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On the interplay of cognitive, tactile, and computational approaches to number theory in teacher education

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ABSTRACT: This paper focuses on topics from elementary number theory used by the author in the mathematical preparation of K-12 teacher candidates by juxtaposing concrete materials, digital technology, and formal reasoning. The topics include triangular numbers, their connection to trapezoidal numbers, and their extension to other figurate numbers. The paper shows how problem solving may be based on the integration of modern-day approaches to mathematics supported by the creation of images, their numeric interpretation, followed by algebraic generalization, and computational verification of general statements in symbolic form. Digital tools used in the paper include spreadsheets, *Wolfram Alpha*, and *Maple*. Solicited comments by teacher candidates about their use of digital tools are shared.

KEYWORDS: number theory; visualization; polygonal numbers; trapezoidal numbers; *Maple*; *Wolfram Alpha*; teacher education

1. Introduction

Elementary number theory is one of the oldest branches of mathematics. It studies the properties of natural numbers (e.g., primality[1]), their representations through other numbers (e.g., through the sums of cubes of integers[2]), or the special behavior of number sequences under certain rules (e.g., the $3x + 1$ problem[3]). There are many number theory problems that, despite the ease of their formulation, are still unsolved. For example, although it is known from the time of Euclid (3rd century B.C.) that the number of primes is infinite, it is still unknown (as of the time of writing this paper) whether this is true for twin primes. Whereas some simple problems, like the impossibility of representing powers of two as sums of consecutive natural numbers, can be proved by a high school student (see Section 6), it remains a puzzle for an elementary school student, who can only discover this fact through trial and error in the context of special cases. All things considered, as Hardy[4] put it, "The elementary theory of numbers should be one of the very best subjects for early mathematical instruction. It demands very little previous knowledge; its subject matter is tangible and familiar; the processes of reasoning that it employs are simple, general, and few; and it is unique among the mathematical sciences in its appeal to natural human curiosity." More recently, a similar point of view was presented by Vavilov[5] who argued that number theory is comprised of ideas that can be taught at any level of learning mathematics, including grade school.

One enjoyable topic in elementary number theory, with little prerequisite knowledge, deals with polygonal (figurate) numbers. The representation of numbers as simple geometric figures goes back to the arithmetic of antiquity, when certain numbers were recognized (most likely, contextually) to have different characteristics from others. As noted by Smith[6] in reviewing topics from the history of elementary mathematics, "being ancient even in Plato's time... [was the game of] guessing odd or even with respect to the number of coins or other objects held in hand". Before arithmetic was conceptually advanced to become the theory of numbers, people used simple visual patterns to portray numbers. The use of concrete materials for mathematical visualization has great educational significance. For example, billiard balls could be placed like pins in a bowling alley to form different triangles, and in such a way, the number becomes triangular. In much the same way, in a pre-college classroom, multicolored counters can form other geometric figures that represent numbers known as trapezoidal, square, pentagonal, hexagonal, etc. All these numbers, with their genesis in geometry, have been studied in elementary number theory. Arithmetic and geometry are the two roots from which the whole mathematics has grown; their mutual relation and consequently the more general interrelation of all mathematical theories have been used both in the study and the teaching of the subject matter.

This paper is written as an extension of an earlier collaborative work by the author on the use of multiple digital tools in the study of elementary number theory by prospective teachers of mathematics through numeric and graphic modeling[7]. This extension allows for the inclusion of more advanced technologies, such as *Wolfram Alpha* and *Maple,* capable of symbolic computations needed for a deeper inquiry into the properties of numbers. The paper reflects on several activities designed by the author for K-12 teacher candidates in order to demonstrate the computational capabilities of modern-day digital tools. The activities, motivated by mathematical situations arising in the context of using concrete materials, are aimed at investigating relationships among triangular and other polygonal numbers. Some of the activities can be described through the lens of TITE (technology-immune/technology-enabled) problem-solving[8], something that is supported by the interplay of formal mathematical reasoning that is at the core of computational thinking[9], tactile approaches to mathematical visualization[10], and digital symbolic computations, the use of which, in the words of Vygotsky[11], "immensely extends the possibilities of behavior by making the results of the work of geniuses available to everyone". Among those "everyone" are teacher candidates and the author's students, whose solicited comments are shared as appropriate. The paper and the comments provide a glimpse into the experience of teaching and learning mathematics in the United States in the era of Common Core[12].

2. Materials and methods

Two types of materials have been used by the author when working on this paper. The first type included "mathematical action technologies"[13]—electronic spreadsheet, computational knowledge engine *Wolfram Alpha* developed by Wolfram Research (www.woframalpha.com; accessed on July 23, 2023), *Maple*[14]—mathematics software for education and STEM fields, and multicolored counters available online.

The second type of materials used by the author included teaching and learning mathematics standards[12,15] and recommendations for mathematics teacher preparation in the United States[16,17]. These materials recommend the appropriate use of technology in the classroom, provide expectations for mathematics teachers, and offer ideas for mathematics education courses. In full agreement with the above-mentioned documents, methods specific to mathematics education used in this paper include computer-based instruction, standards-based mathematics, and problem solving. In particular, those methods are conducive to presenting "teacher candidates with experiences in mathematics relevant to their chosen profession"[17]. These experiences are critical to enabling teacher candidates' professional development as they "use number theory to justify relationships involving whole numbers"[15] and "connect mathematical practices to mathematical content in mathematics instruction"[12]. The most important characteristic of such development in the context of teacher education courses is to provide evidence of how "reasoning and proof... is... intellectually satisfying for all students... [and how] the proper use of technology can make complex ideas tractable"[16].

Finally, the problem-solving methods and conceptual methodology used in this paper follow the TITE (technology-immune/technology-enabled) framework introduced in the work of Abramovich[8]. As was mentioned in the introduction, the framework represents an interplay of cognitive and technological approaches to mathematics that are mutually complimentary in the sense that the accuracy of computations is dependent on the correctness of reasoning, and the correctness of reasoning is verified through computations. Activities to which the TITE framework is applied are discussed in Sections 5 and 6.

3. Mathematics education literature on triangular numbers

Many mathematics education papers published around the world within the last two decades have been devoted to activities with triangular and square numbers. Canadian mathematics teacher educators[18] discussed the emergence of triangular numbers in the context of the so-called toothpicks, or alternatively, matchsticks[19] problem. In the study of Pedemonte[20], Italian mathematics educators described their work with upper secondary school students (17–18 years old) towards finding and proving a general rule for the *n*-th triangular number by counting the number of dots in an equilateral triangle with *n* dots on a side representing a triangular number (see the top part of **Figure 6** below). Researchers from Philippines[21], motivated by the work of Asiru[22] in Nigeria, presented several summation formulas in the context of triangular numbers using the method of mathematical induction for their proof. Bütüner[23], in the context of the upper middle school in Turkey, advocated for the use of history of mathematics and concrete materials in the teaching of formulas for the sums of natural, triangular, and square numbers. German mathematics educators[24] described how fourth graders were developing a formula for triangular numbers in the context of substantial learning environments[25]. A Spanish mathematics educator, Plaza[26] demonstrated the visual development of formulas, including the sum of triangular numbers, using the "proof without words" approach. Australian mathematics educators[27] worked with 10–11-year-old pupils who used concrete materials (e.g., a fruit pyramid) and diagrams to connect triangular and square numbers. Most recently, Demircioglu[28] expressed concern regarding the low ability of Turkish future teachers of fourteen-age students to prove the (ready-made rather than mathematically derived) formula for the sum of triangular numbers using the method of mathematical induction. Despite their worldwide span and the diversity of contexts, none of the above-mentioned papers used digital technology in the context of triangular numbers and their generalizations to polygonal numbers, not to mention trapezoidal numbers. This may serve as a justification for writing the present paper in order to promote the use of commonly available digital (as well as tactile) tools, enabling deeper inquiries into the arithmetic of integers. In no small part, the paper confirms the viewpoint that "mathematics courses that explore elementary school mathematics in depth can be genuinely college-level intellectual experiences, which can be interesting for instructors to teach and for the teachers to take"[16].

