Forum for Education Studies 2025, 3(1), 2310.
https://doi.org/10.59400/fes2310

Article

Cookies on plates: Extending Fibonacci-like numbers to fractions in a 3rd grade classroom

Sergei Abramovich*, **Laura L. Griffin**

Department of Teacher Education, School of Education and Professional Studies, State University of New York, Potsdam, NY 13676, United States

*** Corresponding author:** Sergei Abramovich, abramovs@potsdam.edu

CITATION

Abramovich S, Griffin LL. Cookies on plates: Extending Fibonacci-like numbers to fractions in a 3rd grade classroom. Forum for Education Studies. 2025; 3(1): 2310. https://doi.org/10.59400/fes2310

ARTICLE INFO

Received: 17 December 2024
Accepted: 7 January 2025
Available online: 24 February 2025

Abstract: The paper is written to share the authors' ongoing research project of bringing elements of problem-solving curriculum of elementary mathematical teacher education to grade school. Exploring 3rd graders' understanding of tasks with more than one correct answer, their ability to move from visual to symbolic, and pattern recognition as a problem-solving method using the playful context of cookies on plates structured by Fibonacci-like numbers resulted in several outcomes, both expected and unexpected. Mathematical interpretation of those outcomes provided new ideas to be discussed with teacher candidates. The collateral creativity of a 3rd grader in representing half a cookie made it possible to extend Fibonacci-like numbers to fractions and support this extension with the use of multiple digital instruments, including spreadsheets, Wolfram Alpha, and the Graphing Calculator under the conceptual umbrella of computational triangulation.

Keywords: mathematical problem solving; teacher education; grade school; collateral creativity; Fibonacci-like numbers; technology; computational triangulation

1. Introduction

This paper describes the ongoing project of the authors, the outset of which was published elsewhere [1], on bringing "teachers' mathematics" to a local grade school, a site for teacher candidates (the authors' students) fieldwork. The specific objective of the current part of the project was twofold: to investigate how young children deal with problems their future teachers might find (or even believe—perhaps due to the so-called "math anxiety" [2,3]—are) difficult and to enrich graduate and undergraduate mathematics content and methods courses for elementary teacher candidates taught by the authors with present-day findings from the field. More specifically, the intent of the authors' action research was to see whether 3rd graders at an elementary school in rural Upstate New York are capable, in the words of Vygotsky [4], of moving from the first-order symbols [images of cookies on plates they created] to the second-order symbolism [numeric patterns the cookies on plates formed]. Put another way, the activities were designed to encourage students' recognition of how one can "bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize … and the ability to contextualize" [5]. It turned out that the 3rd graders were able to use their recognition of the patterns, formed by cookies and the corresponding numbers, to solve problems from their future teachers' mathematics education curriculum. As mentioned by one of the 3rd graders in a solicited reflection, "*I used patterns to help me solve the cookies on a plate*.". But as the title of this paper indicates, another 3rd grader, by playing with two-sided counters as a substitute for cookies, demonstrated collateral creativity [6] through representing half a cookie by putting a counter in the vertical position (using its thick edge as the base). This representation and its corresponding numeric description were recognized by the authors as an opportunity to extend Fibonacci-like numbers, according to which cookies on plates develop, from integers to fractions. Whereas the concept of Fibonacci-like numbers (of which Lucas numbers 2, 1, 3, 4, 7, 11, 18, … are the prime example) is well known, the concreteness of the situation with cookies on plates and the opportunity to play with "cookies" provided the 3rd grader with a collateral creativity condition followed by the authors' recognition of the mathematical significance of the situation, which allows for considering the extension of Fibonacci-like numbers to fractions.

As was mentioned elsewhere [1], the problem with cookies on plates (structured by Fibonacci-like numbers) has its origin in a 1995 talk (attended by the first author) at the University of Georgia by Erich Wittmann of the University of Dortmund who was motivated by ideas discussed in the 1990s within the UK Association of Teachers of Mathematics; see also research by Harlen & Lena [7] and Abramovich & Brown [8]. In this talk, concerned with the interplay between problem solving and preparation of mathematics teachers, the following problem was presented by Wittmann: *Given natural number N as the fifth term of a Fibonacci-like sequence, find all possible non-negative initial values.* In terms of cookies on plates, when $N = 8$ the first two plates have one and two cookies (as initial values), respectively; when $N = 11$, there are two options to put cookies on the first two plates: one and three cookies and four and one cookies, respectively. However, when $N = 11$ is the fourth term of a Fibonacci-like sequence, there are five such quadruples, namely, (1, 5, 6, 11), (3, 4, 7, 11), (5, 3, 8, 11), (7, 2, 9, 11), and (9, 1, 10, 11); i.e., contextually, there are five ways to put cookies on the first two plates. This case was used by the authors with the 3rd graders as a warmup activity (**Figure 1**) to motivate their engagement with the learning environment. The combination of two-sided counters as physical manipulatives and the virtual display of cookies on plates created simulated experience of immersive technology [9] for the 3rd graders.

