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Article

Regularity, synthesis, rigidity and analytic classification for linear ordinary differential equations of second order

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Abstract: We study second order linear homogeneous differential equations a(x)y" + b(x)y' + c(x)y = 0 with analytic coefficients in a neighborhood of a regular singularity in the sense of Frobenius. These equations are model for a number of natural phenomena in sciences and applications in engineering. We address questions which can be divided in the following groups: (i) Regularity of solutions. (ii) Analytic classification of the differential equation. (iii) Formal and differentiable rigidity. (iv) Synthesis and uniqueness of ODEs with a prescribed solution. Our approach is inspired by elements from analytic theory of singularities and complex foliations, adapted to this framework. Our results also reinforce the connection between classical methods in second order analytic ODEs and (geometric) theory of singularities. Our results, though of a clear theoretical content, are important in justifying many procedures in the solution of such equations.

Keywords: second order linear ODE; Frobenius method; regular singularity; rigidity

1. Introduction and main results

Second order linear homogeneous differential equations appear in many concrete problems in natural sciences, as physics, chemistry, meteorology and even biology. Throughout this paper we shall be dealing with analytic linear homogeneous second order equations, i.e., ODEs of the form

$$a(x)y'' + b(x)y' + c(x)y = 0 \qquad (1)$$

with analytic (real or complex) coefficients defined in some domain $D \subset \mathbb{R}, \mathbb{C}$, of the form $D\colon |x - x_0| < R$. Solving these equations is an important subject of active study in mathematics. The search for solutions via power series is recently resumed with the aid of scientific computing. A powerful method is due to Frobenius [1–4]. Frobenius method is algorithmic and works pretty well in a suitable class of second order ODEs, so called regular singular ODEs around x_0. There is a number of recent papers in this classical subject showing that, with the recent developments in computer science [5], it may be quite useful as a powerful tool in solving ODEs associate to problems in Physics, Engineering and Sciences in general [6]. A non-exhaustive list is as follows: [7–11]. In Roques [10] for instance, the author uses a variant of Frobenius method in order to study linear Mahler equations.

The class of second order ODEs having a regular singularity is quite wide and interesting, containing many of the classical ODEs found in problems coming from Physics. For instance we can mention Bessel equation and Laguerre equation.

The point x_0 is a singular point of the ODE (1) if $a(x_0) = 0$. It is a regular singularity if $\lim_{x\to x_0}(x-x_0)\frac{b(x)}{a(x)}$ and $\lim_{x\to x_0}(x-x_0)^2\frac{c(x)}{a(x)}$ are finite. Frobenius method consists in associating to the original ODE an Euler equation given by $(x-x_0)^2y''+p_0(x-x_0)y'+q_0y=0$ where $p_0=\lim_{x\to x_0}(x-x_0)\frac{b(x)}{a(x)}$ and $q_0=\lim_{x\to x_0}(x-x_0)^2\frac{c(x)}{a(x)}$. The basic solutions of the equation are then obtained as multiples of the solutions of the Euler equation, by some power series.

We shall refer to a second order analytic ODE (1), analytic in $D\subset\mathbb{K}$, and with a regular singularity (or ordinary point) at $x_0\in D$, as a Frobenius type ODE centered at x_0.

The classical method of Frobenius is found in the books of Hartman [12] and Birkoff-Rota [13]. In this paper, we follow the exposition presented in Boyce and DiPrima [1] and, sometimes the one from Coddington [2]. Next, we have the following classical result from Frobenius, which is stated as Theorem 5.6.1 in Boyce and DiPrima [1] (pages 293, 294), Theorem 3 and 4 in Coddington [2] (pages 158, 175), and in Frobenius [14].

Theorem of Frobenius. Assume that the ODE $a(x)y''+b(x)y'+c(x)y=0$ has a regular singularity at $x=x_0$, where the functions $a(x), b(x)$ and $c(x)$ are analytic with convergent power series in $|x-x_0|<R$. Then there is at least one solution of the form

$$y(x)=|x-x_0|^r\sum_{n=0}^{\infty}d_n\,(x-x_0)^n$$

where r is a suitable root of the indicial equation, $d_0=1$ and the series converges for $|x-x_0|<R$.

Frobenius method actually consists in looking for solutions of the form

$$y_1(x)=|x-x_0|^r\sum_{n=0}^{\infty}d_n\,(x-x_0)^n$$

where r is the zero of the indicial equation having greater real part. Whether there is a second linearly independent solution is related to the roots of the indicial equation. Indeed, there is some zoology and we may have to consider other solutions of the form

$$y_2(x)=ky_1(x)log|x-x_0|+|x-x_0|^{\tilde{r}}\sum_{n=0}^{\infty}\check{d}_n\,(x-x_0)^n.$$

in the special case where there is a second root $\tilde{r}$ of the indicial equation such that $r-\tilde{r}\in\mathbb{N}$.

As mentioned above, Frobenius method is described in the book of Boyce-DiPrima and in the book of E. Coddington. Nevertheless, a more detailed and complete proof of the convergence of the formal part of the solutions in the disc $|x-x_0|<R$ (i.e., in the common disc where the coefficients of the ODE are analytic) seems to be found only in Coddington's book.

Recent research has explored advanced techniques for studying the oscillatory behavior of differential equation solutions, highlighting significant contributions from Omar Bazighifan and Rami Ahmad El-Nabulsi. For instance, Bazighifan and El-Nabulsi developed improved Hille-type oscillation criteria for second-order nonlinear functional dynamic equations on arbitrary time scales, using Riccati transformation and integral averaging methods. These results demonstrate improvements compared to previously established criteria in the literature and are illustrated with examples showcasing their importance [15,16].

Additionally, new oscillation criteria for fourth-order neutral delay differential equations have been proposed, introducing Riccati substitutions to obtain oscillation criteria without requiring unknown functions [17,18]. Further studies have developed enhanced Hille-type oscillation conditions for second-order quasilinear dynamic equations, extending previous research and demonstrating the importance of the obtained results [19]. Novel criteria for the oscillation of solutions in advanced noncanonical dynamic equations have been established, addressing previously unresolved issues in the literature [20]. These advances are directly related to the themes of formal and differentiable rigidity, as well as the synthesis and uniqueness of ODEs with prescribed solutions addressed in this work.

These techniques and recent results complement and generalize previous findings, highlighting the intersection between classical and modern methods in the theory of ODEs and their practical applications [21]. The analysis of the oscillatory properties of solutions of higher-order differential equations, using Riccati transformations and integral averaging techniques, remains a vital research area, with results improving and complementing well-known oscillation criteria [18].

In this paper we present a contribution to the study of second order linear analytic differential equations with a regular singularity. Indeed, this paper is a continuation of the work by León and Scárdua [4] where initiate our study of this special class of ODEs, by using ideas from the Frobenius method together with geometric ingredients. Our method allows the study of higher-order differential equations, as in León et al. [3].

