Journal of AppliedMath 2025, 3(1), 1592.
https://doi.org/10.59400/jam1592

Article

Background seismicity and seismic correlations

Bogdan Felix Apostol

Institute of Earth's Physics, 0777125 Magurele, Romania; afelix@theory.nipne.ro

CITATION

Apostol BF. Background seismicity and seismic correlations. Journal of AppliedMath. 2025; 3(1): 1592. https://doi.org/10.59400/jam1592

ARTICLE INFO

Received: 6 August 2024
Accepted: 22 October 2024
Available online: 3 January 2025

Abstract: The law of energy accumulation in the earthquake focus is presented, together with the temporal, energy and magnitude distributions of regular, background earthquakes. The background seismicity is characterized by two parameters—the seismicity rate and the Gutenberg-Richter parameter, which can be extracted by fitting the empirical earthquake distributions. Time-magnitude and temporal correlations are presented, and the information they can provide is discussed. For foreshocks the time-magnitude correlations can be used to forecast (with limitations) the mainshock. The temporal correlations indicate a decrease of the Gutenberg-Richter parameter for small magnitudes, in agreement with empirical observations for foreshocks. On the other hand, the aftershocks may be viewed as independent earthquakes with changed seismic conditions, so they may exhibit an increase of this parameter, also in accordance with empirical observations. The roll-off effect for small magnitudes and the modified Gutenberg-Richter distribution are discussed for temporal corralations, and the derivation of the Bath's law is briefly reviewed.

Keywords: background seismicity; time-magnitude correlations; temporal correlations; foreshocks for forecasting

1. Introduction

This paper deals with the statistical properties of the earthquakes. We represent a typical earthquake by a spatially-localized focus (called also focal region), *i.e.* a region extending in space over distances much smaller than the distances over which we measure the seismic effects. We may say that the focus is "pointlike". In time, the earthquake energy is accumulated in focus, and suddenly released as seismic waves. These seismic waves propagate in the earth, viewed as an elastic medium, where they are attenuated, scattered by inhomogeneities, may produce damage (ruptures), especially in the near-field region, etc. The particular geometry of the focal region is not very relevant for statistical properties. By means of this simple assumption we are able to derive the law of energy accumulation in the focus, and derive the well-known standard Gutenberg-Richter statistical distrbutions of the earthquakes. Moreover, by the same assumption, we derive the correlations that may exist between earthquakes. These elaborations, together with other special cases are discussed in detail in Apostol [1–4].

From a theoretical point of view the earthquakes imply the knowledge and understanding of three questions at least. First, we need to know the connection between the earthquake energy and the accumulation time in the focal region. Second, the statistical distributions are an important aspect, related to the former, especially including earthquake correlations. These two problems can be viewed as defining the statistical seismology. Third, it is necessary to know the force acting in the seismic focus and the seismic waves it produces, especially at earth's surface (seismological problem). Related to this problem we have also the inverse seismological problem, which aims at deriving the earthquake and the focal parameters from measurements conducted at earth's surface. The seismological problem was recently discussed [5]. We give herein a brief presentation of the statistical seismology, with emphasis on its meaning and information it can provide. A technical Appendix is included.

2. Background seismicity

The background (regular) seismicity views earthquakes as independent events, occurring by accumulating energy in a pointike focus during a certain lapse of time. "By pointlike focus" we mean focal dimensions much smaller than the distances over which we measure the earthquakes' effects. The process of accumulation consists of a succession of small amounts of energy E_0, each produced in a short time t_0; in time $t > t_0$ we may accumulate an energy $E > E_0$. The energy E is the sum of several energies E_0, and the time t is the sum of several short times t_0. The events with E_0 and t_0 are fundamental, E_0-seismic events. In Apostol [4] we established the law of accumulation

$$t/t_0 = (E/E_0)^r \tag{1}$$

where r is a (statistical) parameter which characterizes the geometry of the focus; it takes values in the range $1/3 < r < 1$ (see Appendix).