4. From triangular to polygonal numbers using the interplay of three approaches

Triangular numbers t_n are partial sums of natural numbers: $t_1 = 1, t_2 = 1 + 2 = 3, t_3 = 1 + 2 + 3 = 6, t_4 = 1 + 2 + 3 + 4 = 10$, ..., $t_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$. There are many interesting properties that triangular numbers satisfy. For example, the difference between two consecutive triangular numbers is the rank of the larger number (the top part of **Figure 1**) and the sum of two consecutive triangular numbers is a square of the rank of the larger number (the bottom part of **Figure 1**). Using formula (1), the following two statements (the second one known since the 4th century A.D. as the theorem of Theon) can be immediately verified:

$$t_n = \frac{n(n+1)}{2} \tag{1}$$

$$t_n - t_{n-1} = \frac{n(n+1)}{2} - \frac{(n-1)n}{2} = n, \tag{2}$$

$$t_n + t_{n-1} = \frac{n(n+1)}{2} + \frac{(n-1)n}{2} = n^2. \tag{3}$$

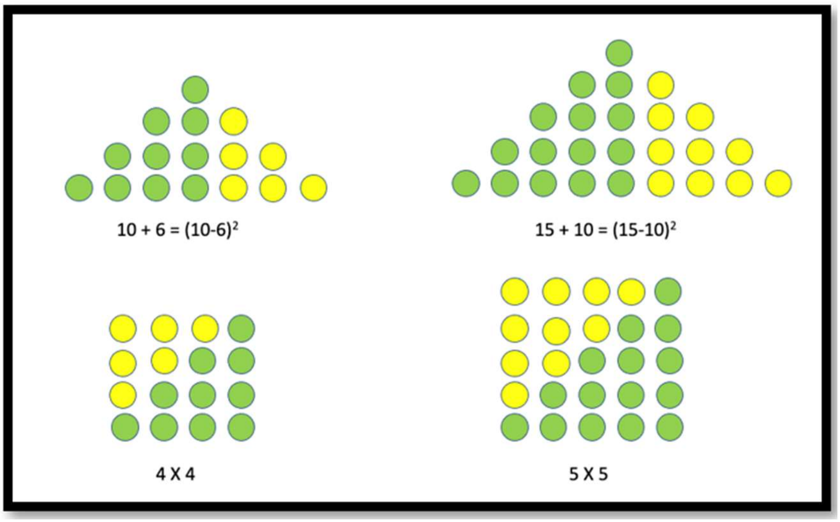

Figure 1. Adding and subtracting two consecutive triangular numbers.

One can note that just as between the natural numbers *m* and *n*, *n* > *m*, there are *n* − *m* − 1 natural numbers, between the triangular numbers t_m and t_n, there are (*n* − *m* − 1) triangular numbers (**Figure 2**, *n* = 7, *m* = 4). Put another way, the quantity of triangular numbers that one skips when moving from a triangular number to the next triangular number, is the same as the quantity of integers between the corresponding two ranks. That is, the quantity in question is the difference between the ranks decreased by one. Visually, as shown in **Figure 2**, there are two counters between seven and four counters representing the ranks of t_7 and t_4. Formally, the statement about the quantity of integers between *m* and *n* can be proved by setting *n* = *m* + *k* and creating the list of integers *m*, *m* + 1, *m* + 2, ..., *m* + (*k* − 1), *m* + *k* with *k* − 1 = *n* − *m* − 1 integers between *m* and *m* + *k* = *n*. To formally prove the statement about the quantity of triangular numbers between t_m and t_n, one has to note that triangular numbers can be put in one-to-one correspondence with their respective ranks, so what is true for counting the ranks is true for counting their more complicated matches. The significance of this note is due to the distinction between tactile (action) and computational (operation): at the bottom row of counters in **Figure 2**, one *skips two* counters when moving from four counters to seven counters but *subtracts three* (or adds three) to get from the number 7 to the number 4 (or from 4 to 7). A 1st grade student was observed asking a question during a lesson on skip counting: Why do we call it skip counting by *two* when we skip *one* number only?

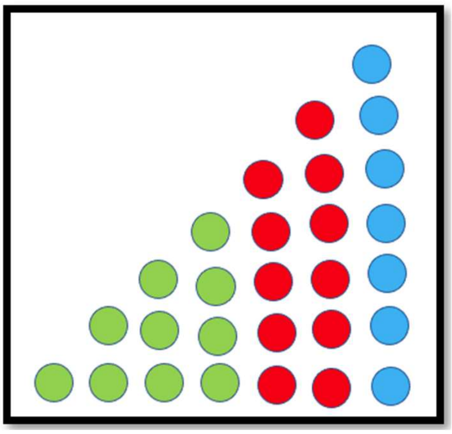

Figure 2. Between t_m and t_n there are $(n - m - 1)$ triangular numbers (m = 4, n = 7).

Leonhard Euler (1707–1783), father of all modern mathematics, found an explicit formula for numbers that are both triangular and square, called square triangular numbers[29]. Whereas this formula is rather complicated and can be generated by *Wolfram Alpha* as a solution (in positive integers) to the equation $\frac{n(n+1)}{2} = m^2$, the tool also provides a few examples of integer solutions (**Figure 3**) one of which (m = 6, n = 8) is shown in **Figure 4**. This is a classic number theory problem known from the time of Diophantus (3rd century A.D.). The connection between triangular and square numbers represented through relation (3) was used in the 18th century by a Dutch minister of church and mathematics teacher, Élie de Joncourt, to compute squares and square roots[30]. **Figure 1** shows how a square (the bottom part) can be constructed from circular objects formed by two right isosceles triangles that differ by a leg of the larger triangle.

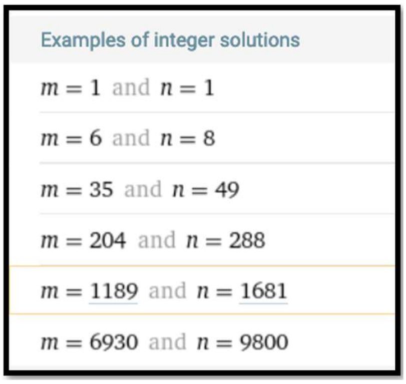

Figure 3. *Wolfram Alpha* generates the first six pairs satisfying $\frac{n(n+1)}{2} = m^2$.