Regarding the method of Frobenius we must mention that there are versions of this method for other classes of differential equations, e.g., Fuchsian equations [22]. A PDE version of this classical method can be found in Martínez and Azevedo [23].

Frobenius method for order n linear homogeneous complex analytic equations can be found in the classical book of Ince [24] throughout Chapter XVI (§16.1 page 396 and on). Indeed, in §16.1 (page 396) the author proceeds in the classical way to prove the existence of a formal solution of the form $z^r \sum_{n=0}^{\infty} a_n z^n$ where r is a (maybe complex) root of the indicial equation. Next, in §16.2 page 398 the same author makes use of Cauchy Integral Formula to prove, always in the case of a complex regular singular point, the existence of a first "convergent" solution of the form $z^r \sum_{n=0}^{\infty} a_n z^n$ where the power series $\sum_{n=0}^{\infty} a_n z^n$ converges in some neighborhood of the singular point. Next, in §16.3 page 400 the author hints the form of the other possible solutions according to the disposition of the roots of the indicial equation. Our results in León et al. [3], León and Scárdua [4], and the present work provide solid confirmation of this statement. Indeed, we discuss more accurately the connection between the disposition of the roots of the indicial equation and the types of the solutions. Moreover, our proof of the convergence is more elementary, without the need of Cauchy Integral Formula, and our estimates in the coefficients of the power series are more clearly and may be used in a computing process in order to control the speed of the convergence, something which is fundamental in applications to engineering. Finally, it is not clear from the argumentation in Ince [24] that the real case, i.e., the case of real ODEs may be treated in the same way. Indeed, the roots of the indicial equation may be complex non-real and therefore the coefficients in the power series in the formal solution may be complex non-real, which would not be useful in the search of real solutions.

1.1. Main results

In the current paper we address some questions which can be divided in the following groups:

- Regularity of solutions;
- Synthesis and analytic classification of these type of differential equations.

One of the main gains in this paper is that, our results also reinforce the connection between classical methods in second order analytic ODEs and (geometric) theory of singularities. Just to exemplify, in León and Scárdua [4] we associate to such an EDO (linear of second order with a Frobenius singularity) a Riccati foliation and its monodromy which we called monodromy of the original ODE. Another geometric object is a codimension one foliation which is associated to the ODE in the three-dimensional space [4]. Following this line we address the questions above.

In what follows we shall give a more detailed description of our main results:

1.1.1. Regularity of solutions

We study the convergence of formal or differentiable solutions defined in a neighborhood or away from the singular point.

Theorem A (analyticity and extension of smooth solutions). *Consider a second order ordinary differential equation given by Equation (1) with a, b, c analytic functions in some interval $I = (x_0 - \delta, x_0 + \delta) \subset \mathbb{R}$. Assume that the point $x_0 \in I$ is an ordinary point or a regular singularity of the ODE. Let $\tilde{y}$ be a smooth solution of the ODE defined in some subinterval $J \subset I$. Then $\tilde{y}$ extends to a convergent solution of the ODE defined in I. If $x_0 \in J$ then the extension of $\tilde{y}$ is analytic in I.*

we also study when a pair of solutions having some asymptotic expansions at the singular point, actually correspond to convergent solutions of the differential equation. We shall prove:

Theorem B (regularity via asymptotic expansion). *Consider a second order ordinary differential Equation (1) with a, b, c analytic functions at $x_0 \in \mathbb{R}$. Suppose that there exist two smooth solutions $\tilde{y}_1(x)$ and $\tilde{y}_2(x)$ defined in an interval $(x_0, x_0 + \delta)$ and having linearly independent asymptotic expansions at x_0. Then x_0 is an ordinary point or a regular singular point of the ODE. Moreover, $\tilde{y}_1(x)$ and $\tilde{y}_2(x)$ extend to analytic solutions in some interval $(x_0 - \epsilon, x_0 + \epsilon)$.*

Existence of smooth solutions versus analytic solutions is addressed. We also prove that the analytic character of a nonsingular ODE is assured by the existence of two analytic solutions.

Theorem C. *A linear ODE of second order*

$$y'' + b(x)y' + c(x)y = 0 \qquad (2)$$

with continuous coefficients $b(x), c(x)$ in some interval $I \subset \mathbb{R}$ is analytic in I if and only if it admits two linearly independent analytic solutions in I.

1.1.2. Analytic classification of the differential equation

In this second part we study possible analytic normal forms of such differential equations. We shall call the form $x^2y'' + xb(0)y' + \tilde{c}(x)y = 0$ of a Frobenius type ODE $x^2y'' + xb(x)y' + c(x)y = 0$, the reduced normal form of the ODE. We prove the existence of the reduced normal form (Lemma 3.1), to which we associate a characteristic function. The characteristic function of a Frobenius ODE is

$$\mu(x) = \frac{\left(x^{\lambda}\varphi'(x)\right)'}{x^{\lambda}\varphi(x)}$$

where $y(x) = x^r\varphi(x)$ is a solution of the ODE in the reduced form $x^2y'' + xb(0)y' + c(x)y = 0$ and $\lambda = 2r + b(0)$ (cf. Definition 3.2). The vanishing of the characteristic function is equivalent to the fact that the equation is of Euler type. Also, by associating to the original differential equation, a Riccati differential equation, we introduce a notion of local monodromy for such the second order differential equation. This local monodromy will be a Moebius map and is strongly related to the analytic classification of the Riccati equation and therefore of the original second order equation. Then, in a theoretical approach, we study the classifying space of such second order equations in terms of elements of a first cohomology group, associate to the group of automorphisms of Euler equations, and some other affine parameters. We are then able to prove the existence of a differential equation admitting a predetermined solution and study the dependence of the equation on the solution.

Theorem D (existence of ODEs with a prescribed solution). *Given $r \in \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and a convergent power series $\varphi(x) \in \mathbb{K}\{x\}$, with $\varphi(0) = 1$, there is a Frobenius type ODE $x^2y'' + xb(x)y' + c(x)y = 0$ with analytic coefficients $b(x), c(x) \in \mathbb{K}\{x\}$, which has $y(x) = x^r\varphi(x)$ as a solution. The coefficients $b(x), c(x)$ are analytic in some disc centered at the origin.*

The above ODE is necessarily unique, except for the case where the solution satisfies a liouvillian relation (see §2.6 for the detailed notion). In the case of polynomial coefficients this is of special interest according to the work of Singer [25]. These results bring some elements from theory of singularities to our setting.