Figure 1. A classroom where the activities took place (standing is the first author).

By looking at the above five quadruples, the reader can immediately recognize a pattern which the number of cookies on the first two plates follow (cookies on the first plate are represented by the first five odd numbers and cookies on the second plate are represented by the first five natural numbers reversed, respectively) as well as the method of solution prompted by seeing the number of cookies on the third plate. One goal of the activities was to investigate whether the 3rd graders could see (or at least appreciate) the patterns that (full) cookies on the first two plates follow. One can also see that the quadruple (1, 5, 6, 11) may be followed by the quadruples (2, 4.5, 6.5, 11) or $(2\frac{1}{3}, 4\frac{1}{3}, 6\frac{2}{3}, 11)$ if fractional parts of cookies may be used.

2. Materials and methods

Two types of materials have been used by the authors when working on this paper. The first type included "mathematical action technologies" [10] such as computer spreadsheets, computational knowledge engines like Wolfram Alpha developed by Wolfram Research [11], the Graphing Calculator [12], Unifix cubes [13], and two-sided counters used in the classroom as substitutes for cookies. The second type of materials used by the authors included teaching and learning mathematics standards [5,14] and recommendations for mathematics teacher preparation in the United States [15–17]. These educational documents recommend the appropriate use of concrete materials and digital tools in the classroom, provide expectations for mathematics teachers and their students, and offer teaching ideas for elementary classrooms and mathematics education courses. In full agreement with the above-mentioned documents, methods specific to mathematics education used in this paper include technology-based instruction, standards-based mathematics, and problem solving. In particular, by reflecting on young students' work through an advanced lens, those methods are conducive to preparing teacher candidates "with broad and deep understanding of fundamental mathematics" [16]. Such field-based reflections are critical to enabling teacher candidates' professional development in support of the notion that "mathematics courses that explore elementary school mathematics in depth can be genuinely college-level intellectual experiences, which can be interesting for instructors to teach and for teachers to take" [15].

The problem-solving methods and conceptual methodology used in this paper follow the ideas of computational triangulation introduced by Abramovich [18] and collateral creativity introduced by Abramovich & Freiman [6]. The term computational triangulation grew out of the first author's experience using computers with future K-12 teachers of mathematics. The use of sophisticated and diverse software tools by teacher candidates' learning mathematics, both as a subject matter and a teaching discipline, allows them to appreciate the role of the intrinsic complexity of the tools in the development of deep mathematical knowledge required nowadays for teaching mathematics across the grades. In the age of technology, when mathematical problem solving involves more than one digital tool, computational triangulation can be used in mathematics education, in general, and in its teacher education component, in particular, as a way of providing rigor through enhancing the credibility of computational experiments [19]. Finally, the notion of collateral creativity is similar to the notion of "collateral learning" [20] in the sense that just as the latter does not result from the immediate goal of the traditional curriculum, the former does not result from the immediate goal of intended problem solving and can be upheld when a teacher is capable of addressing students' emerging creative ideas at least in some way. After all, the goal of an elementary mathematics teacher education course is to nurture schoolteachers who are not afraid of mathematics [21] and look forward "to empower each and every student" [16].

The activities spanned over two sessions, 70 min each. About 20 student-participants were assisted by the authors, two classroom teachers, a student teacher candidate, a teacher assistant and an aid. The first session started with four plates asking students to create images of putting cookies on the first two plates so that the third and the fourth plates have as many cookies as the previous two plates combined to enable exactly 11 cookies on the fourth plate. The students were given 20 min to solve the problem; having seven adults in the classroom to assist students and record their work.

3. The 3rd graders' absorbing ideas about patterns as osmosis

Note that only the authors were familiar with the problems (used with teacher candidates in a mathematics content and methods course); for other adults the problems were new and almost as challenging as for the 3rd graders. The students were told by the second author that, in her experience based on work with 3rd graders in another school, kids are often more creative than adults. Interesting, many things said by the authors to the 3rd graders were absorbed by the kids as one of them even noted in a solicited reflection "*one thing I learned is that creativity is big in kids but small in adults*". Similarly, another student wrote "*that math can be used everywhere and kids have better creativity then [sic.] adults*".

At the end of the first session, the first author, with reference to **Figure 2**, advised the 3rd graders that knowing patterns formed by cookies and their numerical description can help solving problems by using trial and error only one time. For example, knowing that cookies on the first and second plates increase/decrease by two/one, if the solution (1, 5, 6, 11), shown in **Figure 3**, were found, other solutions, (3, 4, 7, 11), (5, 3, 8, 11), (7, 2, 9, 11) and (9, 1, 10, 11), can be found by using the pattern recognized. Even without conceptual understanding of the development of the pattern, the solutions, once found, can be verified numerically. Although this idea was not an easy one for the 3rd graders, it was nonetheless absorbed through osmosis as students shared with their classroom teacher through solicited reflections: "*I learned there are patterns in the different rows of cookies*", and "*math can be used in different ways and in math there are patterns*". The ability of young children to develop through osmosis mediated by adults can be found in the writings of Vygotsky [22]: "Although at an early stage of mathematical development, quantitative reasoning and arithmetical thinking of a child are pretty vague and immature in comparison with those of an adult with whom the child interacts, it is through this interaction that the final forms of reasoning and thinking about numbers, that have to be developed as a result of having an adult in his/her environment, are somehow present at that stage and, not only present but in fact define and guide the child's first steps toward the development of the final forms of understanding quantity and comprehending arithmetic". It appears that this quote assumes the presence of mediated osmosis in the child-adult interaction.