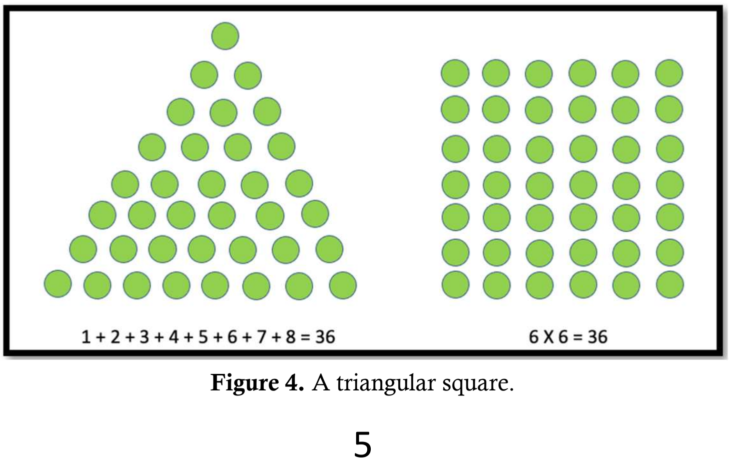

Figure 4. A triangular square.

The context of triangular numbers provides mathematics teacher education with many opportunities for developing skills in formal reasoning. With this in mind, note that whereas any natural number is either a multiple of three or is smaller or greater by one than a multiple of three, any triangular number which is not a multiple of three (e.g., 10) is greater by one than a multiple of three (i.e., 9). Indeed, if the triangular number $\frac{n(n+1)}{2}$ is not a multiple of three, then neither n nor $n + 1$ is a multiple of three, implying that three divides $n + 2$ and, therefore, $\frac{n(n+1)}{2} - 1 = \frac{n^2+n-2}{2} = \frac{(n-1)(n+2)}{2}$ is a multiple of three. One can demonstrate the last property by diminishing the sum $1 + 2 + 3 + \ldots + n$ (assuming it is not a multiple of three) by one so that the sum $2 + 3 + \cdots + n$ (a trapezoidal number[31] with $n - 1$ rows; in other words, the sum of $n - 1$ terms of an arithmetic progression with the first term two and the difference one) being equal to $\frac{(n+2)(n-1)}{2}$ is divisible by three.

Triangular numbers, which are multiples of three, go in pairs, which are separated by multiples of three increased by one. The last statement can be demonstrated numerically by using a spreadsheet (**Figure 5**; numbers of the latter type are marked with the symbol @). Also, because "technology ... without connection to mathematical reasoning, can take up precious course time without advancing learning"[16], the statement can be proved algebraically by considering four consecutive triangular numbers $\frac{n(n+1)}{2}$, $\frac{(n+1)(n+2)}{2}$, $\frac{(n+2)(n+3)}{2}$, $\frac{(n+3)(n+4)}{2}$, and noting that if n is divisible by 3 then the last two numbers are divisible by 3 and the second number $\frac{(n+1)(n+2)}{2} = \frac{n(n+3)}{2} + 1$ (that is, a multiple of 3 plus 1); if $n + 1$ is divisible by 3, then the first two and the fourth numbers are divisible by three and the third number $\frac{(n+2)(n+3)}{2} = \frac{(n+1)(n+4)}{2} + 1$ (that is, a multiple of 3 plus 1). As mentioned by one of the author's students in the context of the joint use of reasoning and computation, "*It is important to make connections between experimental and theoretical knowledge in order to deepen one's understanding of the material*".

	A	B	C	D	E
1	1		1		@
2	2		3		DIV. BY 3
3	3		6		DIV. BY 3
4	4		10		@
5	5		15		DIV. BY 3
6	6		21		DIV. BY 3
7	7		28		@
8	8		36		DIV. BY 3
9	9		45		DIV. BY 3
10	10		55		@
11	11		66		DIV. BY 3
12	12		78		DIV. BY 3
13	13		91		@
14	14		105		DIV. BY 3
15	15		120		DIV. BY 3
16	16		136		@

Figure 5. Two out of three consecutive triangular numbers are divisible by three.

Furthermore, an appropriate (tactile) use of counters allows teacher candidates to notice with a relative ease that any triangular number is a sum of a multiple of three and another triangular number. Indeed, as shown in **Figure 6**,

$$10 = 3 \times 3 + 1, 15 = 3 \times 4 + 3, 21 = 3 \times 5 + 6.$$

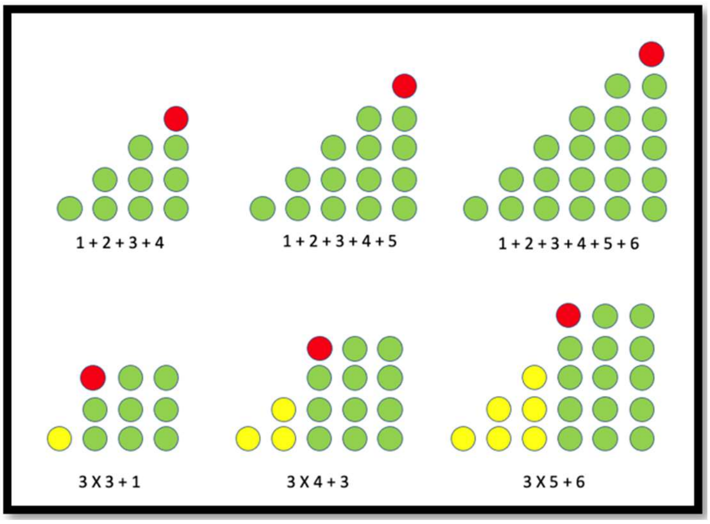

Figure 6. Triangular numbers as sums of a multiple of three and a triangular number.

What might go unnoticed though is that such representations may not be unique for a triangular number. For example,

$$55 = 3 \times 18 + 1 = 3 \times 15 + 10 = 3 \times 9 + 28.$$

Teacher candidates, especially at the elementary level, do not have much experience with questions allowing for more than one correct answer. At the same time, they are expected to "tailor instruction in ways that build on what students understand and consistently encourage students to stretch their mathematical thinking"[17]. However, among such multiple representations, there is always a representation in which the number repeated three times is one smaller than the rank of the triangular number being represented; alternatively, the difference between two triangular numbers is located between the represented one and the addend in the right-hand side of the relation in question. For example, in the representation of the 5th triangular number 15 the number 4 is the rank of the triangular number 10, and, according to relation (2), the equality $4 = 10 - 6$ holds true. That is, one can write $15 = 3 \times (10 - 6) + 3$. In the representation $21 = 3 \times 5 + 6$ of the 6th triangular number 21 the number $5 = 15 - 10$. That is, $21 = 3 \times (15 - 10) + 6$ where 15, 10, and 6 are the 5th, 4th, and 3rd triangular numbers. One can generalize from the diagrams of **Figure 6** to have the relation:

$$t_n = 3 \times (t_{n-1} - t_{n-2}) + t_{n-3}. \tag{4}$$

Proof of relation (4) is based on formula (1) and relation (2). Using (1) and (2) yields $\frac{n(n+1)}{2} = 3 \times (n-1) + \frac{(n-3)(n-2)}{2}$ whence $n^2 + n = 6n - 6 + n^2 - 5n + 6$. The last equality holds true for all n. This proof can also be outsourced to a digital tool, either *Wolfram Alpha* of *Maple*.