Denote by $\mathcal{F}_{\mathbb{R}}$ the space of germs Frobenius real analytic ODEs at the origin $0 \in \mathbb{R}$. We shall consider the Frobenius map $\mathcal{F}: \mathcal{F}_{\mathbb{R}} \to \mathbb{R} \times \mathbb{R}\{x\}$, that associates to each germ of ODE of Frobenius type $F \in \mathcal{F}_{\mathbb{R}}$, the zero with greatest real part $r \in \mathbb{R}$ of the indicial equation and a series $\varphi(x) \in \mathbb{R}\{x\}$ such that $x^r\varphi(x)$ is a solution of F.

Since each Frobenius type ODE centered at $0 \in \mathbb{R}$ admits a solution of the form $x^r\varphi(x)$ with $\varphi(x) \in \mathbb{R}\{x\}$, we can state:

Theorem E (Analytic classification). *The Frobenius map $\mathcal{F}: \mathcal{F}_{\mathbb{R}} \to \mathbb{R} \times \mathbb{R}\{x\}$ is surjective. It is also injective when restricted to the subspace $\mathcal{F}'_{\mathbb{R}} \subset \mathcal{F}_{\mathbb{R}}$ of those Frobenius type ODEs not admitting liouvillian solutions.*

Finally, we shall study the rigidity of equivalences for Frobenius type ODEs. We refer to Definition 3.3 for the precise notion of formal, C^2 or analytical equivalence between two Frobenius equations.

Theorem F (formal-smooth rigidity for Frobenius ODEs). *Any two Frobenius type ODEs which are formally or C^2 equivalent, are analytically equivalent. Indeed, a formal or C^2 equivalence between two Frobenius type ODEs is analytic.*

2. Regularity of the solutions

The idea of regularity is related to intuitive notion that the solutions of an ODE must have the same class of differentiability than the one of (the coefficients of) the ODE. Indeed, this is quite sure in the ordinary (non-singular) case. Nevertheless, for the case of a singularity, this may not happen (cf. Examples 2.1 and 2.2). Let us see how to overcome this difficult in the Frobenius case.

2.1. Regularity of smooth solutions

By a smooth solution of a second order ODE we shall mean a two times differentiable function that satisfies the ODE. Thus, for the case we are dealing with, given by (1) with coefficients defined in some domain $D \subset \mathbb{K}$, by a smooth solution of this ODE we shall mean a function $\varphi: D \to \mathbb{K}$, twice differentiable in D, such that at each point $x \in D$ we have $a(x)\varphi''(x) + b(x)\varphi'(x) + c(x)\varphi(x) = 0$.

In the case of a regular singular point, we have the following regularity theorem for the solutions of an analytic ODE.

In order to establish this result we recall some properties of the wronskian.

2.2. The wronskian

Consider the linear homogeneous second order ODE (1) with a, b, c are differentiable real or complex functions defined in some open domain $D \subset \mathbb{K}$. We may assume that D is an open disc centered at the origin $0 \in \mathbb{K}$. We make no hypothesis on the nature of the point $x = 0$ as a singular or ordinary point of (1). Given two solutions y_1 and y_2 of (1) their wronskian is defined by $W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$.

Lemma 2.1. *The wronskian $W(y_1, y_2)$ satisfies the following first order ODE*

$$a(x)w' + b(x)w = 0 \tag{3}$$

This is a well-known fact and we shall not present a proof, which can be done by straightforward computation. Most important, the above fact allows us to introduce the notion of wronskian of a general second order linear homogeneous ODE as (1) as follows:

Definition 2.1. *The wronskian of (1) is defined as the general solution of (3).*

Hence, in general the wronskian is of the form

$$W(x) = kexp\left(-\int^x \frac{b(\eta)}{a(\eta)} d\eta\right)$$

where k is a constant.

A well-known consequence of the above formula is the following:

Lemma 2.2. *Given solutions $y_1(x), y_2(x)$ the following conditions are equivalent:*

1) $W(y_1, y_2)(x)$ *is identically zero.*
2) $W(y_1, y_2)(x)$ *vanishes at some point* $x = x_0$.
3) $y_1(x), y_2(x)$ *are linearly dependent.*

Proof of Theorem A. Let y_1, y_2 be linearly independent convergent solutions of the ODE defined in I as given by classical Frobenius theorem. Let us first prove the extension of $\tilde{y}$ as a convergent solution defined in I. We know that the Wronskian $W_j := W(y_j, \tilde{y})$ is a solution of the first order ODE $a(x)W' + b(x)W = 0$ in the interval J, so that it writes

$$W_j(x) = k_j exp\left(-\int_{x_1}^{x} \frac{b(\xi)}{a(\xi)} d\xi\right)$$

for some fixed point $x_1 \in J$ and some constant $k_j \in \mathbb{R}$. If $k_j = 0$ then $W_j \equiv 0$ in J and therefore y_j and $\tilde{y}$ are linearly dependent in J. This implies the desired extension in this case.

Assume now that $k_j \neq 0, j = 1,2$. This implies that $W_1 = \frac{k_1}{k_2} W_2$. Therefore

$$\frac{y_1'\tilde{y} - y_1(\tilde{y})'}{(\tilde{y})^2} = \frac{k_1}{k_2}\left(\frac{y_2'\tilde{y} - y_2(\tilde{y})'}{(\tilde{y})^2}\right)$$

This implies

$$\left(\frac{y_1}{\tilde{y}}\right)' = \frac{k_1}{k_2}\left(\frac{y_2}{\tilde{y}}\right)'$$

and therefore $\tilde{y}$ is a linear combination of y_1 and y_2 proving the extension of $\tilde{y}$ to I also in this case.

Finally, if $x_0 \in J$ then $\tilde{y}$ is a linear combination of the solutions y_1 and y_2 with the property that $\tilde{y}$ is smooth (well-defined without indetermination point) at x_0. This shows that $\tilde{y}$ is indeed an analytic function at x_0 (the only possible indetermination points of a solution of a Frobenius equation are of the form $|x - x_0|^r$ and $|x - x_0|^r log|x - x_0|$). It is easy to conclude that $\tilde{y}$ is analytic in I. □

2.3. Solutions admitting asymptotic expansion

We start with some classical definitions. By a formal function centered at $x_0 \in \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ we shall mean a formal power series $\hat{y}(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ with constant coefficients $a_n \in \mathbb{K}$. We recall the classical definition from Wasow [26], page 32, below:

Definition 2.2. *Let $J \subset \mathbb{R}$ be an interval of the form $J = (x_0, x_0 + \delta)$ or $J = (x_0 - \delta, x_0)$. A real function $f: J \to \mathbb{R}$ is said to admit a formal power series $\hat{f}(x) = \sum_{j=0}^{\infty} a_j x^j$ as asymptotic expansion at the point x_0 if for each $k \in \mathbb{N}$, there exists a constant $A_k > 0$ such that*

$$\left|f(x) - \sum_{j=0}^{k-1} a_j x^j\right| \leq A_k |x|^k$$

for all $x \in J$. This will be denoted as $f(x) \simeq \sum_{j=0}^{\infty} a_j x^j$ as $x \to x_0$.