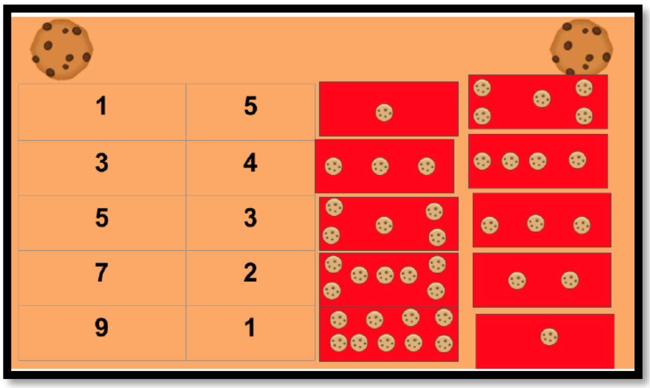

Figure 2. Do you see a pattern?

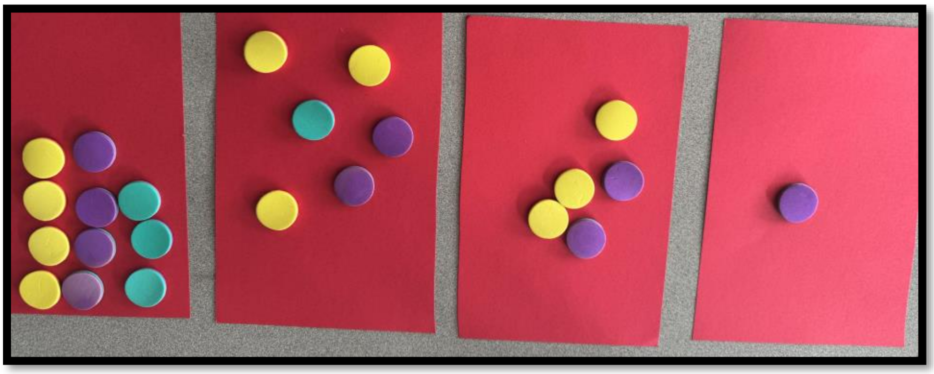

Figure 3. One solution with 11 cookies on the fourth plate.

4. Extending Fibonacci-like numbers to fractions through play

National Council of Teachers of Mathematics [14], in the description of the Representation (process) standard included a pictorial representation by a child of the number $5\frac{1}{2}$ in the form of the digit 5 and its upper part. A similar representation of $\frac{1}{2}$ was offered by a 3rd grader through half a cookie (**Figure 4**), when the child through play with "cookies" had half a cookie on the second and third plates so that two halves make a full cookie. The child imagined a situation when half a cookie may be put on a plate, found a possible instrumental representation, and followed the rule structured by Fibonacci-like numbers as "there is no such thing as play without rules" [4]. When rules are defined by mathematics, young children's engagement with concrete materials, like two-sided counters, does not mean just fun but evolves in learning thereby addressing the dichotomy fun vs. learning discussed by Quigley [23]. Indeed, the child correctly added cookies to end up with 11 cookies on the fourth plate. This, however, required having two cookies on the first plate, i.e., "an imaginary situation can be regarded as a means of developing abstract thought" [4]. With this in mind note, that algebraically, the equation $x + 2y = 11$ is equivalent to $y = \frac{11-x}{2}$ and whereas only odd values of x allow for an integer value of y, if we allow the second plate to have half a cookie among the cookies, x may only be an even number from the set {2, 4, 6, 8, 10} yielding $y \in \left\{4\frac{1}{2}, 3\frac{1}{2}, 2\frac{1}{2}, 1\frac{1}{2}, \frac{1}{2}\right\}$, respectively. That is, to have 11 cookies on the fourth plate, we have the following five quadruples with the first plate having an even number of cookies:

$$\left(2, 4\frac{1}{2}, 6\frac{1}{2}, 11\right), \left(4, 3\frac{1}{2}, 7\frac{1}{2}, 11\right), \left(6, 2\frac{1}{2}, 8\frac{1}{2}, 11\right), \left(8, 1\frac{1}{2}, 9\frac{1}{2}, 11\right), \left(10, \frac{1}{2}, 10\frac{1}{2}, 11\right).$$

That is, with full cookies on each plate we have $x \in \{1, 3, 5, 7, 9\}$ and to allow for half cookies on the second and the third plates we have $x \in \{2, 4, 6, 8, 10\}$. One can see that allowing half a cookie on the first plate would require a quarter of a cookie for which we do not have a physical representation. Bringing "teachers' mathematics" to their future students can motivate activities for teachers themselves by extending what children did. This is what may be referred to as collateral creativity [6]—when such creativity is displayed by a student (whatever their age is), it leads to an interesting mathematical problem to be explored that is within the student's zone of proximal development. In the age of technology, multiple computational tools can be used in the teacher education classroom to demonstrate same phenomenon stemming from a collaterally creative idea.

