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Article

Validation of the practicability of logical assessment formula for evaluations with inaccurate ground-truth labels: An application study on tumour segmentation for breast cancer

Supplementary materials

Preliminary of logical reasoning

We introduce some propositional connectives and rules for proof of propositional logical reasoning, which are respectively shown as **Table S1** and **Table S2**, for the logical reasonings conducted in this paper.

Table S1. Propositional connectives.

Connective	Meaning
$\wedge$	Conjunction
$\rightarrow$	Implication

Table S2. Rules for proof of propositional logical reasoning, $\vdash$ denotes 'bring out'.

Rule	Meaning
$\wedge -$	Reductive law of conjunction: $A \wedge B, \vdash A$ or B.
$\wedge +$	Additional law of conjunction: $A, B, \vdash A \wedge B$.
MP	Modus ponens: $A \rightarrow B,\ A, \vdash B$.
HS	Hypothetical syllogism: $A \rightarrow B,\ B \rightarrow C, \vdash A \rightarrow C$.

Proof of Reasoning 1

Reasoning 1. *If $\tilde{t}_{TSfBC,1}$ is given, then pixels included in negative areas of $\tilde{t}_{TSfBC,1}$ are most probably true tumour negatives.*

Proof. Firstly, with the given $\tilde{t}_{TSfBC,1}$, we have following preconditions for Reasoning 1.

1) If $\tilde{t}_{TSfBC,1}$ is given, then the recall of positive areas of $\tilde{t}_{TSfBC,1}$ to represent true tumour positives is very high.
2) If the recall of positive areas of $\tilde{t}_{TSfBC,1}$ to represent true tumour positives is very high, then almost all of true tumour positives are included in positive areas of $\tilde{t}_{TSfBC,1}$.
3) If almost all of true tumour positives are included in positive areas of $\tilde{t}_{TSfBC,1}$, then true tumour positives included in negative areas of $\tilde{t}_{TSfBC,1}$ are rare.
4) If true tumour positives included in negative areas of $\tilde{t}_{TSfBC,1}$ are rare, then pixels included in negative areas of $\tilde{t}_{TSfBC,1}$ are mostly probably true tumour negatives.

Secondly, we give the propositional symbols for the above preconditions 1–4 for Reasoning 1, which are shown in **Table S3**.

Table S3. Propositional symbols of preconditions for Reasoning 1.

Symbol	Meaning
a	$\tilde{t}_{TSfBC,1}$ is given.
b	The recall of positive areas of $\tilde{t}_{TSfBC,1}$ to represent true tumour positives is very high.
c	Almost all of true tumour positives are included in positive areas of $\tilde{t}_{TSfBC,1}$.
d	True tumour positives included in negative areas of $\tilde{t}_{TSfBC,1}$ are rare
e	Pixels included in negative areas of $\tilde{t}_{TSfBC,1}$ are mostly probably true tumour negatives

Thirdly, referring to **Table S3**, we signify the propositional formalizations of the preconditions 1–4 for Reasoning 1 and Reasoning 1 via the propositional connectives listed in **Table S1** as follows.

1) $a \to b$ Precondition
2) $b \to c$ Precondition
3) $c \to d$ Precondition
4) $d \to e$ Precondition
   $a \to e$ Reasoning 1

Fourthly, we show the validity of Reasoning 1 via the rules for proof of propositional logical reasoning listed in **Table S2** as follows.

$\therefore \boldsymbol{a \to e}$

5) a Hypothesis
6) $a \to c$ 1),2); HS
7) $c \to e$ 3),4); HS
8) $a \to e$ 6),7); HS
9) e 8),5); MP
10) $a \to e$ 5)-9); Conditional Proof

Since the hypothesis a of the 5) step has been fulfilled by the abduced $\tilde{t}_{TSfBC} = \{\tilde{t}_{TSfBC,1}, \tilde{t}_{TSfBC,2}\}$ in section 5.2.2., Reasoning 1 is proved to be valid. □

Proof of Reasoning 2

Reasoning 2. *If $\tilde{t}_{TSfBC,2}$ is given, then pixels included in positive areas of $\tilde{t}_{TSfBC,2}$ are most probably true tumour positives.*

Proof. Firstly, with the given $\tilde{t}_{TSfBC,2}$, we have following preconditions for Reasoning 2.