The proof of Theorem B is based in a series of lemma, starting from the one below:

Lemma 2.3. *Consider a second order ordinary differential equation given by (1) with a, b, c are analytic functions at $x_0 \in \mathbb{R}$. Suppose (1) has at x_0 an ordinary point or a regular singular point. Given any formal solution $\hat{y}(x)$ of the ODE centered at x_0 then this solution converges, indeed this solution is a linear combination of any two smooth linearly independent solutions of the ODE defined in some interval $(x_0 - \delta, x_0 + \delta)$.*

Proof. Let $y_1(x), y_2(x)$ be two convergent solutions defined in some interval $(x_0 - \delta, x_0 + \delta)$. We consider the wronskian $W_j = W(y_j, \hat{y}) = y_j'\hat{y} - y_j(\hat{y})'$, $j =$

1,2. Since $y_j, \hat{y}$ are solutions of the ODE we know that the wronskian is a formal function which is a formal solution of the order one linear homogeneous ODE $a(x)Y' + b(x)Y = 0$. Proceeding as in the proof of Theorem A we must have that (the power series representing) $\hat{y}$ is a linear combination of (the power series of) y_1 and y_2 at x_0. □

We shall need the following couple of results.

Proposition 2.1 *(Theorem A and Theorem B in the study of León and Scárdua [4], page 2102). Consider a second order ordinary differential equation given by (1) with* a, b, c *are real or complex analytic functions at* $x_0 \in \mathbb{R}, \mathbb{C}$.

1) Assume that there two linearly independent formal solutions $\hat{y}_1(x)$ and $\hat{y}_2(x)$ centered at x_0 of equation (1). Then x_0 is an ordinary point or a regular singular point of (1).
2) Suppose that (1) has at x_0 an ordinary point or a regular singular point. Then a formal solution $\hat{y}(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ is always convergent in some neighborhood $|x - x_0| < R$ of the point x_0. Indeed, this solution converges in the same disc type neighborhood where the coefficients $a(x), b(x), c(x)$ are analytic.

Proof of Theorem B. Let us denote by $\hat{y}_1, \hat{y}_2$ the asymptotic expansions of $\tilde{y}_1, \tilde{y}_2$ respectively, at x_0. Since $\tilde{y}_j(x)$ is a smooth solution of the ODE we have $a(x)\tilde{y}_j''(x) + b(x)\tilde{y}_j'(x) + c(x)\tilde{y}_j(x) = 0$. Let $\hat{y}_j(x) = \sum_{j=0}^{\infty} a_j x^j$. Then we can introduce $\hat{y}_j'(x) = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$ and $\hat{y}_j''(x) = \sum_{n=2}^{\infty} n\,(n-1) a_n x^{n-2}$. It is well-known that we also have $y_j'(x) \simeq \hat{y}_j'$ as $x \to x_0$ and $y_j''(x) \simeq \hat{y}_j''(x)$ as $x \to x_0$. Since $a(x), b(x), c(x)$ are analytic we conclude from that $a(x)\hat{y}_j''(x) + b(x)\hat{y}_j'(x) + c(x)\hat{y}_j(x) = 0$ as formal expressions. This shows that the ODE has two linearly independent formal solutions at x_0. By Proposition 2.1 above we conclude that x_0 is an ordinary point or a regular singularity of the ODE. Moreover, we also obtain that the series $\hat{y}_j$ are convergent. Theorem A then implies that the solutions $\tilde{y}_j$ admit extensions as analytic functions in $I = (x_0 - \epsilon, x_0 + \epsilon)$. □

2.4. Convergence and extension of solutions

Our next result implies that the search for solutions of a Frobenius type ODE can be performed at a non-singular point and by formal means only.

Theorem 2.1 *(formal solution at some point* x_1 *with* x_0 *regular singularity). Consider a second order ordinary differential equation (1) with* a, b, c *are analytic functions at* $x_0 \in I \subset \mathbb{R}$. *Suppose that the ODE has at* x_0 *an ordinary point or a regular singular point. Given a formal solution* $\hat{y}(x) = \sum_{n=0}^{\infty} c_n (x - x_1)^n$, *centered at any ordinary point* $x_1 \in I$, *then this solution is convergent, indeed it represents a solution converging in I.*

Proof. It is enough to observe that because the ODE has an ordinary or regular singular point at x_0 then there are two linearly independent solutions y_1, y_2 which are convergent in I, and analytic in $I \backslash \{x_0\}$. These two solutions are linearly independent in any open nonempty interval $J \subset I$. Now, given the point $x_1 \in I$ and a formal solution $\hat{y} = \sum_{n=0}^{\infty} c_n (x - x_1)^n$, centered at x_1, we know from Lemma 2.3 that this solution is a linear combination of the convergent solutions y_1, y_2 once regarded as analytic series centered at x_1. This implies that $\hat{y}$ represents a convergent solution of the ODE defined in I. □

Corollary 2.1 (analytic solution some interval). Consider a second order ordinary differential Equation given by (1) with a, b, c are analytic functions at $x_0 \in I \subset \mathbb{R}$. Suppose (1) has at x_0 an ordinary point or a regular singular point. If $\tilde{y}$ is an analytic solution of the ODE defined in some interval $J \subset I$ (not necessarily containing x_0) then $\tilde{y}$ extends to I as a solution of the ODE.

Proof. Choose any point $x_1 \in J$ and consider the power series of $\tilde{y}$ centered at x_1. Then, by the above Theorem 1 this power series is a linear combination of two linearly independent solutions of the ODE and therefore this solution is convergent in I. □

Corollary 2.2. Consider a second order ordinary differential equation given by (1) with a, b, c are analytic functions at $x_0 \in I \subset \mathbb{R}$. Suppose (1) admits two linearly independent smooth solutions $\tilde{y}_1, \tilde{y}_2$ defined in some interval $(x_0, x_0 + \delta)$ where the ODE has at x_0 an ordinary point or a regular singular point. Then $\tilde{y}_1, \tilde{y}_2$ extend to convergent solutions of the ODE which are linearly independent. In particular, any solution of the ODE is a linear combination of these extensions.

Proof. Indeed, any smooth solution in some interval extends to a convergent solution in I. Such a solution cannot be zero in some interval unless it is identically zero. This shows that the extensions of the solutions are still linearly independent. □

2.5. Regularity of the coefficients of the ODE in terms of solutions

Let us prove a regularity theorem for the coefficients of an ODE.