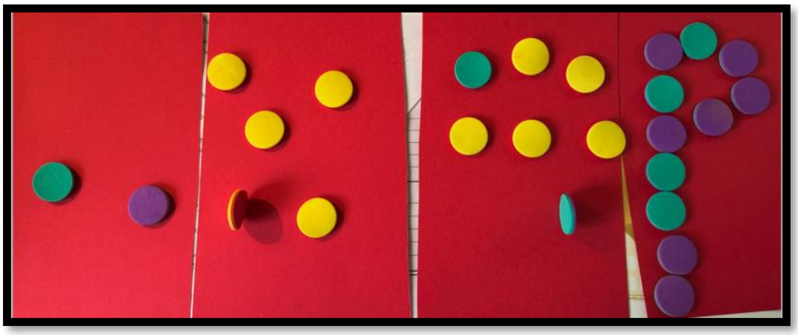

Figure 4. From integer to fractions through play: $2, 4\frac{1}{2}, 6\frac{1}{2}, 11$.

Several problems motivated by collateral creativity of a 3rd grader can be posed for teacher candidates using multiple digital tools in the true spirit of computational triangulation. As shown in **Figure 5** (spreadsheet), **Figure 6** (Graphing Calculator), and **Figure 7** (Wolfram Alpha) when there are only four plates, half a cookie can be seen on the second plate only (thereby requiring half a cookie on a third plate).

	A	B	C	D
1	cookies	11	plates	4
2				
3	1st plate	2nd plate		
4	0.5			
5	1	5	6	11
6	1.5			
7	2	4.5	6.5	11
8	2.5			
9	3	4	7	11
10	3.5			
11	4	3.5	7.5	11
12	4.5			
13	5	3	8	11
14	5.5			
15	6	2.5	8.5	11
16	6.5			
17	7	2	9	11
18	7.5			
19	8	1.5	9.5	11
20	8.5			
21	9	1	10	11
22	9.5			
23	10	0.5	10.5	11
24	10.5			

Figure 5. Ten ways to have 11 cookies on the fourth plate if half a cookie is allowed.

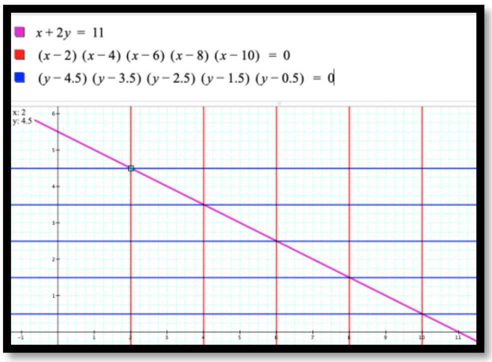

Figure 6. Five ways of putting 11 cookies on 4 plates using half a cookie.

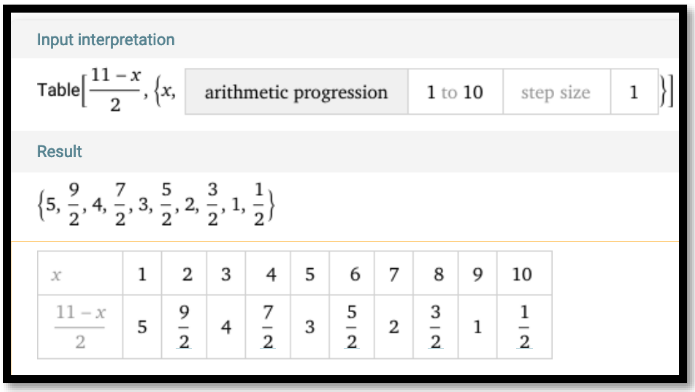

Figure 7. Wolfram alpha confirming spreadsheet solution of **Figure 5**.

Problem 1. Assuming that the fourth plate must have a whole number of cookies, is it possible not to have a whole number of cookies on the first plate? Why or why not?

Problem 2. Assuming that the fifth plate must have a whole number of cookies, is it possible not to have a whole number of cookies on the first plate? Why or why not?

Whereas solving the above two problems by trial and error, just as the 3rd graders did with one of them demonstrating collateral creativity due to the use of counters as substitute for cookies and even half a cookie, might be difficult and cannot be recommended, teacher candidates can use algebra of two-variable linear equations. Already students in grade 6 have been taught to “reason about and solve one-variable equations and inequalities” [5] and it is quite appropriate to engage their future teachers in reasoning involving two-variable linear inequalities. In their mathematics education course, the word inequality is often used in the context of using three integers as the side lengths of a triangle the existence of which depends on whether the integers satisfy the triangle inequality.