By definition the (moment) magnitude M of an earthquake is given by

$$E/E_0 = e^{bM} \tag{2}$$

where $b = 3.45$ (in decimal base $b = 3/2$) [6–11]. By combining the above two equations we can write the accumulation law as

$$t = t_0 e^{\beta M} \ , \ \beta = br \tag{3}$$

this is the mean recurrence time.

The probability for a fundamental event to occur in time t is t_0/t. If $P(t)dt$ is the probability of having an earthquake in the time interval t and $t + dt$, its occurrence probability is $\int_{t_0}^{t} dt' P(t')$, and we must have

$$\frac{t_0}{t} + \int_{t_0}^{t} dt' P(t') = 1 \tag{4}$$

this equation tells that we have either an E_0-event, or an $E > E_0$-earthquake in time t. Hence, we get the probability

$$P(t)dt = \frac{t_0}{t^2} dt \tag{5}$$

for an earthquake to occur in the short duration from t to $t + dt$; it is the probability of having an earthquake at time t, with an accumulated energy E given by Equation (2). Making use of Equation (3) we get the magnitude probability

$$P(M)dM = \beta e^{-\beta M} dM \tag{6}$$

and the cumulative (excedence) probability for an earthquake with magnitude greater than M

$$P_{ex} = e^{-\beta M} \tag{7}$$

Equations (6) and (7) are known as Gutenberg-Richter (or Hanks-Kanamori) laws. We can see that the law of energy accumulation and the definition of the probabilty lead to the Gutenberg-Richter empirical distributions.

If, in a given seismic region, we have N_0 earthquakes in a long time T and N earthquakes with magnitude greater than M, then $P_{ex} = N/N_0$, or $P_{ex} = Nt_0/T$, such that the logarithmic form of Equation (7) reads

$$\ln(N/T) = -\ln t_0 - \beta M \tag{8}$$

By fitting this law to the empirical distribution of earthquakes, we get the parameters t_0 and β (and $r = \beta/b$) of the background seismicity; $1/t_0$ is the rate of seismicity. We analyzed a set of 3640 Vrancea earthquakes with magnitude $M \geq 3$, occurred during $1981 - 2018$ [12], and get the parameters $-\ln t_0 = 11.32$ (t_0 measured in years) and $\beta = 2.26$ ($r = 0.65$; with an estimated 15% error). A completeness magnitude $M = 2.2$ to $M = 2.8$ is usually accepted for Vrancea (a more conservative figure would be $M = 3$) [13], and the magnitude average error is $\Delta M = 0.1$. Similar parameters are obtained for 8455 Vrancea earthquakes with magnitude $M \geq 2$ (period $1980 - 2019$).

The value $\beta = 2.26$ for Vrancea is close to $\beta = 2.3$ (1 in decimal base), corresponding to $r = 2/3$, which is accepted as reference value [14–17].

3. Time-magnitude correlations

It is reasonable to assume that an earthquake may affect the occurrence time and the characteristics of another earthquake. In this case we say that the earthquakes are correlated. Any pair, or any sequence of earthquakes, may be correlated, but it is likely that correlations appear especially between a mainshock and its acompanying, smaller foreshocks and aftershocks, in the same seismic region (relatively close to the focus of the mainshock) and in a limited period of time, before and after the mainshock. We note that we use the term "mainshock" in this paper as distinct from the mainshock recorded by seismograms, usually associated with Rayleigh-Love waves. In the sense used herein a mainshock is a big earthquake accompanied by foreshocks and aftershocks. We cannot exclude regular earthquakes from foreshocks and aftershocks, but it is likely that a mainshock modifies the seismic conditions, such as to produce correlations, and correlated foreshocks and aftershocks. If two (or several) earthquakes share their energy, or their accumulation time, we have correlations; other conditions (constraints) which can be imposed upon earthquakes may also lead to correlations. The laws governing correlated earthquakes are different from the laws described above for background, regular earthquakes. The Bath's law and the roll-off effect have been derived from temporal correlations [4].