1) If $\tilde{t}_{TSfBC,2}$ is given, then the precision of positive areas of $\tilde{t}_{TSfBC,2}$ to represent true tumour positives is very high.
2) If the precision of positive areas of $\tilde{t}_{TSfBC,2}$ to represent true tumour positives is very high, then the positive areas of $\tilde{t}_{TSfBC,2}$ are almost all true tumour positives.
3) If the positive areas of $\tilde{t}_{TSfBC,2}$ are almost all true tumour positives, then false tumour positives included in positive areas of $\tilde{t}_{TSfBC,2}$ are rare.
4) If false tumour positives included in positive areas of $\tilde{t}_{TSfBC,2}$ are rare, then pixels included in positive areas of $\tilde{t}_{TSfBC,2}$ are most probably true tumour positives.

Secondly, we give the propositional symbols for the above preconditions 1–4 for Reasoning 2, which are shown in **Table S4**.

Table S4. Propositional symbols of preconditions for Reasoning 2.

Symbol	Meaning
f	$\tilde{t}_{TSfBC,2}$ is given
g	The precision of positive areas of $\tilde{t}_{TSfBC,2}$ to represent true tumour positives is very high.
h	The positive areas of $\tilde{t}_{TSfBC,2}$ are almost all true tumour positives.
i	False tumour positives included in positive areas of $\tilde{t}_{TSfBC,2}$ are rare.
j	Pixels included in positive areas of $\tilde{t}_{TSfBC,2}$ are most probably true tumour positives.

Thirdly, referring to **Table S4**, we signify the propositional formalizations of the preconditions 1–4 for Reasoning 2 and Reasoning 2 via the propositional connectives listed in **Table S1** as follows.

1) $f \to g$ Precondition
2) $g \to h$ Precondition
3) $h \to i$ Precondition
4) $i \to j$ Precondition
   $f \to j$ Reasoning 2

Fourthly, we show the validity of Reasoning 2 via the rules for proof of propositional logical reasoning listed in **Table S2** as follows.

$\therefore \boldsymbol{f \to j}$

5) f Hypothesis
6) $f \to h$ 1), 2); HS
7) $h \to j$ 3), 4); HS
8) $f \to j$ 6), 7); HS
9) j 8), 5); MP
10) $f \to j$ 5)–9); Conditional Proof

Since the hypothesis f of the 5) step has been fulfilled by the abduced $\tilde{t}_{TSfBC} = \{\tilde{t}_{TSfBC,1}, \tilde{t}_{TSfBC,2}\}$ in section 5.2.2., Reasoning 2 is proved to be valid. □

Proof of Reasoning 3

Reasoning 3. *If t_{TSfBC} is given and $LF_{TSfBC,1}$ is given, then the intersection of pixels of t_{TSfBC} that are predicted as tumour positives (t^p_{TSfBC}) and pixels included in negative areas of $\tilde{t}_{TSfBC,1}$ ($\tilde{t}^n_{TSfBC,1}$) can be considered as logically false positives.*

Proof. Firstly, with the given t_{TSfBC} and $LF_{TSfBC,1}$, we have following preconditions for Reasoning 3.

1) If $LF_{TSfBC,1}$ is given, then $\tilde{t}_{TSfBC,1}$ is given.
2) If $\tilde{t}_{TSfBC,1}$ is given, then pixels included in negative areas of $\tilde{t}_{TSfBC,1}$ ($\tilde{t}^n_{TSfBC,1}$) are most probably true tumour negatives. (Reasoning 1)
3) If t_{TSfBC} is given, then pixels of t_{TSfBC} that are predicted as tumour positives (t^p_{TSfBC}) exist.
4) If pixels included in negative areas of $\tilde{t}_{TSfBC,1}$ ($\tilde{t}^n_{TSfBC,1}$) are most probably true tumour negatives and pixels of t_{TSfBC} that are predicted as tumour positives (t^p_{TSfBC}) exist, then the intersection of pixels included in t^p_{TSfBC} and pixels included in $\tilde{t}^n_{TSfBC,1}$ can be considered as most probably predicted false tumour positives.
5) If the intersection of pixels included in t^p_{TSfBC} and pixels included in $\tilde{t}^n_{TSfBC,1}$ can be considered as most probably predicted false tumour positives, then the intersection of pixels included in t^p_{TSfBC} and pixels included in $\tilde{t}^n_{TSfBC,1}$ can be considered as logically false positives.