Proof of Theorem C. The only nontrivial part is the if part. Assume then that there are two linearly independent analytic solutions $y_1(x), y_2(x)$ defined in I. The Wronskian $W = y_1 y_2' - y_1' y_2$ is then an analytic non-vanishing function defined in I. Moreover, this is a solution of the ODE $w' + b(x)w = 0$. Therefore, we have $W(x) = kexp\left(-\int^x b\,(\xi)d\xi\right)$ for some constant k. Since $W(x)$ is never zero, the quotient $\frac{W'(x)}{W(x)} = -b(x)$ is analytic in I. This shows that b is analytic in I. Since y_j is a solution of the ODE (2) we have $y_j''(x) + b(x)y_j'(x) + c(x)y_j(x) = 0$ and then $c(x) = -\frac{a(x)y_j''(x)+b(x)y_j'(x)}{y_j(x)}$. Since both $y_1(x), y_2(x)$ are analytic and do not vanish at the same time, and since b is analytic we conclude that c is analytic in I. □

2.6. Liouvillian solutions

We refer to Camacho and Scárdua [27] and Singer [25] for the precise notion of liouvillian function, and liouvillian solution first integrals and equations of second order ODEs.

We shall now give a sufficient condition in order to have a liouvillian solution:

Let $L_j[y] = a_j(x)y'' + b_j(x)y' + c_j(x)y = 0$, $j = 1,2$; be two linearly independent ODEs with analytic coefficients in a neighborhood of $0 \in \mathbb{K}$. Assume that there is common solution $y(x)$. Then we have $y''(x) = -\frac{b_j(x)y'(x)+c_j(x)y}{a_j(x)}$, $j = 1,2$. Therefore, $y(x)$ is an integral curve of the following closed differential 1-form

$$\Omega = \frac{dy}{y} - \frac{a_2(x)c_1(x) - a_1(x)c_2(x)}{a_2(x)b_1(x) - a_1(x)b_2(x)}dx.$$

The 1-form Ω writes

$$\Omega = d\left[lny - \int^x \frac{a_2(\xi)c_1(\xi) - a_1(\xi)c_2(\xi)}{a_2(\xi)b_1(\xi) - a_1(\xi)b_2(\xi)}d\xi\right].$$

Thus the common solution writes $y(x) = kexp\left(\int^x \frac{a_2(\xi)c_1(\xi)-a_1(\xi)c_2(\xi)}{a_2(\xi)b_1(\xi)-a_1(\xi)b_2(\xi)}d\xi\right)$ for some constant k. In particular each ODE admits a liouvillian solution as described above.

Summarizing we promptly have the proof of the following proposition.

Proposition 2.2. *If two independent analytic second order linear ODEs* $L_j[y] = a_j(x)y'' + b_j(x)y' + c_j(x)y = 0, j = 1,2$*; have a common solution then this solution is a liouvillian function.*

2.7. Examples

If we are not in the Frobenius case, even for analytic ODEs, the solutions may not be analytic or convergent. Indeed, we have:

Example 2.1. *This example shows that we may have* C^∞ *solutions but no analytic solutions. Consider the following ODE*

$$x^{\ell+2}y'' - \left(kx^{\ell+1} + \ell x\right)y' + \left[\ell(\ell+1) + kx^\ell\right]y = 0.$$

This ODE admits the solution $\varphi(x) = x^k exp\left(-\frac{1}{x^\ell}\right)$ which is C^∞ in $\mathbb{R}\backslash\{0\}$ for $k, \ell \in \mathbb{N}$ and ℓ even. In this case admits a C^∞ extension to $\mathbb{R}$. It is C^∞ in $\mathbb{R}\backslash\{0\}$ for ℓ odd. In this last case it admits a C^∞ extension to $[0, +\infty)$. For instance $x^4y'' - 2xy' + 6y = 0$ admits the C^∞ solution $\varphi(x) = exp\left(-\frac{1}{x^2}\right)$. By reduction of order we find a second linearly independent solution $\psi(x) = exp\left(-\frac{1}{x^2}\right)\int^x exp\left(\frac{1}{\xi^2}\right)d\xi$. This second solution is C^∞ in $\mathbb{R}\backslash\{0\}$, but clearly not at the origin.

The next examples show how sharp are the statements of Theorem A and B.

Example 2.2. *Consider the equation*

$$x^4y'' - xy' + y = 0.$$

The origin $x = 0$ is a singular point which is not regular. We have a first solution given by $\varphi_1(x) = x$. A second solution may be obtained by the method of reduction of order and writes $\varphi_2(x) = x\int^x \frac{exp\left(-\frac{1}{2\xi^2}\right)}{\xi^2}d\xi$. Thus the solutions are C^∞ in $\mathbb{R}$. This example can be extended to a more general form

$$a(x)y'' + xy' - y = 0$$

for any analytic function $a(x)$. Indeed, $y(x) = x$ is still a solution and the second solution may be obtained by reduction of order. Then, according to the choice of $a(x)$ we may have more C^∞ non-analytic solutions.

3. Synthesis and analytic classification

In this section we address various aspects of the classification of second order linear homogeneous analytic ODEs. We discuss from analytic normal forms to geometrical-analytic aspects.

3.1. Synthesis

We address now the following problem: given any pair of typical solutions y_1, y_2 of a Frobenius type ODE, is there a Frobenius type ODE having y_1, y_2 as solutions? Let us begin with the case of a single solution:

Proof of Theorem D. Write $y(x) = x^r \varphi(x)$ with $\varphi(0) = 1$ and then $y'(x) = rx^{r-1}\varphi(x) + x^r \varphi'(x)$ and $y''(x) = r(r-1)x^{r-2}\varphi(x) + 2rx^{r-1}\varphi'(x) + x^r \varphi''(x)$. Then

$$x^2 y''(x) = r(r-1)x^r \varphi(x) + 2rx^{r+1}\varphi'(x) + x^{r+2}\varphi''(x),$$

and

$$xy'(x) = rx^r \varphi(x) + x^{r+1}\varphi'(x).$$

Let us now choose $b_0, c_0 \in \mathbb{K}$ such that $r(r-1) + rb_0 + c_0 = 0$. Then we have

$$x^2 y''(x) + xb_0 y'(x) + c_0 y(x) = (2r + b_0)x^{r+1}\varphi'(x) + x^{r+2}\varphi''(x).$$

Put now $b(x) = b_0 + xb_1(x)$ and $c(x) = c_0 + xc_1(x)$. Then for $y(x) = x^r \varphi(x)$ as above we have

$$x^2 y''(x) + xb(x)y'(x) + c(x)y(x) = x^2 y''(x) + xb_0 y'(x) + c_0 y(x) + x^2 b_1(x)y'(x) + xc_1(x)y(x).$$