Let us suppose that two earthquakes share their energies $E_{1,2}$, *i.e.* their accumulation law reads

$$t/t_0 = [(E_1 + E_2)/E_0]^r \tag{9}$$

If we view the earthquake with energy E_1 as a mainshock and the earthquake with energy $E_2 < E_1$ as a foreshock or an aftershock, the above equation leads to

$$M \simeq \frac{1}{b} \ln \frac{\tau}{\tau_0} \tag{10}$$

where M is the magnitude of the accompanying event, τ is the time elapsed from the foreshock to the mainshock, or from the mainshock to the aftershock, and the small cutoff τ_0 is given by

$$\tau_0 = rt_0 e^{-b(1-r)M_0} \tag{11}$$

where M_0 is the magnitude of the mainshock ($t_1 = t_0 e^{\beta M_0}$) [4]. The law given above is valid for $\tau > \tau_0$ and τ smaller than a cutoff time, in order to have $M < M_0$. We can see that the magnitude of the foreshocks decreases abruptly in the proximity of the mainshock, and the magnitude of the aftershocks increases abruptly immediately after a mainshock (see Appendix).

By writing $\tau = t_{ms} - t$, where t_{ms} is the occurrence time of the mainshock, we can use Equation (10) to forecast a mainshock, by fitting this equation to a sequence of magnitude-descending correlated foreshocks, occuring at times $t < t_{ms}$. The fitting parameters are the occurrence time t_{ms} of the mainshock and the cutoff time τ_0. This latter parameter gives the magnitude M_0 of the mainshock, by using Equation (11), providing we know the parameters t_0 and r (β) (background seismicity). The best fit determines also the period of time over which the correlated foreshocks (or aftershocks) are present. This procedure (together with its limitations) is described in detail elsewhere [18,19], with many specific examples.

Although they are governed by the same time-magnitude correlation law given by Equation (10), the aftershocks and the foreshocks have a distinct character.

Like the foreshocks, the aftershocks share their energy with the mainshock. On the other hand, Equation (10) can be used as if the aftershocks might be viewed as independent earthquakes, occurring after the mainshock, with an accumulation time

$$\tau = \tau_0 e^{bM} \tag{12}$$

By using the procedure described above for background earthquakes, this accumulation law leads to the distribution

$$P_{after} dM = be^{-bM} dM \tag{13}$$

By comparing these results with Equations (3)–(6), we can see that the Gutenberg-Richter parameter increased from β to b. Indeed, it is reasonable to assume that the mainshock changes the seismicity conditions of the focal region. The increase from β to b is valid until $P_{after} = \beta e^{-\beta M}$, which indicates a range of magnitudes from 0 up to $M_c = 0.36$ for $r = 2/3$, $b = 3.45$ and $\beta = 2.3$. An estimation of the average increase in β is $(b - \beta)/2\beta = 25\%$ for $\beta = 2.3$, which agrees quantitatively with data recently reported [20–22]. Within this interpretation, the Gutenberg-Richter parameter increases for aftershocks. Of course, it is not necessary to have changed seismicity conditions after a mainshock, in which case the aftershocks cannot be viewed as independent earthquakes; they remain, simply, correlated events to the mainshock. We cannot know apriori which possibility is the actual one. These time variations of the Gutenberg-Richter parameter for foreshocks and aftershocks are analyzed in detail elsewhere [19].

A similar interpretation is not valid for foreshocks, because we cannot define a reference moment of time for foreshocks, when they start to accumulate energy, as we could for aftershocks. We cannot disentangle the foreshocks from the mainshock, in order to view them as independent earthquakes. The mechanism of occurrence of the foreshocks is different from the mechanism corresponding to the aftershocks. For aftershocks the focal region accumulates an energy which is released successively as the mainshock and the remainder as aftershocks. For foreshocks, the focal region accumulates a higher energy at the moment of the foreshocks, the small excess is released as foreshocks, and the larger amount is released later as the mainshock. In order to accumulate a higher energy at a previous moment, the accumulation law requires a decrease in the parameter r at that moment. This change indicates an instantaneous modification in the seismic conditions of the focal region, though of a distinct nature than the modification corresponding to the aftershocks. The background value of the parameter r is recovered immediately after the foreshock.