An emphasis on divisibility by three often prompts teacher candidates to wonder whether the number three, as a divisor, stems from geometry as triangular numbers represent equilateral triangles (top part of **Figures 1** and **6**) and that a triangular number is either a multiple of three or greater by another triangular number than a multiple of three. However, in the case of square numbers 1, 4, 9, 16, 25, 36, 49 ... (partial sums of the sequence 1, 3, 5, 7, 9, ...), one can recognize similar relationships such as $16 = 3 \times 5 + 1 = 3 \times (9 - 4) + 1$, $25 = 3 \times 7 + 4 = 3 \times (16-9) + 4$, $36 = 3 \times 9 + 9 = 3 \times (25 - 16) + 9$, $49 = 3 \times 11 + 16 = 3 \times (36 - 25) + 16$, which show that a square number is not only a sum of a multiple of three plus another square number but each multiple of three has difference between two consecutive squares as the second factor. Furthermore, as one teacher candidate noted in the classroom, all four square numbers involved are consecutive ones, not only those the difference of which is multiplied by three. One can check to see that the same type of relationships can be developed for pentagonal numbers 1, 5, 12, 22, 35, 51, ... (partial sums of the sequence 1, 4, 7, 10, ...). For example, $51 = 3 \times 13 + 12 = 3 \times (35 - 22) + 12$. In general, among four consecutive polygonal numbers of side m the following relation holds true

$$P(m,n) = 3 \times [P(m,n-1) - P(m,n-2)] + P(m,n-3). \quad (5)$$

Here $P(m,n) = \frac{n^2(m-2)-n(m-4)}{2}$—an m-gonal number of rank n.

Once this (technology-immune) observation has been made, the proof of relation (5) can be outsourced to *Maple* (**Figure 7**; the percentage symbol "%" in the *Maple* code means "the latter"). The use of *Maple* by prospective mathematics teachers requires some training, as the tool is very sensitive to the code used. As mentioned by one of the secondary teacher candidates, "*I have worked extensively with Maple, but I don't feel enough where I could utilize it extensively in a classroom without some training*". An answer to this comment can be found in recommendations for teacher preparation by the Conference Board of the Mathematical Sciences[16], "Teachers must have opportunities to engage in the use of a variety of technological tools, including those designed for mathematics and for teaching mathematics, even if these tools are not the same ones they will eventually use with children".

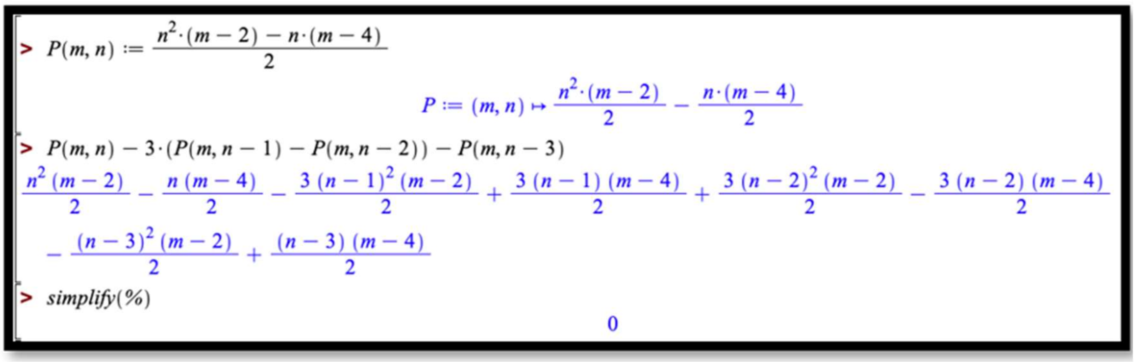

Figure 7. Using *Maple* in proving relation (5).0.

To conclude this section, note that the development of formula (5) and its proof can be referred to as a TITE problem-solving activity. As an extension of this activity, it will be shown in the next section that the factor three appearing in relations (4) and (5) is due to the length of a string of the polygonal numbers considered. Quite unexpectedly, the change of an (even) length of a string yields a new (odd) divisor.

5. Generalizing from formula (5)

If one considers the first six triangular numbers 1, 3, 6, 10, 15, 21, then, as shown at the left-hand side of **Figure 8**, the number 21 (= 1 + 2 + ... + 6) is one greater than a multiple of five, i.e., $21 = 5 \times 4 + 1 = 5 \times (10 - 6) + 1$. Likewise, if one considers the first eight triangular numbers 1, 3, 6, 10, 15, 21, 28, 36, then, as shown at the right-hand side of **Figure 8**, the number 36 (= 1 + 2 + ... + 8) is one greater than a multiple of seven, i.e., $36 = 7 \times 5 + 1 = 7 \times (15 - 10) + 1$. One can check to see that similar relationships hold true in the case of other strings of six and eight consecutive triangular numbers, respectively. Note that whatever even the length of a string is, one can always select a pair of numbers in the middle of the string; thus, from any string of an even length, the first, the last, and two numbers in the middle can be selected to satisfy the corresponding relationship.

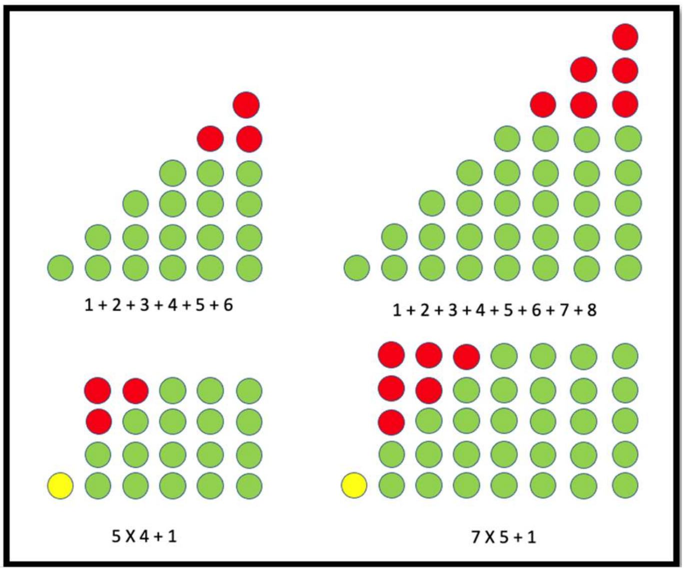

Figure 8. Triangular numbers as sums of multiples of five and seven plus one.