Note that if x and y stand for the number of cookies on the first and second plates, respectively, the number of cookies on the fourth and fifth plates are $x + 2y$ and $2x + 3y$, respectively. Assuming that only half a cookie may be used (as one can see in **Figure 4**), the equation $x + 2y = n$ (Problem 1) yields $x = n – 2y$ and, therefore, when $y = \frac{k}{2}$, $k < n$, we have $x = n – k$, an integer; that is, the first plate (as well as the fourth plate) may have only a whole number of cookies. The situation with five plates is different though. The equation $2x + 3y = n$ (Problem 2) yields $x = \frac{n-3y}{2}$. First, if $y = \frac{k}{2}$, k being an odd number, we have $x = \frac{n-\frac{3k}{2}}{2} = \frac{2n-3k}{4}$ and $2n - 3k$ must be a multiple of two, in order to avoid quarters of a cookie. This implies that k is a multiple of two, thus making $y = \frac{k}{2}$ a whole number. Second, the relation $x = \frac{n-3y}{2}$ shows that the first plate may include half a cookie (whereas the second plate may not). For example, when the fifth plate have 11 cookies ($n = 11$) we have the following three solutions: $y = 1$, $x = 4$; $y = 2$, $x = 2\frac{1}{2}$; $y = 3$, $x = 1$. The only case with having half a cookie on the first plate is represented by the quintuple $\left(2\frac{1}{2}, 2, 4\frac{1}{2}, 6\frac{1}{2}, 11\right)$. That is, it is impossible not to have a whole number of cookies on the first plate (Problem 2). Note that rewriting the relation for the fifth plate in the form $y = \frac{n-2x}{3}$, the case $n = 11$ yields the following five quintuples (three of which include thirds):

$$(1, 3, 4, 7, 11), \left(2, 2\frac{1}{3}, 4\frac{1}{3}, 6\frac{2}{3}, 11\right),$$

$$\left(3, 1\frac{2}{3}, 4\frac{2}{3}, 6\frac{1}{3}, 11\right), (4, 1, 5, 6, 11), \left(5, \frac{1}{3}, 5\frac{1}{3}, 5\frac{2}{3}, 11\right).$$

However, there is no physical representation for 1/3 of a cookie, unless one uses virtual manipulatives as shown in **Figure 8**. Thus, we can extend Fibonacci-like numbers to rational numbers. **Figure 9** shows Wolfram Alpha confirming the number of cookies on the first two plates if 1/3 of a cookie is allowed. An interesting problem to explore is whether other fractions can be used for x and y to reach 11 on the fifth plate.

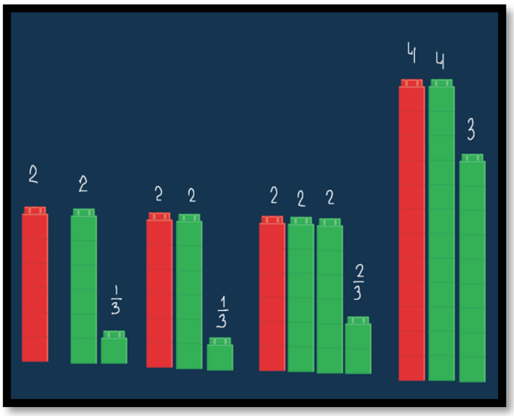

Figure 8. The quintuple $\left(2, 2\frac{1}{3}, 4\frac{1}{3}, 6\frac{2}{3}, 11\right)$.

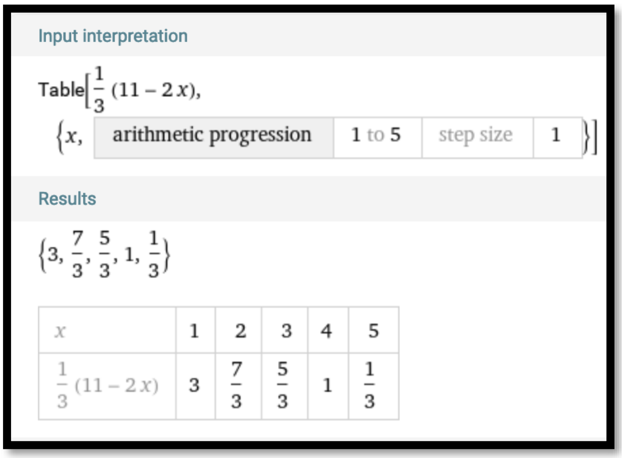

Figure 9. Wolfram alpha confirming manipulative solution with one-third of a cookie.