The only possible interpretation for the foreshocks is their correlations with the mainshock. The same interpretation holds also for the aftershocks, but the aftershocks accept also another interpretation, as independent earthquakes, in changed seismicity conditions.

By using the time-magnitude correlations the maximal information we can get is that described above. However, while sharing their energy, the accompanying events share also their occurrence time with the mainshock. Consequently, another aspect of the correlations can be seen in the temporal correlations.

4. Temporal correlations

We consider now that an earthquake occurs in time t_1, and another earthquake occurs in time t_2 after the former; the total time is $t = t_1 + t_2$. These earthquakes share their accumulation time, which means correlations. We call them temporal (or dynamical) correlations. By a similar procedure which leads to Equations (5) and (6) we get the probability densities

$$\begin{gathered} P(t_1, t_2) = \frac{2t_0}{(t_1+t_2)^3} \ , \\ d^2P(M_1, M_2) = 4\beta^2 \frac{e^{\beta(M_1+M_2)}}{\left(e^{\beta M_1}+e^{\beta M_2}\right)^3} dM_1 dM_2 \end{gathered} \tag{14}$$

which are pair (bivariate) distributions ($t_{1,2} = t_0 e^{\beta M_{1,2}}$) (Apostol [4]). By integrating the second equation with respect to M_2, we get the marginal distribution of a correlated earthquake

$$dP = \beta e^{-\beta M_1} \frac{2}{(1 + e^{-\beta M_1})^2} dM_1 \tag{15}$$

A further integration from $M_1 = M$ to $+\infty$ leads to the correlated cumulative distribution

$$P_{ex}(M) = \int_M^{\infty} dP = e^{-\beta M} \frac{2}{1 + e^{-\beta M}} \tag{16}$$

(see Appendix). It is worth noting that if we use the law of energy accumulation $t_1/t_0 = e^{\beta M_1}$ in Equation (15) we get the time probability of a correlated earthquake $dP = 2t_0 dt_1/(t_1 + t_0)^2$, which differs from the law of a regular (background) earthquake (Equation (5)) by the presence of t_0 in the denominator and a factor 2. We can see that the correlated distributions given by these equations differ from the standard Gutenberg-Richter distributions (Equations (6) and (7)). If we look at the cumulative distribution given by Equation (16) we can see that the difference consists in a flattening for small magnitudes and an increase by $\simeq \ln 2$ for larger magnitudes, while the slope β is practically preserved in the region of larger magnitudes. According to Equation (16), the Gutenberg-Richter parameter tends to $\beta/2$ for correlated earthquakes. The small-magnitude flattening is known as the roll-off effect [23,24]. Usually, this effect is assigned to an insufficiency of data for small-magnitude earthquake. We can see that it is given, at least partially, by time correlations. The data reported for southern California earthquakes recorded between 1945–1985 and 1986–1992 exhibit such an effect [25].

The pair distribution given by Equation (14) can be applied to an earthquake and the magnitude difference between that earthquake and another correlated earthquake. From the resulting distribution we can extract the distribution of the magnitude difference between two correlated earthquakes. For small values of the magnitude difference this distribution is affected by the roll-off effect, which, according to Equation (16), amounts to an effective $\beta/2$ Gutenberg-Richter parameter. By making use of these results the Bath law has been derived [4]. This empirical law tells that, in certain conditions, the average magnitude difference between a mainshock and its largest aftershock (foreshock) is approximately 1.2 (though deviations are known [4]).

We can apply Equation (16) to foreshocks (as well as to all earthquakes). A convenient way to use it is to introduce a modified Gutenberg-Richter parameter B by

$$e^{-\beta M} \frac{2}{1 + e^{-\beta M}} = e^{-BM} \tag{17}$$

which leads to

$$B \simeq \beta - \frac{\ln 2}{M} \tag{18}$$

for a reasonable range of magnitudes $M > 1$. A similar formula can be obtained for the modified parameter $R = B/b$ and its temporal dependence, by using Equation (10). Equation (18) shows the decrease of the Gutenberg-Richter parameter in a foreshock sequence. For instance, a 10% decrease is achieved for $M = 3$, or $\tau/\tau_0 \simeq 3.6 \times 10^4$ ($\beta = 2.3$, $r = 2/3$), which agrees quantitatively with recently reported data [19–22].