Given that $x^2 y''(x) + xb_0 y'(x) + c_0 y(x) = (2r + b_0)x^{r+1}\varphi'(x) + x^{r+2}\varphi''(x)$ we have

$$x^2 y''(x) + xb(x)y'(x) + c(x)y(x) \quad = \left[(2r + b_0)\varphi'(x) + \big(rb_1(x) + c_1(x)\big)\varphi(x)\right]x^{r+1}$$

$$+x^{r+2}[\varphi''(x) + b_1(x)\varphi'(x)].$$

Thus a solution is assured if we have

$$\begin{cases} \varphi''(x) + b_1(x)\varphi'(x) = 0, \\ (2r + b_0)\varphi'(x) + \big(rb_1(x) + c_1(x)\big)\varphi(x) = 0. \end{cases}$$

Assume that $\varphi'(0) \neq 0$. We may consider therefore $b_1(x)$ as defined by $b_1(x) = -\frac{\varphi''(x)}{\varphi'(x)}$ and $c_1(x) = -(2r + b_0)\frac{\varphi'(x)}{\varphi(x)} - rb_1(x)$. With this definition the functions $b_1(x), c_1(x)$ are analytic or have poles of order one at the origin. Indeed, because $\varphi(0) = 1$ and $\varphi'(0) \neq 0$. Indeed, let us now consider the general case, that only requires that $\varphi(0) = 1$.

We write $\varphi(x) = x^\ell \xi(x)$ for some $\ell \in \{0,1,\dots\}$ and some analytic function $\xi(x)$ with $\xi(0) \neq 0$. Then

$$\frac{\varphi''(x)}{\varphi'(x)} = dln\big(\varphi'(x)\big) = \ell\frac{dx}{x} + \frac{\xi'(x)}{\xi(x)}.$$

Then we shall have $b_1(x) = -\left(\ell\frac{dx}{x} + \frac{\xi'(x)}{\xi(x)}\right)$ and then $b(x) = b_0 + xb_1(x) = b_0 - x\left(\ell\frac{dx}{x} + \frac{\xi'(x)}{\xi(x)}\right)$ which is analytic.

Now we consider $c_1(x) = -(2r + b_0)\frac{\varphi'(x)}{\varphi(x)} - rb_1(x)$. Then $c(x) = c_0 + xc_1(x) = c_0 - x\left(-(2r + b_0)\frac{\varphi'(x)}{\varphi(x)} - rb_1(x)\right)$ is analytic at $x = 0$ because $\varphi(0) = 1$ and $b_1(x)$ has a simple pole at the origin.

Take $b_1(x) = 0$ and then we obtain

$$[(2r + b_0)\varphi'(x) + c_1(x)\varphi(x)]x^{r+1} + x^{r+2}\varphi''(x) = 0.$$

This last is equivalent to

$$[(2r+b_0)\varphi'(x)+c_1(x)\varphi(x)]+x\varphi''(x)=0$$

and therefore has solution given by

$$c_1(x)=-\frac{x\varphi''(x)+(2r+b_0)\varphi'(x)}{\varphi(x)}.$$

Since $\varphi(0)=1$ this solution is analytic. We obtain therefore a Frobenius ODE of the form $x^2y''+xb_0y'+\left(c_0+xc_1(x)\right)y=0$. Thus, $b(x),c(x)$ define the coefficients of a Frobenius type ODE $x^2y''+xb(x)y'+c(x)y=0$ that admits $x^r\varphi(x)$ as a solution, defined in some disc centered at the origin. □

As an immediate consequence of Theorem D and Proposition 2.2 we have:

Proof of Theorem E. We already know that to each $r\in\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$ and each convergent series $\varphi(x)\in\mathbb{K}\{x\}$ we can associate a Frobenius type ODE that admits $y(x)=x^r\varphi(x)$ as a solution. Moreover, according to Proposition 2.2 this ODE is unique, except if $\varphi(x)$ is a liouvillian function. □

3.2. Reduced form

Let us now investigate the relation between two solutions of the same Frobenius ODE. Let us consider $y_1(x)=x^{r_1}\varphi_1(x)$ and $y_2(x)=x^{r_2}\varphi_2(x)$, solutions of the same Frobenius type ODE $x^2y''+xb(x)y'+c(x)y=0$ as above. From the above considerations we have

$$\left[\left(2r_j+b_0\right)\varphi_j'(x)+r_jb_1(x)\varphi_j(x)+c_1(x)\varphi_j(x)\right]x^{r_j+1}+x^{r_j+2}\left(\varphi_j''(x)+b_1(x)\varphi_j'(x)\right)=0.$$

Lemma 3.1. *Given a Frobenius type ODE* $x^2y''+xb(x)y'+c(x)y=0$*, there is an analytic change of coordinates* $\Phi(x,y)=\left(x,y\varphi(x)\right)$ *that transforms this ODE into a Frobenius type ODE of the form* $x^2y''+xb_0y'+\tilde{c}(x)y=0$.

Proof. Put $y=\tilde{y}\varphi(x)$. Then $y'=\tilde{y}'\varphi(x)+\tilde{y}\varphi'(x)$ and $y''=\tilde{y}''\varphi(x)+2\tilde{y}'\varphi'(x)+\tilde{y}\varphi''(x)$. Then

$$x^2y''+xb(x)y'+c(x)y=x^2\varphi(x)\tilde{y}''+[2x^2\varphi'(x)+xb(x)\varphi(x)]\tilde{y}'+[x^2\varphi''(x)+xb(x)\varphi'(x)+c(x)\varphi(x)]\tilde{y}.$$

Dividing by $\varphi(x)$ we conclude that $x^2y''+xb(x)y'+c(x)y=0$ is equivalent to

$$x^2\tilde{y}''+\left[\frac{2x^2\varphi'(x)+xb(x)\varphi(x)}{\varphi(x)}\right]\tilde{y}'+\left[\frac{x^2\varphi''(x)+xb(x)\varphi'(x)+c(x)\varphi(x)}{\varphi(x)}\right]\tilde{y}=0.$$

Now we require that the coefficient of $\tilde{y}'$ is of the form cx for some constant c, therefore equal to b_0x. Then we must have

$$\frac{2x^2\varphi'(x)+xb(x)\varphi(x)}{\varphi(x)}=xb_0$$

and therefore

$$2x\frac{\varphi'(x)}{\varphi(x)}=b_0-b(x).$$

Since $b(0)=b_0$, we have $b(x)=b_0+xb_1(x)$ for some analytic $b_1(x)$ so that $2\frac{\varphi'(x)}{\varphi(x)}=-b_1(x)$. This is equivalent to $\left(ln\left(\varphi^2(x)\right)\right)'=-b_1(x)$ and therefore we have solutions of the form

$$\varphi(x)=k exp\left(-\frac{1}{2}\int_0^x b_1(\xi)d\xi\right)$$

for some constant $k\in\mathbb{K}\backslash\{0\}$.□

Definition 3.1 (reduced form of a Frobenius ODE). *We shall call the form* $x^2y'' + xb_0y' + \tilde{c}(x)y = 0$ *of a Frobenius type ODE* $x^2y'' + xb(x)y' + c(x)y = 0$, *the reduced normal form of the ODE.*