Generalizing from such numeric equalities, yields the symbolic relations

$$t_n = 5(t_{n-2} - t_{n-3}) + t_{n-5} \text{ and } t_n = 7(t_{n-3} - t_{n-4}) + t_{n-7}$$

that hold true among four numbers selected out of, respectively, six and eight consecutive triangular numbers. Furthermore, the following generalization can be inductively formulated for four numbers (the first, the last, and two in the middle) selected out of $2k + 2$ consecutive m-gonal numbers:

$$P(m, n) = (2k + 1)[P(m, n - k) - P(m, n - k - 1)] + P(m, n - 2k - 1). \quad (6)$$

Once again, the proof of relation (6) can be outsourced to *Maple* (**Figure 9**). The tool checks whether the difference between the left- and right-hand sides of relation (6) is equal to zero by carrying out symbolic computations, which in the pre-digital era were too complicated unless special interest in such computations by hand was developed through successful yet time-consuming drill and practice. As Langtangen and Tveito[32] put it, "Much of the current focus on algebraically challenging, lengthy, error prone paper and pencil work can be significantly reduced. The reason for such an evolution is that the computer is simply much better than humans on any theoretically phrased, well-defined repetitive operation". In the digital era, the use of *Maple* is what may be considered a technology-enabled part of TITE problem solving. Just as the development and proof of formula (5), its generalization may also be considered a TITE problem, which combines the mathematical reasoning necessary for the development of formula (6) and its technology-enabled proof. The development of formula (6) was not supported by technology (unless someone used a calculator to do simple calculations). Yet the proof of formula (6) was not possible without using technology capable of symbolic computations, or, at the very least, "would require large amounts of computational time if done by hand"[15]. Furthermore, in the words of a secondary teacher candidate, "*Using Maple ... allows us to easily manipulate and visualize concepts that may otherwise be difficult to comprehend*". The use of diagrams created from counters can also be considered a TE part of the activity, and the interpretation of the diagrams can be considered a TI part. One can see that a TITE activity may have several TI and TE parts, while the order in which the parts are used depends on the problem situation.

```
> P(m, n) := (n^2·(m − 2) − n·(m − 4))/2
```

$$P := (m, n) \mapsto \frac{n^2 \cdot (m-2)}{2} - \frac{n \cdot (m-4)}{2}$$

```
> P(m, n) − (2 k + 1 )·(P(m, n − k) − P(m, n − k − 1)) − P(m, n − 2 k − 1)
```

$$\frac{n^2\,(m-2)}{2} - \frac{n\,(m-4)}{2} - (2\,k+1)\left(\frac{(n-k)^2\,(m-2)}{2} - \frac{(n-k)\,(m-4)}{2} - \frac{(n-k-1)^2\,(m-2)}{2} + \frac{(n-k-1)\,(m-4)}{2}\right) - \frac{(n-2\,k-1)^2\,(m-2)}{2} + \frac{(n-2\,k-1)\,(m-4)}{2}$$