Applied to aftershocks, the law $\tau = \tau_0 e^{bM}$ (Equation (12)) shows that these events may be viewed as independent earthquakes with accumulation time τ and changed seismicity conditions ($\tau_0 \neq t_0$). Since b is greater than β, the ratio τ/τ_0 is greater than the background ratio $t/t_0 = e^{\beta M}$. Consequently, the higher-magnitude earthquakes are disfavoured, the Gutenberg-Richter parameter increases, and the magnitude distribution gets flattened for small magnitudes, in accordance with the predictions of the temporal correlations (as expected). As discussed above, though valid for foreshocks, the law $\tau = \tau_0 e^{bM}$ cannot be viewed as indicating independent events, since we cannot define a starting moment of time when the foreshocks begin to accumulate energy. On the other hand, the temporal correlations show that the higher-magnitude events are favoured in this case, due to the factor $\ln 2$ in Equation (16). Consequently, the Gutenberg-Richter parameter decreases and the magnitude distribution becomes again flattened for small magnitudes, in accordance with the temporal correlations prediction. We note that the flattening of the magnitude distribution for small magnitudes is valid irrespective of viewing the aftershocks as correlated independent events; indeed, the change in the seismicity conditions, necessary for viewing the aftershocks as independent earthquakes, is derived by assuming correlations.

The flattening of the correlated distribution given by Equation (16) for small magnitudes requires a special attention in fitting the empirical distributions (for getting the background-seismicity parameters). We should note that the modified distribution is valid for a certain amount of correlated earthquakes (including foreshocks and aftershocks), while the regular Gutenberg-Richter distribution remains valid for the other fraction of background seismicity. Therefore, an aditional fitting parameter must be used in dealing with these fitting formulae.

5. Concluding remarks

We have briefly presented above the laws of background seismicity and correlated earthquakes, with the aim of bringing a further clarification in their meaning and applications. We emphasized that the background seismicity is characterized by two basic parameters, namely the inverse rate of seismicity t_0 and the Gutenberg-Richter parameter β. These parameters can be obtained by fitting the cumulative earthquake (statistical) distribution corresponding to a long period of time in a seismic region. The background seismicity views the earthquakes as independent events. However, the earthquakes may be correlated, in the sense that they may be affected by one another. For instance, any condition (constraint) imposed upon a pair of earthquakes leads to correlations between those two earthquakes. Most likely, the correlations occur between foreshocks and the mainshock and between the mainshock and its aftershocks, in a relatively limited seismic region and time window. The time-magnitude correations are defined for earthquakes which share their energy. For foreshocks these correlations can be used to forecast (with limitations) the mainshock. Temporal correlations are defined for earthquakes which share their accumulation time. These correlations predict a decrease of the Gutenberg-Richter parameter for small magnitudes, as indicated by some empirical studies of the foreshocks. The aftershocks may also be viewed as independent earthquakes with changed seismicity conditions, so they exhibit an increase of this parameter, also in agreement with some empirical observations. The temporal correlations flattens the Gutenberg-Richter distribution for small magnitudes, also in agreement with empirical observations (the roll-off effect).

Acknowledgments: The author is indebted to the colleagues in the Institute of Earth's Physics, Magurele-Bucharest, for many enlightening discussions, and the anonymous Reviewers for very useful comments and suggestions.

Data and resources: All data used in this paper came from published sources listed in References. Data used for the Vrancea region have been extracted from the Romanian Earthquake Catalog, 2018, http://www.infp.ro/data/romplus.txt, 10.7014/SA/RO.

Funding: This work was carried out within the Program Nucleu SOL4RISC, funded by the Romanian Ministry of Research, Innovation and Digitization, contract no. 24N/03.01.2023, project no. PN23360202.

Conflict of interest: The author declares no conflict of interest.