3.3. Characteristic function and analytic classification

Let us consider two solutions $y_1(x) = x^{r_1}\varphi_1(x)$ and $y_2(x) = x^{r_2}\varphi_2(x)$, solutions of the same Frobenius type ODE in the reduced normal form $x^2y'' + xb_0y' + c(x)y = 0$ as above. From the above considerations we have $b_1 = 0$ and then

$$\frac{x\varphi_1''(x) + (2r_1 + b_0)\varphi_1'(x)}{\varphi_1(x)} = \frac{x\varphi_2''(x) + (2r_2 + b_0)\varphi_2'(x)}{\varphi_2(x)}.$$

Let us put $\lambda_j = 2r_j + b_0$ then

$$\frac{x\varphi_1''(x) + \lambda_1\varphi_1(x)}{\varphi_1(x)} = \frac{x\varphi_2''(x) + \lambda_2\varphi_2(x)}{\varphi_2(x)}$$

and then

$$\frac{x^{\lambda_1}\varphi_1''(x) + \lambda_1 x^{\lambda_1 - 1}\varphi_1(x)}{x^{\lambda_1 - 1}\varphi_1(x)} = \frac{x^{\lambda_2}\varphi_2''(x) + \lambda_2 x^{\lambda_2 - 1}\varphi_2(x)}{x^{\lambda_2 - 1}\varphi_2(x)}.$$

This last equation can be rewritten as

$$\frac{\left(x^{\lambda_1}\varphi_1'(x)\right)'}{x^{\lambda_1}\varphi_1(x)} = \frac{\left(x^{\lambda_2}\varphi_2'(x)\right)'}{x^{\lambda_2}\varphi_2(x)}.$$

The above equation holds information about the analytic classification of the ODE.

Definition 3.2 (characteristic function of the ODE). *The characteristic function of a Frobenius ODE is*

$$\mu(x) = \frac{\left(x^{\lambda}\varphi'(x)\right)'}{x^{\lambda}\varphi(x)}$$

where $y(x) = x^r\varphi(x)$ *is a solution of the ODE in the reduced form* $x^2y'' + xb_0y' + c(x)y = 0$ *and* $\lambda = 2r + b_0$.

The characteristic function also writes

$$\mu(x) = \left[ln\left(x^{\lambda}\varphi'(x)\right)\right]' [ln\varphi(x)]'$$

or else as

$$\mu(x) = \frac{x\varphi''(x) + \lambda\varphi'(x)}{\varphi(x)}.$$

Lemma 3.2. *A reduced Frobenius ODE that admits a monomial solution* $y = x^r$ *is of Euler type.*

Proof. Indeed, given the equation $x^2y'' + xb_0y' + c(x)y = 0$ if we have a solution of the form $y(x) = x^r$ then $[r(r-1) + rb_0 + c(x)]x^r = 0$. Write now $c(x) = c_0 + xc_1(x)$. Then for $x = 0$ we obtain $r(r-1) + rb_0 + c_0 = 0$, i.e., r is a zero of the indicial equation, and therefore $xc_1(x) = 0$. This implies $c_1 = 0$ and the ODE is of Euler type. □

Proposition 3.1. *A Frobenius ODE is of Euler type if, and only if, the characteristic function is zero.*

Proof. Assume that we are in the reduced form and the characteristic function is zero.

Then $\frac{\left(x^{\lambda}\varphi'(x)\right)'}{x^{\lambda}\varphi(x)} = 0$ so that $\left(x^{\lambda}\varphi'(x)\right)' = 0$. This implies $x^{\lambda}\varphi'(x) = constant$ and therefore $x^{\lambda}\varphi'(x) = 0$. Hence $\varphi'(x) = 0$ and $\varphi(x) = constant$. Since we assume $\varphi(0) = 1$ we have $\varphi(x) = 1$ and therefore $y(x) = x^r$. Thanks to the Lemma 3.2 the ODE is of Euler type.

The converse is straightforward. □

Next we show how to compute the characteristic function in a pair of classical equations. Recall that, from the above Proposition, the characteristic function measures how much the equation differs from its corresponding Euler equation.

Example 3.1 (Airy equation). *The Airy equation $y'' - xy = 0$ is non-singular. Therefore it has indicial equation $r(r-1) = 0$ and indexes $r = 0$ and $r = 1$. The characteristic function is $\mu(x) = -c_1(x) = x^2$. On the other hand, we know that given a solution of the form $y(x) = x^r\varphi(x)$ we have $\mu(x) = \frac{\left(x^{\lambda}\varphi'(x)\right)'}{x^{\lambda}\varphi(x)}$ where $\lambda = 2r + b_0 = 2r$. For $r = 0$ we have therefore $\lambda = 0$ and the solution must be of the form $\frac{\left(\varphi'(x)\right)'}{\varphi(x)} = x^2$. Thus we must have $\varphi''(x) - x^2\varphi(x) = 0$ which will be called first Airy characteristic equation. For $r = 1$ we have $\lambda = 2$. In this case the equation has solutions of the form $y(x) = x\varphi(x)$ for $\varphi(x)$ satisfying the ODE $\frac{\left(x^2\varphi'(x)\right)'}{x^2\varphi(x)} = x^3$. This last equation is equivalent to*

$$x\varphi''(x) + 2\varphi'(x) - x^3\varphi(x) = 0.$$

This equation is regular and in a reduced normal form. It will be called second Airy characteristic equation.

Example 3.2 (Bessel). *The Bessel equation $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ will be first considered for $\nu = 0$. Thus we have $x^2y'' + xy' + x^2y = 0$ and then $b_0 = 1, c(x) = x^2$ so that $c_0 = 0$ and $c_1(x) = x$. This gives $r(r-1) + r = 0$ as indicial equation. The indexes are then $r = 0$. For $r = 0$ then we have $\lambda = 2r + b_0 = 1$. The characteristic function is $\mu(x) = -c(x) = -x$. The characteristic equation is then $\frac{\left(x\varphi'(x)\right)'}{x\varphi(x)} = -x$. This last rewrites as*

$$x\varphi''(x) + \varphi'(x) + x^2\varphi(x) = 0$$

which is the Bessel characteristic equation.

3.4. Rigidity of equivalence for Frobenius type ODEs

We shall now prove Theorem F. Let us first make clear our framework.