6) If the intersection of pixels included in t^{p}_{TSfBC} and pixels included in $\tilde{t}^{n}_{TSfBC,1}$ can be considered as logically false positives, then the intersection of pixels of t_{TSfBC} that are predicted as tumour positives (t^{p}_{TSfBC}) and pixels included in negative areas of $\tilde{t}_{TSfBC,1}$ ($\tilde{t}^{n}_{TSfBC,1}$) can be considered as logically false positives.

Secondly, we give the propositional symbols for the above preconditions 1–6 for Reasoning 3, which are shown in **Table S5**.

Table S5. Propositional symbols of preconditions for Reasoning 3.

Symbol	Meaning
k	$LF_{TSfBC,1}$ is given.
l	$\tilde{t}_{TSfBC,1}$ is given.
m	Pixels included in negative areas of $\tilde{t}_{TSfBC,1}$ ($\tilde{t}^{n}_{TSfBC,1}$) are most probably true tumour negatives.
n	t_{TSfBC} is given.
o	Pixels of t_{TSfBC} that are predicted as tumour positives (t^{p}_{TSfBC}) exist.
p	The intersection of pixels included in t^{p}_{TSfBC} and pixels included in $\tilde{t}^{n}_{TSfBC,1}$ can be considered as most probably predicted false tumour positives.
q	The intersection of pixels included in t^{p}_{TSfBC} and pixels included in $\tilde{t}^{n}_{TSfBC,1}$ can be considered as logically false positives.
r	The intersection of pixels of t_{TSfBC} that are predicted as tumour positives (t^{p}_{TSfBC}) and pixels included in negative areas of $\tilde{t}_{TSfBC,1}$ ($\tilde{t}^{n}_{TSfBC,1}$) can be considered as logically false positives.

Thirdly, referring to **Table S5**, we signify the propositional formalizations of the preconditions 1–6 for Reasoning 3 and Reasoning 3 via the propositional connectives listed in **Table S1** as follows.

1) $k \to l$ Precondition
2) $l \to m$ Precondition
3) $n \to o$ Precondition
4) $(m \wedge o) \to p$ Precondition
5) $p \to q$ Precondition
6) $q \to r$ Precondition

$(n \wedge k) \to r$ Reasoning 3

Fourthly, we show the validity of Reasoning 3 via the rules for proof of propositional logical reasoning listed in **Table S2** as follows.

$\therefore (\boldsymbol{n} \wedge \boldsymbol{k}) \to \boldsymbol{r}$

7) $n \wedge k$ Hypothesis
8) n 7); $\wedge -$
9) k 7); $\wedge -$
10) l 1), 9); MP
11) m 2), 10); MP
12) o 3), 8); MP
13) $m \wedge o$ 11), 12); $\wedge +$
14) $(m \wedge o) \to q$ 4), 5); HS
15) $(m \wedge o) \to r$ 14), 6); HS
16) r 15), 13); MP
17) $(n \wedge k) \to r$ 7)–16); Conditional Proof

Since the hypothesis $n \wedge k$ of the 7) step has been fulfilled by the prediction of the image semantic segmentation

model for tumour segmentation for breast cancer (t_{TSfBC}) in section 5.2.3. and the two narrated logical facts $LF_{TSfBC} = \{LF_{TSfBC,1}, LF_{TSfBC,2}\}$, Reasoning 3 is proved to be valid. □