References

1. Apostol BF. A model of Seismic Focus and Related Statistical Distributions of Earthquakes. Phys. Lett. 2006; A357: 462–466. doi: 10.1016/j.physleta.2006.04.080
2. Apostol BF. Elastic waves inside and on the surface of a half-space. Quart. J. Mech. Appl. Math. 2017; 70: 289–308.
3. Apostol BF. An inverse problem in seismology: derivation of the seismic source parameters from P and S seismic waves. J. Seismol. 2019; 23: 1017–1030.
4. Apostol BF. Correlations and Bath's law. Results in Geophysical Sciences. 2021; 5: 100011.
5. Apostol BF. Seismological problem, seismic waves and the seismic mainshock. Mathematics. 2023; 11: 3777.
6. Utsu T, Seiki A. A relation between the area of aftershock region and the energy of the mainshock (Japanese). J. Seism. Soc. Japan. 1955; 7: 233–240.
7. Gutenberg B, Richter C. Frequency of earthquakes in California. Bull. Seism. Soc. Am. 1944; 34: 185–188.
8. Gutenberg B, Richter C. Magnitude and energy of earthquakes. Annali di Geofisica. 1956; 9: 1–15.
9. Utsu T. Aftershocks and earthquake statistics (I, II): Some parameters which characterize an aftershock sequence and their interaction. J. Fac. Sci. Hokkaido Univ. Ser. VII (Geophysics). 1969; 3: 129–266.
10. Kanamori H. The energy release in earthquakes. J. Geophys. Res. 1977; 82: 2981–2987.
11. Hanks TC, Kanamori H. A moment magnitude scale. J. Geophys. Res. 1979; 84: 2348–2350.
12. Popa M, Chircea A, Dinescu R, et al. Romanian Earthquake Catalog. Available online: http://www.infp.ro/data/romp lus.txt (accessed on 2 June 2024).
13. Enescu B, Struzik Z, Kiyono K. On the recurrence time of earthquakes: insight from Vrancea (Romania) intermediate-depth events. Geophys. J. Int. 2008; 172: 395–404.
14. Stein S, Wysession M. An Introduction to Seismology, Earthquakes, and Earth Structure. Blackwell. 2003.
15. Udias A. Principles of Seismology. Cambridge University Press; 1999.
16. Frohlich C, Davis SD. Teleseismic b values; or much ado about 1.0. J. Geophys. Res. 1993; 98: 631–644.
17. Lay T, Wallace TC. Modern Global Seismology. Academic Press; 1995.
18. Apostol BF, Cune LC. On the relevance of the foreshocks in forecasting seismic mainshocks. Ann. Geophys. 2023; 66: SE635.
19. Apostol BF. Time-magnitude correlations and time variation of the Gutenberg-Richter parameter in foreshock sequences. Pure Appl. Geophys. 2024; 18: 27–36.
20. Gulia L, Tormann T, Wiemer S, et. al. Short-term probabilistic earthquake risk assessment considering time-dependent b values. Geophys. Res. Lett. 2016; 43: 1100–1108.
21. Gulia L, Rinaldi AP, Tormann T, et al. The effect of a mainshock on the size distribution of the aftershocks. Geophys. Res. Lett. 2018; 45: 13277–13287.
22. Gulia L, Wiemer S. Real-time discrimination of earthquake foreshocks and aftershocks. Nature. 2019; 574: 193–199.
23. Bhattacharya P, Chakrabarti CK, Kamal KD, et. al. Fractal models of earthquake dynamics. In: Schuster HG (editor).

Reviews of Nolinear Dynamics and Complexity. Wiley; 2009. pp.107–150.
24. Pelletier JD. Spring-block models of seismicity: review and analysis of a structurally heterogeneous model coupled to the viscous asthenosphere. In: Rundle JB, Turcotte DL, Klein W (editors). Geocomplexity and the Physics of Earthquakes. AGU press; 2000. Volume 120.
25. Jones LM. Foreshocks, aftershocks and earthquake probabilities: accounting for the Landers earthquake. Bull. Seism. Soc. Am. 1994; 84: 892–899.