5. From the 3rd grade classroom to a course for teacher candidates

Several questions can be discussed with teacher candidates that are motivated by the activities with young children were mentioned in the previous section. Although "play creates a zone of proximal development of the child" [4], those questions were not asked by the latter group of students, yet the former group would benefit from being prepared to answer those and like questions or at least to know how to answer them. As mentioned by the National Council of Teachers of Mathematics [14], "Given their primary role in shaping the mathematical learning of their students, teachers in grades 3–5 often must seek ways to advance their own understanding". Furthermore, Common Core State Standards [5] expect students in grade 3 "identify and explain patterns in arithmetic". To help young learners of mathematics to meet such ambitions expectations about explaining arithmetical patterns, Association of Mathematics Teacher Educators [16] call on teacher candidates to recognize and appreciate the presence of "depth and complexity" within elementary mathematics concepts. To this end, "teachers need opportunities to look for and use regularity and structure by seeking to explain the phenomena they observe" [15].

For example, the 3rd graders found the following five solutions for 11 cookies on the fourth plate: (1, 5, 6, 11), (3, 4, 7, 11), (5, 3, 8, 11), (7, 2, 9, 11) and (9, 1, 10, 11). Likewise, with 12 cookies on the fourth plate one can find another four solutions: (2, 5, 7, 12), (4, 4, 8, 12), (6, 3, 9, 12), (8, 2, 10, 12) and (10, 1, 11, 12). When encouraged to look for patterns some children reported that with four plates the first and the fourth plates may only both have either odd number of cookies or an even number of cookies. Also, in both cases, the second plates have the same number of cookies. How can these observations be explained?

The explanation is not beyond the elementary mathematics level, but it requires that teachers have skills in algebraic thinking. If unknowns x and y represent the number of cookies on the first and second plates, respectively, then the sums $x + y$ and $x + 2y$ represent cookies on the third and fourth plates, respectively. Therefore, if x is even or odd, then $x + 2y$ is also even or odd, respectively, due to the second addend being even regardless of y. Explanation that the same number of cookies appear on the second plate for both 11 and 12 cookies on the fourth plate, note that when $x + 2y =$ 11 we have x an odd number, $1 \leq x \leq 9$, $y = \frac{11-x}{2}$. Then $-9 \leq -x \leq -1$, $2 = 11 - 9 \leq 11 - x \leq 11 - 1 = 10$ and $1 \leq \frac{11-x}{2} \leq 5$ or $1 \leq y \leq 5$. Likewise, when $x + 2y = 12$ we have x an even number, $2 \leq x \leq 10$, $y = \frac{12-x}{2}$. Then $-10 \leq -x \leq -2$, $2 = 12 - 10 \leq 12 - x \leq 12 - 2 = 10$ and $1 \leq \frac{12-x}{2} \leq 5$ or $1 \leq y \leq 5$. That is, in both cases, y is an integer, $1 \leq y \leq 5$. **Figure 10** shows that in any quadruple of integers developed like cookies on plates the first and the last numbers are of the same parity.

1	5	6	11	17	28	45	73	118	191	309	500	809	1309	2118	3427	5545	8972	14517
2	5	7	12	19	31	50	81	131	212	343	555	898	1453	2351	3804	6155	9959	16114

Figure 10. In any quadruple of integers, the first and the last terms have same parity.

The 3rd graders found that when the fifth plate has 11 cookies, the first two plates have either 1 and 3 cookies or 4 and 1 cookies, respectively. When the fifth plate has 13 cookies, the first two plates have either 2 and 3 cookies or 5 and 1 cookies, respectively. When the fifth plate has 15 cookies, the first two plates have either 3 and 3 cookies or 6 and 1 cookies, respectively. **Figure 11** was displayed on the screen and students were asked if they see any pattern. The students recognized that one of the two solutions in all three cases has always second plate either with one cookie or with three cookies. When asked for more patterns, they noticed that when the second plate has one cookie, the first plate is the sum of the number of cookies on the first and the second plate from "the different rows of cookies". These observations can be brought to the attention of teacher candidates as a motivation for interpreting the phenomena recognized at the 3rd grade level. To this end, the sequence x, y, $x + y$, $x + 2y$ and $2x + 3y$ representing the number of cookies on the first five plates must be considered. It was observed that when this sequence represents the first solution, then the sequence $x + y$, 1, $x + y + 1$, $x + y + 2$, $2x + 2y + 3$ represents another solution. Consequently, $2x + 2y + 3 = 2x + 3y$ whence $y = 3$ and $x = 1$ —the number of cookies on the second and the first plates of one of the two solutions (or "rows of cookies", as a child put it). One can see that whenever, in the first solution, the number of cookies on the second plate is three, the number of cookies on the first plate of the second solution is equal to as many cookies as we have on the first two plates of the first solution. All these observations are worthy to be discussed with teacher candidates in an asynchronous elementary mathematics content and method course using forums that provide instructors (the authors' included) with noninvasive teaching modalities [24,25] immune from time constraints of the curriculum to be covered.

Figure 11. What do you notice?