Proof of Reasoning 4

Reasoning 4. *If t_{TSfBC} is given and $LF_{TSfBC,2}$ is given, then the intersection of pixels of t_{TSfBC} that are predicted as tumour positives (t^p_{TSfBC}) and pixels included in positive areas of $\tilde{t}_{TSfBC,2}$ ($\tilde{t}^p_{TSfBC,2}$) can be considered as logically true positives, and the intersection of pixels of t_{TSfBC} that are predicted as tumour negatives (t^n_{TSfBC}) and pixels included in positive areas of $\tilde{t}_{TSfBC,2}$ ($\tilde{t}^p_{TSfBC,2}$) can be considered as logically false negatives.*

Proof. Firstly, with the given t_{TSfBC} and $LF_{TSfBC,2}$, we have following preconditions for Reasoning 4.

1) If $LF_{TSfBC,2}$ is given, then $\tilde{t}_{TSfBC,2}$ is given.
2) If $\tilde{t}_{TSfBC,2}$ is given, then pixels included in positive areas of $\tilde{t}_{TSfBC,2}$ ($\tilde{t}^p_{TSfBC,2}$) are most probably true tumour positives. (Reasoning 2).
3) If t_{TSfBC} is given, then pixels of t_{TSfBC} that are predicted as tumour positives (t^p_{TSfBC}) exist and pixels of t_{TSfBC} that are predicted as tumour negatives (t^n_{TSfBC}) exist.
4) If pixels included in positive areas of $\tilde{t}_{TSfBC,2}$ ($\tilde{t}^p_{TSfBC,2}$) are most probably true tumour positives and pixels of t_{TSfBC} that are predicted as tumour positives (t^p_{TSfBC}) exist, then the intersection of pixels included in t^p_{TSfBC} and pixels included in $\tilde{t}^p_{TSfBC,2}$ can be considered as most probably predicted true tumour positives.
5) If pixels included in positive areas of $\tilde{t}_{TSfBC,2}$ ($\tilde{t}^p_{TSfBC,2}$) are most probably true tumour positives and pixels of t_{TSfBC} that are predicted as tumour negatives (t^n_{TSfBC}) exist, then the intersection of pixels included in t^n_{TSfBC} and pixels included in $\tilde{t}^p_{TSfBC,2}$ can be considered as most probably predicted false tumour negatives.
6) If the intersection of pixels included in t^p_{TSfBC} and pixels included in $\tilde{t}^p_{TSfBC,2}$ can be considered as most probably predicted true tumour positives, then the intersection of pixels included in t^p_{TSfBC} and pixels included in $\tilde{t}^p_{TSfBC,2}$ can be considered as logically true positives.
7) If the intersection of pixels included in t^n_{TSfBC} and pixels included in $\tilde{t}^p_{TSfBC,2}$ can be considered as most probably predicted false tumour negatives, then the intersection of pixels included in t^n_{TSfBC} and pixels included in $\tilde{t}^p_{TSfBC,2}$ can be considered as logically false negatives.
8) If the intersection of pixels included in t^p_{TSfBC} and pixels included in $\tilde{t}^p_{TSfBC,2}$ can be considered as logically true positives, then the intersection of pixels of t_{TSfBC} that are predicted as tumour positives (t^p_{TSfBC}) and pixels included in positive areas of $\tilde{t}_{TSfBC,2}$ ($\tilde{t}^p_{TSfBC,2}$) can be considered as logically true positives.
9) If the intersection of pixels included in t^n_{TSfBC} and pixels included in $\tilde{t}^p_{TSfBC,2}$ can be considered as logically false negatives, then the intersection of pixels of t_{TSfBC} that are predicted as tumour negatives (t^n_{TSfBC}) and pixels included in positive areas of $\tilde{t}_{TSfBC,2}$ ($\tilde{t}^p_{TSfBC,2}$) can be considered as logically false negatives.

Secondly, we give the propositional symbols for the above preconditions 1-9 for Reasoning 4, which are shown in **Table S6**.

Table S6. Propositional symbols of preconditions for Reasoning 4.