Appendix

I. Energy accumulation law

Typically, the focal region of the earthquakes have small dimensions in comparison with the distances over which we measure the effects of the earthquakes (pointlike focus). The energy accumulation should obey the continuity equation

$$\frac{\partial E}{\partial t} = -\boldsymbol{v} gradE \tag{A1}$$

where t denotes the time, $\boldsymbol{v}$ is the accumulation energy and E denotes the energy. We may use small differences for the derivatives in Equation (A1), like $\Delta E/\Delta x$ for $\partial E/\partial x$. The energy may be taken equal to zero at the borders, such that we have $\Delta E = -E$. The coordinates of the borders may be viewed as being in uniform motion, such that we may write $\Delta x = u_x t$, where $\boldsymbol{u}$ is the velocity of the background. Under these condition Equation (A1) becomes

$$\frac{\partial E}{\partial t} = \left(\frac{v_x}{u_x} + \frac{v_y}{u_y} + \frac{v_z}{u_z}\right)\frac{E}{t} \tag{A2}$$

If we leave aside the energy loss, and consider a uniform displacement, the two velocities are equal, $\boldsymbol{v} = \boldsymbol{u}$, such that we get the value 3 for the bracket. This is a limiting case. The other limiting case is a one-dimensional motion, for which the bracket is equal to 1. A two-dimensional accumulation process will give the value 2. Consequently, Equation (A2) becomes

$$\frac{\partial E}{\partial t} = \frac{1}{r}\frac{E}{t} \tag{A3}$$

where $1/3 < r < 1$. For a shearing fault we have both a motion alng the x-direction and a motion along two opposite y-directions (mass conservation), such that $u_x = v_x$ and $u_y = 2v_y$, $v_z = 0$. Then, the r-value is the reference value $r = 2/3$, *i.e.* $\beta = br = 2.3$ ($b = 3.45$).

In order to integrate Equation (A3) we need two cutoffs, one for energy, another for time. They are provided by E_0 and t_0, corresponding to the fundamental processes. By integrating Equation (A3), we get the accumulation law

$$t/t_0 = (E/E_0)^r \tag{A4}$$

which shows that in time t an energy E is released in an earthquake.

II. Time probability

If we denote by $\overline{M}$ the seismic moment and by M the moment magnitude, the well-known Hanks-Kanamori law

$$\ln \overline{M} = const + bM \tag{A5}$$

holds, where $b = 3.45$ (3/2 in decimal base). The seismic moment can be viewed as the mean seismic moment, $\overline{M} = \left(\sum_{ij} M_{ij}^2\right)^{1/2}$, where M_{ij} is the seismic-moment tesor. The relationship $\overline{M} = 2\sqrt{2}E$ is known [3], where E is the earthquake energy, such that we can write

$$\ln E = const + bM \tag{A6}$$

or

$$E/E_0 = e^{bM} \tag{A7}$$

where E_0 is a threshold energy (see above). By making use of Equation (A4), we get

$$t = t_0 e^{brM} = t_0 e^{\beta M} \tag{A8}$$

where $\beta = br$. This equation serves to obtain the important relations $dt = \beta t_0 e^{\beta M} dM$, or $dt = \beta t dM$. Since

$$dP = \beta e^{-\beta M} dM \tag{A9}$$

(the Gutenberg-Richter law), we get the time probability

$$dP = \beta \frac{t_0}{t} \frac{1}{\beta t} dt = \frac{t_0}{t^2} dt \tag{A10}$$

This is the probability of an earthquake to occur between time t and time $t + dt$, with energy E and magnitude M. The definition of the probability for E_0-seismic events leads to the same law ($dP = -\frac{\partial}{\partial t} \frac{t_0}{t} dt$ [4]). It is worth noting that in the derivation of this law the earthquakes are assumed to be independent.

III. Time-magnitude correlations

Two, or several, earthquakes are correlated if there exists an inter-dependence of their parameters. We discuss only two-earthquake (pair) correlations. It is reasonably to assume that correlations appear in the same region and over relatively short intervals of time, like mainshocks, foreshocks and aftershocks. We call time-magnitude correlations those correlations where the earthquakes share their energy. If the earthquakes share their accumulation time, the corelations are temporal. These correlations affect the statistical distributions of the earthquakes. Also, constraints upon the statistical variables may lead to statistical correlations.

