geometry, the linear transformation is the connection operator (10). Corresponding to the variation of metric, the variation of the frame is given by Equation (23) or Equation (24). In a different context, the definitions of differential of the frame are different, so the corresponding linear transformation is also different. This unified view of the frame or basis vectors will bring great convenience to the research and application of Clifford algebra.

Acknowledgments: I am very grateful to James M. Nester for his enlightening discussions and suggestions.

Conflict of interest: The author declares no conflict of interest.

References

1. Clifford WK. Applications of Grassmann's Extensive Algebra. American Journal of Mathematics. 1878; 1(4):350-358, doi: 10.2307/2369379
2. Grassmann HG. Extension Theory, (Die Ausdehnungslehre von 1862) Lloyd C. Kannenberg, trans. American Mathematical Society, Rhode Island. London Mathematical Society Publishing; 2000.
3. Hamilton WR. Elements of quaternions. Longmans, Green, & Company Publishing; 1866.
4. Doran C, Lasenby A. Geometric algebra for physicists. Cambridge University Press Publishing; 2003.
5. Macdonald A. Linear and geometric algebra. Nottingham: Alan Macdonald Publishing; 2010.
6. Pozo JM, Sobczyk G. Geometric algebra in linear algebra and geometry. Acta Applicandae Mathematica. 2002; 71: 207-244. doi:10.1023/A:1015256913414
7. Hestenes D. Observables, operators, and complex numbers in the Dirac theory. J. Math. Phys. 1975; 16:556-572. doi:10.1063/1.522554

8. Keller J, Rodriguez-Romo S. Multivectorial representation of Lie groups. Int. J. Theor. Phys. 1991; 30: 185-196. doi:10.1007/BF00670711
9. Shirokov DS. On inner automorphisms preserving subspaces of Clifford algebras. Adv. Appl. Clifford Algebras. 2021; 31: 30-52. doi:10.1007/s00006-021-01135-6
10. Keller J. Spinors and Multivectors as a Unified Tool for Spacetime Geometry and for Elementary Particle Physics. Int. J. Theor. Phys. 1991; 30: 137-184. doi:10.1007/BF00670710
11. Ablamowicz R, Sobczyk G. Lectures on Clifford (geometric) algebras and applications. Available online: https://link.springer.com/book/10.1007/978-0-8176-8190-6 (accessed on 20 June 2024)
12. Ablamowicz R. Clifford Algebras: Applications to Mathematics, Physics, and Engineering. Available online: https://link.springer.com/book/10.1007/978-1-4612-2044-2 (accessed on 20 June 2024)
13. Lounesto P. Clifford Algebras and Spinors. Cambridge Univ. Press; 2001.
14. Breuils S, Tachibana K, Hitzer, E. New applications of Clifford's geometric algebra. Adv. Appl. Clifford Algebras. 2022; 32:17-55. doi:10.1007/s00006-021-01196-7
15. Gu Y-Q. A Note on the Representation of Clifford Algebras. J. Geom. Symmetry Phys. 2021; 62: 29-52. doi:10.7546/jgsp-62-2021-29-52
16. Snygg J. Clifford Algebra-A Computational Tool For Physicists. Oxford University Press; 1997.
17. Snygg J. A new approach to differential geometry using Clifford's geometric algebra. Springer Science & Business Media Publishing; 2011.
18. Hehl FW, Von der Heyde P, Kerlick GD, Nester JM. General relativity with spin and torsion: Foundations and prospects. Rev. Mod. Phys. 1976; 48: 393-416. doi:10.1103/RevModPhys.48.393.
19. Nester JM. Special orthonormal frames. Journal of mathematical physics. 1992; 33: 910-913,
20. Dimakis A, Muller-Hoissen F. Clifform calculus with applications to classical field theories. Classical and Quantum Gravity. 1991; 8: 2093. doi:10.1088/0264-9381/8/11/018
21. Nester JM, Tung RS, Zhytnikov VV. Some Spinor-Curvature Identities. Classical and Quantum Gravity. 1994; 11: 983. doi:10.1088/0264-9381/11/4/014
22. Gu Y-Q. Theory of Spinors in Curved Space-Time. Symmetry. 2021; 13: 1931, doi:10.3390/sym13101931
23. Gu Y-Q. Clifford A. Hypercomplex Numbers and Nonlinear Equations in Physics. Geometry Integrability and Quantization. 2023; 25: 47-72. doi:10.7546/giq-25-2023-47-72
24. Gu Y-Q. Clifford Algebra and Hypercomplex Number as well as Their Applications in Physics. J. Appl. Math. Phys. 2022; 10: 1375-1393. doi:10.4236/jamp.2022.104097
25. Gu Y.-Q. Miraculous Hypercomplex Numbers. Mathematics and Systems Science. 2023; 1: 2258. doi:10.54517/mss.v1i1.2258
26. Calvet RG. On Matrix Representations of Geometric (Clifford) Algebras. J. Geom. Symmetry Phys. 2017; 43: 1-36. doi:10.7546/jgsp-43-2017-1-36
27. Cartan É. Riemannian Geometry in an Orthogonal Frame (translated into English by V. Goldberg). World Scientific Publishing; 2001.
28. Hestenes D, Sobczyk G. Clifford algebra to geometric calculus: a unified language for mathematics and physics (Vol. 5). Springer Science & Business Media Publishing; 2012
29. Morozov O. Moving coframes and symmetries of differential equations. J. Phys. Am. 2002; 35: 2965- 2977. doi: 10.1088/0305-4470/35/12/317
30. Kogan IA, Olver PJ. Invariant Euler-Lagrange equations and the invariant variational bicomplex. Acta Appl. Math. 2003; 76: 137-193. doi:10.1023/A:1022993616247
31. Arnaldsson Ö. Involutive moving frames. Diff. Geom. Appl. 2020; 69: 101603. doi:10.1016/j.difgeo.2020.101603