6. Results

Whereas the authors expected the results of the project to address its initial objectives—checking out "teachers' mathematics" via the lens of 3rd grade students and collecting field data for the enrichment of teacher education courses—the most unexpected result of the project was somewhat collateral to the objectives. Indeed, nobody expected a 3rd grader, through play, to be instrumental in putting a counter in the vertical position and to describe such position as half of a cookie. This unexpected perspective on the play with "cookies" was collateral to what the students were expected to do but it resulted from the use of technology, physical in that case. Indeed, as mentioned elsewhere [6], "collateral creativity does not result from the immediate goal of traditional problem solving … [and it] emerges as a collateral outcome of using technology". It appears that Fibonacci-like numbers involving non-integer initial values were not previously considered, at least in mathematics education. Some interesting computational problems can be explored in the context of this extension. As far as initial objectives of the project are concerned, it was found that the 3rd graders were able to absorb the idea about problems with more than one correct answer, to recognize patterns formed by "cookies", and to be receptive to the notion that the grasp of patterns can be used as a tool replacing trial-and-error with formal reasoning when solving problems. All things considered, the authors collected quite a few interesting things to enrich mathematics education content and methods courses taught both at the undergraduate and graduate levels.

7. Conclusion

This paper was written to reflect on the authors' work with 3rd grade students at the elementary school in rural upstate New York. Whereas the original intent of activities that the authors designed was to investigate how children can see patterns formed by cookies on plates (first order symbols) and their numerical description (second order symbolism), an unexpected outcome of the activities was the display of collateral creativity by a 3rd grader who, through play, used his own representation of the half of a cookie, something that was within the rules of the activities requiring the number of cookies on the last (fourth of fifth) plate to be a whole number. This made it possible for the authors to recognize the potential of collateral creativity for extending the concept of Fibonacci-like numbers to fractions. It was due to the context of cookies on plates within which collateral creativity of a student was displayed that made such existence possible. As noted by Hilbert [26] in his 1900 address to the International Congress of Mathematicians, "the first and the oldest problems in every branch of mathematics spring from experience and are suggested by the world of external phenomena". That is, cookies on plates and two-sided counters represented external context for the collateral creativity to take place during the project and the associated mathematical problems to be discussed as its extension to teacher education.

Towards the end of using the results of work with young children in the preparation of future teachers of the primary school mathematics, the use of multiple digital tools was demonstrated through the concept of computational triangulation towards using the physically external context of cookies on plates for the demonstration of rigorous extension of activities to the mathematically internal context of Fibonacci-like rational numbers. That is, the interplay between two educational settings—a teacher education course and the 3rd grade classroom—bear fruit. One of the students when asked to reflect on activities with cookies on plates admitted, "I learned that you can have different answers to the same problem". Having such students in grade school requires knowledgeable teachers who are aware of the significance of the problems with multiple solutions [27] and can take an intellectual risk to "change their plans and follow unexpected learning trajectories initiated by the students" [28]. Often, however, mathematics teachers, especially those with "math anxiety" [2,3], when facing students whose thinking is different from what is expected "may inadvertently seek to remedy those differences rather than seeing them as strength and resources upon which to build" [16]. This distinction between seeking remediation of and strengthening incongruity in students' thinking is one of the most important aspects of collateral creativity as the modern-day psychological phenomenon that celebrates diversity. Encouraging the use of various technology tools within the same context and accepting multiple solutions to a single question enhances the continuity of student-teacher productive conversation both in grade school and teacher preparation programs.

Future research directions can be outlined. Because elementary teacher candidates are certified to teach up to grade six, the context of cookies on plates can be extended to include fractions as described through Problem 1 and Problem 2. Manipulatives for the fractions of cookies may be either created in the form of (small) paper fraction circles or pattern blocks in the form of hexagons (full cookie), trapezoids (one-half a cookie), rhombuses (one-third a cookie), and triangles (one-sixth a cookie) can be used. That is, the collaterally creative idea of a 3rd grade student can be put to work within another action research at the upper elementary (or middle school) level. Furthermore, the ideas of computational triangulation can be discussed with 6th graders pursuing new collaterally creative ideas that might emerge from their use of digital technology. A limitation of the current study was the presence of only one teacher candidate in the classroom. In the future, the authors plan to invite more teacher candidates interested in participating in a project as informed assistants to students.

Author contributions: Conceptualization, SA and LLG; methodology, SA and LLG; validation, SA and LLG; formal analysis, SA and LLG; writing—original draft preparation, SA and LLG; writing—review and editing, SA and LLG. All authors have read and agreed to the published version of the manuscript.

Acknowledgments: The authors express their gratitude to Amy Cutler and Julie Jenkins, 3rd grade teachers, and to their students at Lawrence Avenue Elementary School, Potsdam, New York, United States, for cooperation during the project. Special thanks are extended to Sharlee Thomas, Principal, and Joanna Cross, Assistant Principal of the School as well as to Jerry Griffin, Superintendent of Potsdam Central School District, for allowing us to conduct our action research.

Conflict of interest: The authors declare no conflict of interest.