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> simplify(%)
```

$$0$$

Figure 9. Using *Maple* to prove relation (6).

6. Connecting triangular and trapezoidal numbers

As was mentioned above, Pólya[31] introduced the term trapezoidal number—a sum of k consecutive natural numbers starting from n—a generalization of a triangular number for which n = 1. This generalization makes the following important distinction between triangular and trapezoidal numbers: whereas every triangular number has only one representation as a sum of consecutive natural numbers because the sum starts with one, a representation of an integer as a sum of consecutive natural numbers may not be unique as the first addend in the sum may vary, and only powers of two cannot be represented as a sum of consecutive natural numbers. For example, 7 = 3 + 4, 15 = 7 + 8 = 4 + 5 + 6 = 1 + 2 + 3 + 4 + 5, and 16 does not have such representation. Contrary to the last note, one can assume that the equality $2^n = m + (m+1) + \cdots + (m+k-1)$ holds true. Carrying out the summation of an arithmetic sequence on the right-hand side of the last relation leads to the equality $\frac{(2m+k-1)k}{2} = 2^n$ which, however, is not possible. Indeed, whereas its right-hand side has no odd factors, one of the factors k or $2m + k - 1$ in its left-hand side is always odd because either k is odd or, if k is even, the factor $2m + k - 1$ is odd. Obviously, this part of the emerging mathematical activity connecting triangular and trapezoidal numbers is technologically immune.

Observing the top diagrams of **Figure 8**, one can see clearly that a triangular number is the sum of a smaller triangular number and a trapezoidal number. It is also clear (both visually and numerically) that a trapezoidal number may be represented as a sum of other trapezoidal numbers. However, what does not immediately follow from the diagrams in **Figure 8** is that a trapezoidal number (augmenting a triangular number to a larger triangular number) can also be represented as a sum of one (or more) triangular numbers and one (or more) products of two integers. For example, one can develop the representation $78 = 15 + 28 + 5 \times 7$, noting that 5 and 7 are the ranks of triangular numbers 15 and 28, respectively. Such a property of a triangular number can first be presented in a diagrammatic form. To this end, the following problem, requiring, as will be shown below, the use of the TITE framework, can be formulated.

Problem 1. *Is it possible to represent an equilateral triangle as a sum of two equilateral triangles and a rectangle? Alternatively: Is it possible to represent a triangular number as a sum of two triangular numbers and a product of two integers?*

A positive answer to this problem is shown in **Figure 10**. It was offered by one of the author's students, an elementary teacher candidate (Problem 1 was formulated as one of the prompts at the discussion forum of an asynchronous elementary mathematics teacher education course). When the author, responding to the finding shown in **Figure 10**, asked whether there is more than one way to represent 28 as a sum of two triangular numbers and a product of two integers, another candidate noted that such representation is not unique and, in a follow-up post, presented a diagram shown in **Figure 11**. One can interpret the process of answering the questions of Problem 1 as a TITE activity in which one creates diagrams by using tactile technology tools (nowadays available free online) and then interprets the mathematical meaning of the diagrams as a TI part of the activity. One can also see (**Figure 10**) that the triangular number 28 consists of the triangular number 10 (of rank 4) plus the number 18, which can be represented as a trapezoid with three rows: 18 = 5 + 6 + 7. Likewise, in **Figure 11**, the triangular number 28 consists of the triangular number 21 (of rank 6) plus the number 7, which can be represented as a trapezoid with two rows: 7 = 3 + 4, where the first and second addends represent, respectively, an equilateral triangle and a rectangle (square-shaped). Note that the sum 5 + 6 + 7 = 6 + (5 + 7) = 6 + 12 also represents the number 18 as a sum of two addends, which geometrically are an equilateral triangle and a rectangle. But the difference between the two representations of the triangular number 28 shown in **Figures 10** and **11** is that the side lengths of the 3 by 4 rectangle (**Figure 10**) coincide with the lengths of the bases that the corresponding triangles (expressed through the numbers 6 and 10) have.

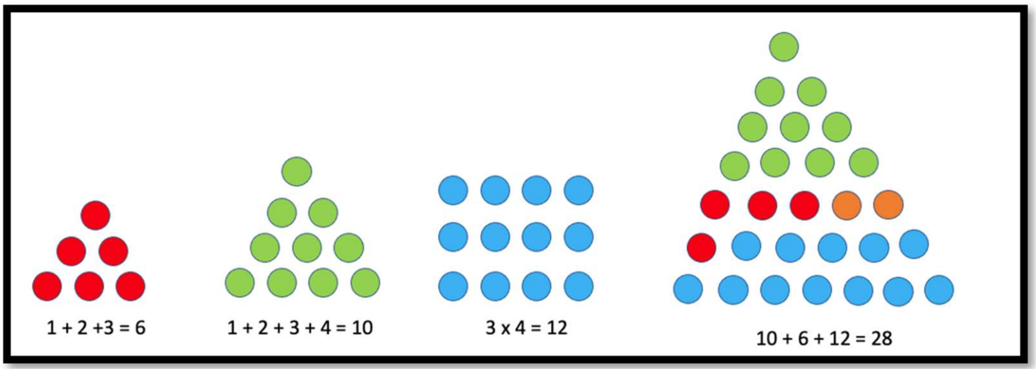

Figure 10. Representing 28 as 6 + 10 + 12.

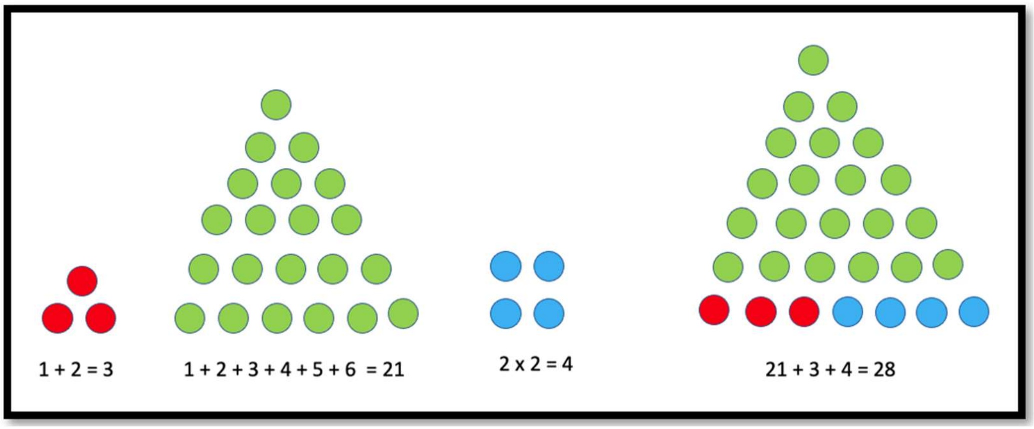

Figure 11. Representing 28 as 3 + 21 + 4.

Using *Wolfram Alpha,* one can prove that the sum $\frac{n(n+1)}{2}+\frac{m(m+1)}{2}+nm$ is a triangular number of rank $n + m$ (**Figure 12**) and that the sum $\frac{m(m+1)}{2}+nm$ is a trapezoidal number with m rows the smallest of which is $n + 1$ (**Figure 13**). When $n = 7$ and $m = 5$ we have

$$78=28+15+35=\frac{7\times 8}{2}+\frac{5\times 6}{2}+7\times 5=\frac{7\times 8}{2}+50=\frac{7\times 8}{2}+8+9+10+11+12.$$

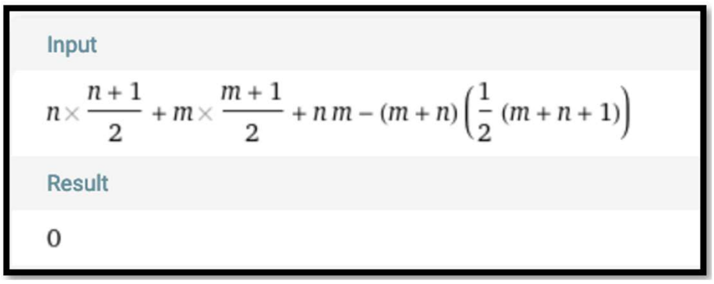

Figure 12. Generalizing from the equality 28 + 15 + 35 = 78 using *Wolfram Alpha*.

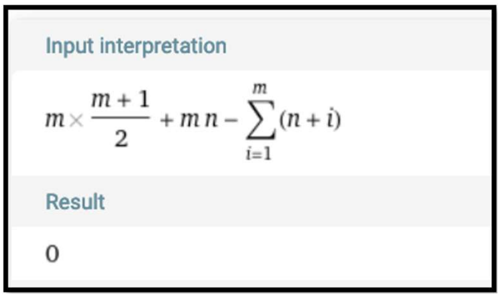

Figure 13. Generalizing from the equality $\frac{5\times(5+1)}{2} + 5 \times 7 = 8 + 9 + 10 + 11 + 12$.

One can see that *Wolfram Alpha* provides (elementary) teacher candidates, in the words of one of them, "*with answers that would not have been easy to find on my own*". The candidate continued: "*When I first encountered Wolfram Alpha it was with a sense of dread, as I was not sure how to approach it and how it would work. As soon as I started using it though I was impressed by the variety of functions it possessed and how much it could help me, not only with my assignments but also with my future classes as an educator*".

In general, as shown in the study of Asiru[22], one can formulate and prove

Problem 2. *Show that the sum of k triangular numbers plus the pairwise products of their ranks is a triangular number the rank of which is the sum of their ranks. That is, the sum* $\sum_{i=1}^{k} \frac{n_i(n_i+1)}{2} + \sum_{i,j=1, i\neq j}^{k}(n_i \times n_j)$ *is a triangular number of the rank* $\sum_{i=1}^{k} n_i$.

A proof of the statement in Problem 2 is pretty straightforward and may be considered a TI part of the activity because doing such a proof using symbolic computations of software is more challenging than carrying out algebraic transformations through "the old school" approach. Indeed,