Let us consider two successive earthquakes with energies $E_{1,2}$ (magnitudes $M_{1,2}$) , such that $E = E_1 + E_2$ (energy sharing); the energy E is accumulated in time t. Then, the accumulation law (Equation (A4)) gives

$$\begin{aligned} t/t_0 = (E/E_0)^r = (E_1/E_0 + E_2/E_0)^r = \\ = (E_1/E_0)^r (1 + E_2/E_1)^r \end{aligned} \tag{A11}$$

or

$$t = t_1 \left[1 + e^{b(M_2 - M_1)}\right]^r \tag{A12}$$

where $t_1 = t_0 (E_1/E_0)^r$ is the accumulation time corresponding to the earthquake with energy E_1. From this Equation (A12) we get

$$b(M_2 - M_1) = \ln \left[(1 + \tau/t_1)^{1/r} - 1\right] \tag{A13}$$

where $t = t_1 + \tau$ and τ is the time between the occurence of the two earthquakes. Since $\tau/t_1 \ll 1$ for foreshocks-mainshock-aftershocs, this equation gives

$$M_2 \simeq \frac{1}{b} \ln \frac{\tau}{\tau_0} \ , \ \tau_0 = r t_0 e^{-b(1-r)M_1} \tag{A14}$$

It is worth noting that τ is different from the accumulation time of an earthquake with magnitude M_2 (Equation (A8)); it depends on the parameters of the M_1-earthquake. The M_1-earthquake may be viewed as as a mainshock, in which case the M_2-earthquake is a foreshock or an aftershock. We may say that such accompanying earthquakes are correlated with the mainshock.

IV. Temporal correlations

Let us suppose that an earthquake occurs in time t_1 and in the next duration of time t_2 appears another earthquake follows in time t_2. These earthquakes share their accumulation time in the total duration $t = t_1 + t_2$. From Equation

(A10), such an event has the probability density

$$-\frac{\partial}{\partial t_2}\frac{t_0}{(t_1+t_2)^2}=\frac{2t_0}{(t_1+t_2)^3} \tag{A15}$$

(where $t_0 < t_1 < +\infty$, $0 < t_2 < +\infty$). By passsing to magnitude distributions ($t_{1,2} = t_0 e^{\beta M_{1,2}}$), we get

$$d^2P = 4\beta^2\frac{e^{\beta(M_1+M_2)}}{\left(e^{\beta M_1}+e^{\beta M_2}\right)^3}dM_1 dM_2 \tag{A16}$$

(where $0 < M_{1,2} < +\infty$, corresponding to $t_0 < t_{1,2} < +\infty$; this explains the factor 2 in Equation (A15)). Usually, this is called a pair (bivariate) statistical distribution (Apostol [4]). By integrating it with respect to M_2, we get the marginal distribution

$$dP = \beta e^{-\beta M_1}\frac{2}{\left(1+e^{-\beta M_1}\right)^2}dM_1 \tag{A17}$$

by a further integration from $M_1 = M$ to $+\infty$, we get the correlated cumulative distribution

$$P(M) = \int_M^\infty dP = e^{-\beta M}\frac{2}{1+e^{-\beta M}} \tag{A18}$$

We note that for $M \gg 1$ this distribution becomes $P(M) \simeq 2e^{-\beta M}$ and $\ln P(M) \simeq \ln 2 - \beta M$. Therefore, the slope β of the logarithm of the cumulative distribution of independent earthquakes (Gutenberg-Richter, standard distribution $e^{-\beta M}$) is not changed (for large magnitudes); the correlations introduce only an upward shift of $\ln 2$. In the region of the small magnitudes ($M \ll 1$) the slope of the correlated distribution is $\beta/2$ (by series expansion $P(M) \simeq 1 - \frac{1}{2}\beta M + ...$), which is different from the slope β of the Gutenberg-Richter distribution ($e^{-\beta M} \simeq 1 - \beta M + ...$). This is a roll-off effect.