References

1. Abramovich S, Griffin LL. From a teacher education course to upper elementary classrooms and back: Revealing innate abilities of children to do teachers' mathematics. Advances in Educational Research and Evaluation. 2024; 4(1): 239-249. doi: 10.25082/aere.2023.01.004
2. Barroso C, Ganley CM, McGraw AL, et al. A meta-analysis of the relation between math anxiety and math achievement. Psychological Bulletin. 2021; 147(2): 134-168. doi: 10.1037/bul0000307
3. Schaeffer MW, Rozek CS, Maloney EA, et al. Elementary school teachers' math anxiety and students' math learning: A large-scale replication. Developmental Science. 2021; 24(4). doi: 10.1111/desc.13080
4. Vygotsky LS. Mind in Society. Cambridge, MA: MIT Press; 1978.
5. Common Core State Standards. Common Core State Standards Initiative: Preparing American students for college and career. Available online: http://www.corestandards.org (accessed on 10 December 2024).
6. Abramovich S, Freiman V. Fostering Collateral Creativity in School Mathematics. Springer International Publishing; 2023.
7. Harlen, W, Léna, P. The Legacy of the Fibonacci Project to Science and Mathematics Education. Available online: https://fondation-lamap.org/sites/default/files/pdf_res_int/fibonacci_book.pdf (accessed on 10 December 2024).
8. Abramovich S, and Brown G. Integrating problem solving, technology, and the experience of mathematical discovery in teacher education. Journal of Computers in Mathematics and Science Teaching. 1996; 15(4): 287-302.
9. Molloy C, Farrell R. Cultivating Positive Classroom Environments: Exploring the Efficacy of Immersive Technologies in Removing Barriers to Learning Among Primary School Students. Computers in the Schools. 2024; 41(2): 164-192. doi: 10.1080/07380569.2024.2325441
10. National Council of Teachers of Mathematics. Principles to Actions: Ensuring Mathematical Success for All. National Council of Teachers of Mathematics; 2014.
11. Wolfram Research. Available online: www.woframalpha.com (accessed on 10 December 2024).
12. Avitzur R. Graphing Calculator. In: Computer software, 4.1 ed. Pacific Tech; 2011.
13. Unifix cubes. Available online: https://www.didax.com/apps/unifix/ (accessed on 10 December 2024).
14. National Council of Teachers of Mathematics. Principles and Standards for School Mathematics. National Council of Teachers of Mathematics; 2000.
15. Conference Board of the Mathematical Sciences. The mathematical education of teachers II. Washington, DC: The Mathematical Association of America; 2012.
16. Association of Mathematics Teacher Educators. Standards for Preparing Teachers of Mathematics. Available online: https://amte.net/standards (accessed on 10 December 2024).
17. Council for the Accreditation of Educator Preparation. CAEP 2018 K-6 Elementary Teacher Preparation Standards. Council for the Accreditation of Educator Preparation; 2018.
18. Abramovich S. Computational Triangulation in Mathematics Teacher Education. Computation. 2023; 11(2): 31. doi: 10.3390/computation11020031
19. Freudenthal H. Weeding and Sowing. Kluwer; 1978.
20. Dewey J. How We Think: A Restatement of the Relation of Reflective Thinking to the Educational Process. Boston, MA: D.C. Heath & Co Publishers; 1933.
21. Artemenko C, Masson N, Georges C, et al. Not all elementary school teachers are scared of math. Journal of Numerical Cognition. 2021; 7(3): 275-294. doi: 10.5964/jnc.6063

22. Vygotsky LS. Lectures on the dynamic of child development (Russian). Available online: https://www.marxists.org/russkij/vygotsky/pedologia/lektsii-po-pedologii.pdf (accessed on 10 December 2024).
23. Quigley MT. Concrete materials in primary classrooms: Teachers' beliefs and practices about how and why they are used. Mathematics Teacher Education and Development. 2021; 23(2): 59-78.
24. Mitra S, Rana V. Children and the Internet: experiments with minimally invasive education in India. British Journal of Educational Technology. 2001; 32(2): 221-232. doi: 10.1111/1467-8535.00192
25. Tolkacheva E, Ivanov S, Pozdniakov S. Implementation of Horizontal Connections in the Course of Mathematics by Combining Pedagogical and Digital Technologies. Mathematics. 2022; 10(13): 2352. doi: 10.3390/math10132352
26. Hilbert D. Mathematical problems. Bulletin of the American Mathematical Society. 1902; 8(10): 437-479. doi: 10.1090/s0002-9904-1902-00923-3
27. Abramovich S. Using Wolfram Alpha with Elementary Teacher Candidates: From More Than One Correct Answer to More Than One Correct Solution. Mathematics. 2021; 9(17): 2112. doi: 10.3390/math9172112
28. Leikin R, Dinur S. Teacher flexibility in mathematical discussion. The Journal of Mathematical Behavior. 2007; 26(4): 328-347. doi: 10.1016/j.jmathb.2007.08.001