$$\frac{n_1(n_1+1)}{2} + \frac{n_2(n_2+1)}{2} + \cdots + \frac{n_k(n_k+1)}{2} + n_1n_2 + n_1n_3 + \cdots + n_{k-1}n_k$$
$$= \frac{1}{2}({n_1}^2 + {n_2}^2 + \cdots + {n_k}^2 + 2n_1n_2 + 2n_1n_3 + \cdots + 2n_{k-1}n_k + n_1 + n_2 + \cdots + n_k)$$
$$= \frac{(n_1 + n_2 + \cdots + n_k)^2 + (n_1 + n_2 + \cdots + n_k)}{2}$$
$$= \frac{(n_1 + n_2 + \cdots + n_k)(n_1 + n_2 + \cdots + n_k + 1)}{2}.$$

An important aspect of the TITE framework is that there are cases when mathematical reasoning is more efficient than the use of computing technology. For example, often in grade school, mental computations may be more time-efficient than the use of a calculator. For example, one does not need a calculator to multiply a number by its reciprocal, as the use of a calculator may not display unity as the answer[33].

Furthermore, not mentioned in the study of Asiru[22], is the following

Problem 3. *Show that the sum* $\sum_{i=2}^{k}\frac{n_i(n_i+1)}{2}+\sum_{i,j=1,i\neq j}^{k}(n_i\times n_j)$ *is a trapezoidal number with* $\sum_{i=2}^{k} n_i$ *rows the smallest of which is* n_1+1.

Indeed,

$$\frac{(n_1+n_2+\cdots+n_k)(n_1+n_2+\cdots+n_k+1)}{2}-\frac{n_1(n_1+1)}{2}$$
$$=\frac{[(n_1+(n_2+\cdots+n_k)][(n_1+1)+(n_2+\cdots+n_k)]}{2}-\frac{n_1(n_1+1)}{2}$$
$$=\frac{(n_2+\cdots+n_k)[2n_1+1+(n_2+\cdots+n_k)]}{2}$$
$$=(n_1+1)+(n_1+2)+\cdots+[n_1+(n_2+\cdots+n_k)].$$

In the case of consecutive triangular numbers, we have the formula

$$\sum_{i=1}^{n}\frac{i(i+1)}{2}+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(i\times j)=\frac{\frac{n(n+1)}{2}\times[\frac{n(n+1)}{2}+1]}{2} \tag{7}$$

being a triangular number of the rank $1+2+3+\cdots+n=\frac{n(n+1)}{2}$. **Figure 14** shows the use of *Maple* in (almost) confirming formula (7). Indeed, the expression

$$\frac{n(n+1)(n^2+n+2)}{8}=\frac{\frac{n(n+1)}{2}\times[\frac{n(n+1)}{2}+1]}{2} \tag{8}$$

represents the triangular number of rank $\frac{n(n+1)}{2}$. Note that the result generated by *Maple*, i.e., the left-hand side of formula (8) requires the use of uncomplicated algebra to arrive at its right-hand side which confirms the statement about the rank of the triangular number defined by the left-hand side of (7). As an aside, note that following formula (8), one can pose and solve the following

Problem 4. *Prove that the product* $n(n+1)(n^2+n+2)$ *is a multiple of eight.*

The use of technology provides ample opportunities for problem posing[34].

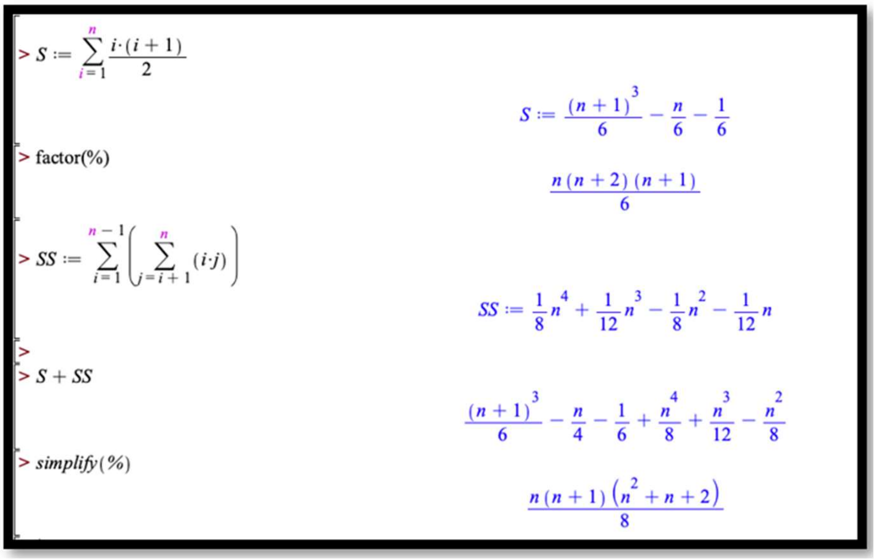

Figure 14. Using *Maple* in exploring the case of consecutive triangular numbers.

7. Using the OEIS® as a source of information

As shown in the spreadsheet in **Figure 15**, triangular numbers, the ranks of which are consecutive triangular numbers, are as follows: 1, 6, 21, 55, 120, 231,... This sequence is mentioned under the number A002817 in the Online Encyclopedia of Integer Sequences (OEIS®, https://oeis.org/) and is called Doubly triangular numbers. The general form of this subsequence of triangular numbers generated by *Wolfram Alpha* (**Figure 16**) is

$$t_{\frac{n(n+1)}{2}} = \frac{n^4+2n^3+3n^2+2n}{8}.$$

1	@	1
6	*	3
21	*	6
55	@	10
120	*	15
231	*	21
406	@	28
666	*	36
1035	*	45
1540	@	55
2211	*	66
3081	*	78
4186	@	91
5565	*	105
7260	*	120
9316	@	136
11781	*	153
14706	*	171
18145	@	190
22155	*	210

Figure 15. Doubly triangular numbers (left).

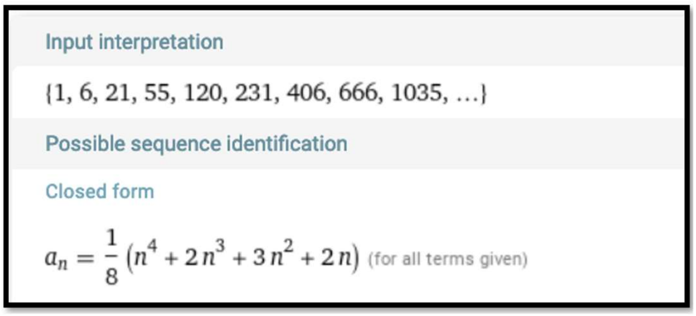

Figure 16. Closed formula for Doubly triangular numbers generated by *Wolfram Alpha*.

Note that unlike formula (6), which defines the same property for any polygonal number sequence, the above property of triangular numbers expressed through (7) does not hold for other polygonal numbers. For example, in the case of square numbers, we have (using *Maple*)

$$\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(i \times j) = \frac{n(n+1)(n+2)(3n+1)}{24},$$

and, as shown in **Figure 17**, $n = 3$ is the only possibility for the right-hand side of the last equality to be a perfect square. Indeed, $1^2 + 2^2 + 3^2 + 1 \times 2 + 1 \times 3 + 2 \times 3 = 5^2$. In the general case of m-gonal numbers, using *Maple* (**Figure 18**), we have

$$\sum_{i=1}^{n} \frac{i^2(m-2)-i(m-4)}{2} + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(i \times j) = \frac{n(n+1)(3n^2+4m(n-1)-9(n-2))}{24}. \quad (9)$$

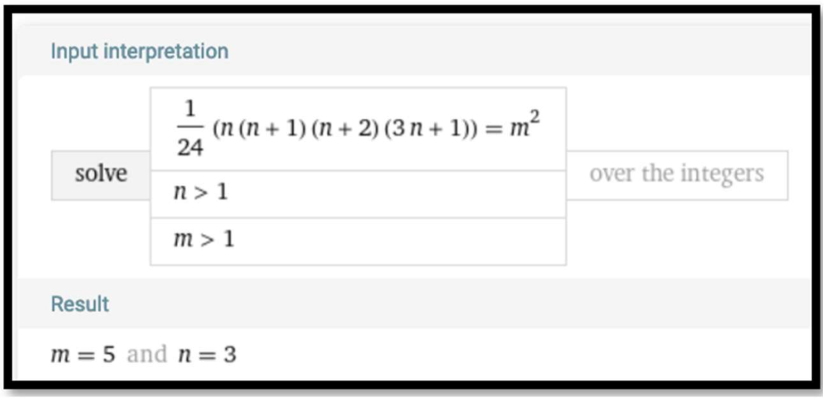

Figure 17. Verifying analogue of (7) in the case of square numbers.

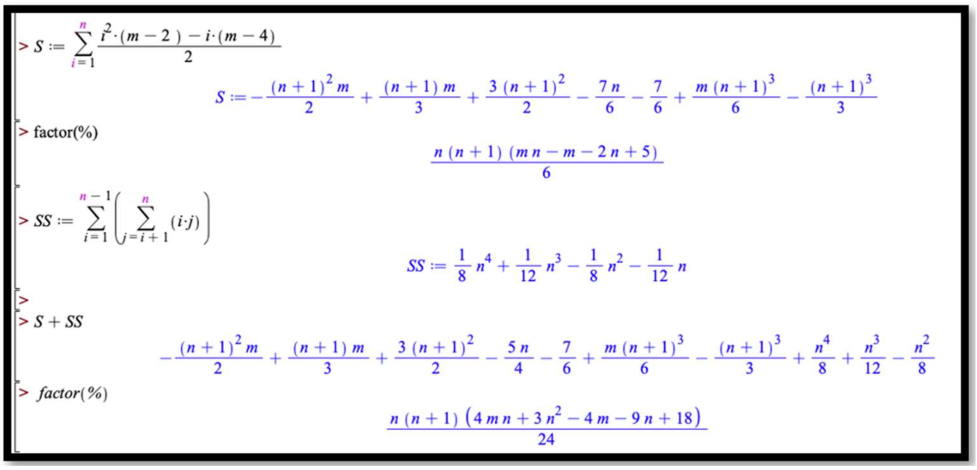

Figure 18. Using *Maple* in proving relation (9).

To conclude this section, note that one can check computationally that the right-hand side of the last relation may not have the form of $\frac{l^2(m-2)-l(m-4)}{2}$ (an m-gonal number of rank l) except the case $m = 4$, $l = 4$ and $n = 3$.

8. Conclusion

The paper was written as a reflection on the author's work with K-12 teacher candidates within several mathematics education graduate level courses that included explorations of number theory concepts using technology. Among such 3-credit courses were *Topics and research in mathematics education* (a capstone secondary education class of about 10 students evaluated via the final (collaborative) project selected by students from a list offered), *Problem solving* (a mixed level class of about 10 students evaluated via portfolio including multiple assignments), *Spreadsheets in education* (a mixed level class of about 10 students evaluated via the final (collaborative) project selected by students from a list offered), *Topics in mathematics for elementary teachers* (a class of about 15 students evaluated via portfolio including multiple assignments), *Elementary mathematics content and methods* (a class, most recently asynchronous, of about 20 students evaluated via tests and the final project selected by students from three types offered), and other courses taught by the author at the State University of New York at Potsdam. The above-mentioned courses and activities described in the paper were technologically enhanced through different educational means, including conference-type presentations in computer labs, online forums, individual final projects, and weekly assignments to work on, for which students had access to digital and other tools either at home or in a university computer lab.