Definition 3.3 (Equivalence for ODEs). *Two ODEs (E_j): $a_j(x)y'' + b_j(x)y' + c_j(x)y = 0$, $j = 1,2$; are (fibered) equivalent if there is a bijective map $\Phi(x,y) = \left(x, g(x,y)\right)$, such that given a solution $y_1(x)$ of (E_1), then $y_2(x) := g\left(x, y_1(x)\right)$ is a solution of (E_2). If Φ is analytic (respectively formal, of class C^r), we shall say that the equivalence is analytic (respectively, formal, of class C^r).*

Proof of Theorem F. Let us study under which conditions a diffeomorphism takes a Frobenius ODE into a Frobenius ODE. Let us start with a Frobenius type ODE

$$x^2\tilde{y}'' + \tilde{b}(x)x\tilde{y}' + \tilde{c}(x)\tilde{y} = 0$$

where $\tilde{b}(x), \tilde{c}(x)$ are analytic functions. Take a fibered diffeomorphism of the form $\Phi(x,y) = (x, \varphi(x,y))$ and put $\tilde{y} = \varphi(x,y)$. Let us see the effect of Φ on the ODE. We have

$$\tilde{y}' = \varphi_x(x,y) + \varphi_y(x,y)y'$$

and

$$\begin{aligned}\tilde{y}'' &= \varphi_{xx}(x,y) + \varphi_{xy}(x,y)y' + y'\left[\varphi_{xy}(x,y) + \varphi_{yy}(x,y)y'\right] + \varphi_y(x,y)y'' \\ &= \varphi_{xx}(x,y) + 2\varphi_{xy}(x,y)y' + \varphi_{yy}(x,y)(y')^2 + \varphi_y(x,y)y''\end{aligned}$$

Then

$$\begin{aligned}0 = x^2\tilde{y}'' + \tilde{b}(x)x\tilde{y}' + \tilde{c}(x)\tilde{y} &= x^2\varphi_y(x,y)y'' + y'\left[2x^2\varphi_{xy}(x,y) + x\tilde{b}(x)\varphi_y(x,y)\right] \\ &+x^2\varphi_{yy}(x,y)(y')^2 + x\tilde{b}(x)\varphi_x(x,y) + x^2\varphi_{xx}(x,y) + \tilde{c}(x)\varphi(x,y).\end{aligned}$$

If we assume that the resulting ODE is linear in the variable y then we must have

$$\varphi_{yy} = 0$$

and therefore $\varphi(x,y) = \lambda(x)y + \mu(x)$ for some analytic $\lambda(x), \mu(x)$. This gives

$$\begin{aligned}0 = x^2\tilde{y}'' + \tilde{b}(x)x\tilde{y}' + \tilde{c}(x)\tilde{y} = \quad & x^2\lambda(x)y'' + y'\left[2x^2\lambda'(x) + x\tilde{b}(x)\lambda(x)\right] \\ & +y\left[x^2\lambda''(x) + x\tilde{b}(x)\lambda'(x) + \tilde{c}(x)\lambda(x)\right] \\ & +\left[x^2\mu''(x) + x\tilde{b}(x)\mu'(x) + \tilde{c}(x)\mu(x)\right].\end{aligned}$$

Assuming that the result ODE is homogeneous we must have

$$x^2\mu''(x) + x\tilde{b}(x)\mu'(x) + \tilde{c}(x)\mu(x) = 0.$$

This means:

Conclusion 3.1. $\tilde{y} = \mu(x)$ must be a solution of the original Frobenius ODE $x^2\tilde{y}'' + x\tilde{b}(x)\tilde{y}' + \tilde{c}(x)\tilde{y} = 0$.

Our resulting ODE can be written as

$$x^2y'' + \left[2x^2\frac{\lambda'(x)}{\lambda(x)} + x\tilde{b}(x)\right]y' + \left[\frac{x^2\lambda''(x) + x\tilde{b}(x)\lambda'(x) + \tilde{c}(x)\lambda(x)}{\lambda(x)}\right]y = 0.$$

Therefore, the coefficients of y in the new ODE $x^2y'' + xb(x)y' + c(x)y = 0$ are given by $xb(x) = 2x^2\frac{\lambda'(x)}{\lambda(x)} + x\tilde{b}(x)$ and $c(x) = \frac{x^2\lambda''(x)+x\tilde{b}(x)\lambda'(x)+\tilde{c}(x)\lambda(x)}{\lambda(x)}$. The first equation gives

$$\lambda(x) = kexp\left(\int^x \frac{b(\xi) - \tilde{b}(\xi)}{2\xi}d\xi\right)$$

for some constant k. This shows that:

Conclusion 3.2. $\lambda(x) = kexp\big(u(x)\big)$ for some analytic function $u(x)$.

Summarizing, we have $\Phi(x,y) = \big(x, \lambda(x)y + \mu(x)\big)$ where $\lambda(x)$ and $\mu(x)$ are described by Conclusions 3.1 and 3.2. Assume that $\Phi(x,y)$ is formal, then $\mu(x)$ is a formal solution of a Frobenius type ODE, therefore $\mu(x)$ must be convergent [4]. This shows the rigidity in the formal case. Assume now that $\Phi(x,y)$ is of class C^2. Then $\mu(x)$ is a C^2 solution of a Frobenius type ODE. This implies that $\mu(x)$ is analytic (cf. Theorem A). This ends the proof of Theorem F. □

Remark 3.1 (Complement to Theorem F). *Let us proceed using the notation from the proof above. Assume now that the resulting ODE is Euler type. Then we must have*

$2x^2\frac{\lambda'(x)}{\lambda(x)}+x\tilde{b}(x)=b_0x$ *for some constant* b_0 *and* $x^2\lambda''(x)+x\tilde{b}(x)\lambda'(x)+\tilde{c}(x)\lambda(x)=c_0$ *for some constant* c_0. *From the first equation we have* $2x\frac{\lambda'(x)}{\lambda(x)}+\tilde{b}(x)=b_0$. *This implies that* λ *satisfies the linear equation* $\lambda'(x)=-\frac{1}{2x}\big(\tilde{b}(x)-b_0\big)\lambda(x)$, *which implies only* $\tilde{b}(0)=b_0$ *and has non-constant solutions for* λ *iff* $\tilde{b}(x)\not\equiv b_0$. *The second equality is* $x^2\lambda''(x)+x\tilde{b}(x)\lambda'(x)+\tilde{c}(x)\lambda(x)=c_0$ *which is a non-homogeneous Frobenius type ODE.*

The above shows that there may be fibered diffeomorphism of the form $\Psi(x,y)=\big(x,\lambda(x)y+\mu(x)\big)$ transforming a Frobenius type ODE into its associate Euler equation. This is equivalent to find solutions $\lambda(x)$ and $\mu(x)$ satisfying the above conclusions.

We would like to end this manuscript by mentioning some questions that we shall address in the near future and we think are interesting to the subject. The first is whether there is an equivalent to the theory of Frobenius in the framework of differential equations in positive characteristic p (see Van der Put [28] for an introduction to the subject). Another interesting question is the study of Frobenius method for Fuchsian equations [22] under the light of our results. Other questions that come naturally when one considers the PDE versions of our results [23].

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