Symbol	Meaning
s	$LF_{TSfBC,2}$ is given.
t	$\tilde{t}_{TSfBC,2}$ is given.
u	Pixels included in positive areas of $\tilde{t}_{TSfBC,2}$ ($\tilde{t}^{p}_{TSfBC,2}$) are most probably true tumour positives.
v	t_{TSfBC} is given.
w	Pixels of t_{TSfBC} that are predicted as tumour positives (t^{p}_{TSfBC}) exist.
x	Pixels of t_{TSfBC} that are predicted as tumour negatives (t^{n}_{TSfBC}) exist.
y	The intersection of pixels included in t^{p}_{TSfBC} and pixels included in $\tilde{t}^{p}_{TSfBC,2}$ can be considered as most probably predicted true tumour positives.
z	The intersection of pixels included in t^{n}_{TSfBC} and pixels included in $\tilde{t}^{p}_{TSfBC,2}$ can be considered as most probably predicted false tumour negatives.
a	The intersection of pixels included in t^{p}_{TSfBC} and pixels included in $\tilde{t}^{p}_{TSfBC,2}$ can be considered as logically true positives.
b	The intersection of pixels included in t^{n}_{TSfBC} and pixels included in $\tilde{t}^{p}_{TSfBC,2}$ can be considered as logically false negatives.
c	The intersection of pixels of t_{TSfBC} that are predicted as tumour positives (t^{p}_{TSfBC}) and pixels included in positive areas of $\tilde{t}_{TSfBC,2}$ ($\tilde{t}^{p}_{TSfBC,2}$) can be considered as logically true positives.
d	The intersection of pixels of t_{TSfBC} that are predicted as tumour negatives (t^{n}_{TSfBC}) and pixels included in positive areas of $\tilde{t}_{TSfBC,2}$ ($\tilde{t}^{p}_{TSfBC,2}$) can be considered as logically false negatives.

Thirdly, referring to **Table S6**, we signify the propositional formalizations of the preconditions 1–9 for Reasoning 4 and Reasoning 4 via the propositional connectives listed in **Table S1** as follows.

1) $s \to t$ Precondition
2) $t \to u$ Precondition
3) $v \to (w \wedge x)$ Precondition
4) $(u \wedge w) \to y$ Precondition
5) $(u \wedge x) \to z$ Precondition
6) $y \to a$ Precondition
7) $z \to b$ Precondition
8) $a \to c$ Precondition
9) $b \to d$ Precondition

$(v \wedge s) \to (c \wedge d)$ Reasoning 4

Fourthly, we show the validity of Reasoning 4 via the rules for proof of propositional logical reasoning listed in **Table S2** as follows.

$\therefore (\boldsymbol{v} \wedge \boldsymbol{s}) \to (\boldsymbol{c} \wedge \boldsymbol{d})$

10) $v \wedge s$ Hypothesis
11) v 10); $\wedge -$
12) s 10); $\wedge -$
13) $s \to u$ 1), 2); HS
14) u 13), 12); MP
15) $w \wedge x$ 3), 11); MP
16) w 15); $\wedge -$

17) x	15); $\wedge -$
18) $u \wedge w$	14), 16); $\wedge +$
19) $(u \wedge w) \rightarrow a$	4), 6); HS
20) $u \wedge x$	14), 17); $\wedge +$
21) $(u \wedge x) \rightarrow b$	5), 7); HS
22) $(u \wedge w) \rightarrow c$	19), 8); HS
23) $(u \wedge x) \rightarrow d$	21), 9); HS
24) c	22), 18); MP
25) d	23), 20); MP
26) $c \wedge d$	24), 25); $\wedge +$
27) $(v \wedge s) \rightarrow (c \wedge d)$	10)–26); Conditional Proof

Since the hypothesis $v \wedge s$ of the 10) step has been fulfilled by the prediction of the image semantic segmentation model for tumour segmentation for breast cancer (t_{TSfBC}) in section 5.2.3. and the two narrated logical facts $LF_{TSfBC} = \{LF_{TSfBC,1}, LF_{TSfBC,2}\}$ Reasoning 4 is proved to be valid.