The students in the courses had quite different backgrounds: recent undergraduates, practicing teachers, former physicians, lawyers, and militaries (to name just a few professions), all pursuing their master's degree in education. Due to the university's geographical location in close proximity to the Ontario province of Canada, many students of the author are Canadians. With the advent of education programs offered online, the number of students increased significantly, including international students. Using elementary number theory as a backdrop for the paper, its goal was to demonstrate possible uses of technological tools, both physical and digital, in the preparation of future teachers of mathematics. The paper demonstrated the use of spreadsheets, *Wolfram Alpha,* and *Maple* in exploring ideas of number theory that were first presented through hands-on activities for creating manipulative-based diagrams. Diagrammatic representations of relationships among integers were used as prompts in developing their numeric interpretations, followed by algebraic generalizations requiring either computational verification (**Figures 7** and **9**) or symbolic simplification (**Figures 12–14**). In that way, the interplay of hands-on, cognitive, and computational approaches to the ideas of elementary number theory was the focus of the paper.

Through this interplay, the TITE (technology-immune/technology-enabled) problem-solving framework was discussed, demonstrating that whereas argument and computation go hand-in-hand in doing mathematics, computational techniques do require the use of argumentation, the accuracy of which can be verified through computation. This kind of verification works in the case of proving identities because if an identity was conjectured in error, computing would defy its correctness. In other cases, the accuracy of arguments leading to symbolic computation of cognitively developed mathematical statements can only be verified through the recourse to special cases. Sometimes, the results of symbolic computations (e.g., **Figure 14**) require refinement by hand, prompted by having an idea of what the final result should look like[35].

The paper demonstrated that, just as mental computation in grade school may often be more efficient and even less prone to errors than the use of a calculator, there are cases when doing algebraic transformations by hand is more efficient than using a computer algebra system. The use of the OEIS® was demonstrated in the context of exploring possible connections with already known results about integer sequences that can be generated by software through computer-enhanced number theory explorations. However, the introduction of online sources of mathematical information like OEIS® into various courses for teacher candidates should go with the advice that the use of such sources requires teacher candidates' mastery of managing the abundance of information provided online[36]. At the same time, the following comment by a candidate is indicative of the value of knowledge and experience with tools not necessarily to be used in the classroom but rather retained as part of professional expertise, especially when answering questions by curious students: "*It is of the outmost importance that educators know much more than their students do so that they can help them in the knowledge acquisition process. If a teacher knows only the things that his or her students are trying to learn, then they cannot answer any questions the students might have with certainty. That is because their limited knowledge stops them from knowing the 'why' things work the way they do.*" With this in mind, note that whereas national standards in the United States see teacher preparation being "the foundation for mathematics teaching,... [mathematics teacher educators are encouraged] to address issues [of high-quality education] in a systemic way, providing teachers with the resources they need for professional growth"[15].

The didactic idea behind connecting geometric representations of number theory relationships to their symbolic formulations stems from Vygotsky's perspective on the development of knowledge through the transition from dealing with the "first-order symbols ... directly denoting objects or actions ... [to] the second-order symbolism, which involves the creation of written signs for the spoken symbols of words"[37]. In other words, connecting manipulative diagrams (the first-order symbols) to algebraic formulas (the second-order symbolism) is based on the de-contextualization of hands-on activities by requiring one "to probe into the referents for the symbols involved"[12]. At the beginning of an asynchronous elementary mathematics methods course, many teacher candidates, especially non-traditional students, confessed that they were not familiar with this approach to the teaching and learning of mathematics. As one of the candidates noted, "*When I learned math, visual representations were not a part of mathematical problem solving and therefore was not part of my education. Instead, our learning centered around rote memorization of multiplication tables and addition of single numbers*." Nonetheless, the Vygotskian approach to mathematics learned in the course teacher candidates used with confidence already during their practicum, as evinced in the following comment: "*With the 5th Graders I am working with, it is helpful to have 5 cookies and 6 cookies and subtract the 5 cookies from the 6 cookies leaving the one cookie behind. This equation would be* $6 - 5 = 1$ *however with the visual aids it is easier to see the remaining cookies*". An apparent applicability of using the physical as a precursor to the symbolic in the teaching of mathematics in grade school opens a field of investigation in other mathematical contexts for future mathematics education researchers.

The paper briefly mentioned a few famous contributors, from Euclid to Euler, to the number theory topics discussed. This opened a window into the value of integrating the history of mathematics in teacher education courses in the form of "historical snippets"[38] thereby presenting "mathematics as a living and evolving subject"[16]. A review of current educational literature on polygonal numbers across five continents was provided to indicate the paucity of digital technology application to the teaching of these topics. This makes it possible to conclude that the contribution of the present paper to education research about basic ideas connecting arithmetic and geometry was to demonstrate how mathematical concepts with ancient roots can be explored by means of contemporary discourse.

Conflict of interest

The author declares no conflict of interest.

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