

Existence of normalized solutions for elliptic equation with Hardy potential and nonautonomous term

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Abstract: The present work is devoted to the existence of normalized solutions for elliptic equations with a Hardy potential and a nonautonomous term. Such equations are closely connected with standing wave solutions of nonlinear Schrödinger equations with singular potentials, which arise widely in mathematical physics. Here, normalized solutions are understood as solutions satisfying a prescribed L^2 norm constraint, this constraint fundamentally distinguishes the problem from classical unconstrained variational problems. The L^2 supercritical case is considered, for which the energy functional is unbounded from below on $\mathcal{N}(\kappa)$. To overcome this difficulty, a suitable manifold is constructed via an L^2 -norm-preserving scaling transformation, and the analysis is carried out on this natural constraint. Several analytical tools, including the Hardy inequality, the Gagliardo–Nirenberg inequality, and variational methods, are combined to systematically examine the variational structure of the constrained minimization problem. Since the problem is set on $\mathbb{R}^N$, a primary difficulty arises from the lack of compactness. To address this issue, the Schwarz symmetric decreasing rearrangement technique is employed, and the problem is reduced to the radial symmetric function space. This space possesses a favorable compact embedding property in the subcritical range $2 < q < \frac{2N}{N-2}$ which compensates for the loss of compactness in the whole space and allows us to extract a strongly convergent subsequence from the constructed radially rearranged minimizing sequence.

Keywords: singular potential; Schrödinger equations; L^2 constraint; scaling transformation; variational method

1. Introduction

Over the past few years, in the context of elliptic equations with specified L^2 norm constraints, the academic community has conducted extensive research on the existence and uniqueness of their normalized solutions. This field has now become a rapidly developing hot topic within the field of nonlinear partial differential equations. The prescribed L^2 norm constraint typically carries an explicit physical meaning, thereby making such elliptic equations particularly important in numerous physical and engineering applications. For instance, in the theoretical investigation of Bose-Einstein condensation, the L^2 norm of the wave function corresponds to the total number of condensed atoms in a quantum system. As a conserved fundamental physical quantity, this physical parameter plays a crucial role in determining the macroscopic dynamics, the stability of standing wave solutions, and the associated phase transitions of the system. In the study of matter waves, normalized solutions

can be used to describe various nonlinear phenomena, including self-focusing, solitons, and the stability of standing waves [1–3]. Normalized solutions are likewise of considerable importance in nonlinear optics; for optical models governed by nonlinear Schrödinger-type equations, the L^2 norm is commonly interpreted as the total optical power carried by the beam. Thus, investigating the existence and stability of normalized solutions subject to a prescribed L^2 norm constraint can reveal the localization and propagation of optical beams and determine stable modes and their power thresholds. This research has substantial value for high-speed optical communications, long-haul fiber transmission, and integrated nonlinear optics [4,5]. From the perspective of pure mathematics, the study of solutions subject to a prescribed L^2 norm constraint is of significant and distinctive theoretical value. Such investigations not only contribute to a deeper understanding of the fundamental properties and dynamical evolution of these solutions, but also help uncover the dynamical features inherent in stationary solutions of the equations, thereby laying a solid theoretical foundation for the further development of the theory of nonlinear partial differential equations and its applications in interdisciplinary fields. The constraint imposed by a fixed L^2 norm renders the problem substantially distinct from classical variational problems free of constraints. Due to the presence of the L^2 norm constraint, a new type of critical exponent arises: the L^2 critical exponent is given by $\bar{m} = 2 + \frac{4}{N}$. For the equation given below,

$$\begin{cases} -\Delta u = |u|^{m-2}u + \lambda u, \\ \|u\|_2^2 = k, \quad u \in H^1(\mathbb{R}^N). \end{cases}$$

When $2 < m < 2 + \frac{4}{N}$, the equation is said to be L^2 subcritical. In view of the Gagliardo-Nirenberg inequality (as shown in Lemma 2), the corresponding energy functional admits a lower bound on the constraint manifold

$$\mathfrak{N}(\kappa) = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = k \right\}.$$

When $2 + \frac{4}{N} < m < 2^*$, the equation is called L^2 supercritical. At this point, on the manifold $\mathfrak{N}(\kappa)$, the energy functional associated with the equation is not bounded from below. Typically, the solutions can be found by seeking critical points of the energy functional under the constraint imposed by the L^2 -sphere. The resulting Lagrange multiplier then plays the role of an unknown frequency in the associated Euler–Lagrange equation. In particular, the geometry and boundedness of the constrained energy functional depend strongly on the growth characteristics of nonlinearities. In the L^2 subcritical regime, minimization methods can often be applied directly, whereas in the L^2 critical and L^2 supercritical regimes, the energy may be unbounded from below on $\mathfrak{N}(\kappa)$, making it necessary to use more refined techniques such as constrained minimax arguments, Pohozaev manifolds, scaling analysis, and concentration–compactness methods.

Recently, neural-network-assisted symbolic methods have provided new approaches for solving nonlinear differential equations. Wang et al. [6] proposed the fSENNs method for obtaining exact solutions of fractional PDEs, while Zhao

and Zhang [7] developed the AI-EMD framework by combining neural networks with symbolic computation. These methods show advantages in solution diversity and exact solution construction, but their performance depends on the choice of network structures, activation functions, and symbolic solvability. In contrast, the present work focuses on a purely analytical variational framework for establishing the existence, compactness, and variational characterization of normalized states for nonautonomous Schrödinger equations. This framework does not aim to construct explicit solutions; rather, it provides rigorous theoretical guarantees and reveals the underlying variational structure of the problem.

Substantial progress has been achieved concerning the existence of normalized solutions for elliptic equations involving only autonomous nonlinearities [3, 8–11]. Soave [3] investigated the elliptic equation below that contains two nonlinear terms:

$$\begin{cases} -\Delta v - \lambda v = |v|^{q-2}v + \mu|v|^{r-2}v, \\ \int_{\mathbb{R}^N} |v|^2 dx = a^2. \end{cases}$$

The paper focused on the existence and stability of normalized ground states when the two nonlinear terms lie in the L^2 subcritical, L^2 critical, and L^2 supercritical regimes. The author employed constrained variational methods and defined the associated energy functional on the mass constraint, thereby reformulating normalized solutions as constrained critical points. To analyze the geometric structure of the energy functional, an L^2 norm-preserving scaling was introduced to construct the fiber maps and the Pohozaev manifold. The local minimization and mountain pass structures of the energy functional were then characterized according to the sign of the second derivative of the fiber maps. Fan and Li [8] employed variational methods and subcritical approximation techniques to prove the existence of normalized solutions when the nonlinearity is in the L^2 supercritical regime. Boulaaras and Guefaifia [9] investigated the existence of normalized solutions for an elliptic equation involving one Sobolev critical nonlinearity and two mass subcritical nonlinearities. They treated the Sobolev critical term with a sufficiently small coefficient as a perturbation of the double mass subcritical structure. By constructing a Pohozaev manifold and combining the fibering map method, constrained variational techniques, and concentration compactness analysis, they handled the loss of compactness brought about by the critical exponent. They ultimately proved that, when the coefficient of the critical term is sufficiently small, the equation admits a positive normalized ground state, and the corresponding Lagrange multiplier is negative.

Compared with simplified models involving only autonomous nonlinearities, general nonautonomous nonlinearities possess a more complex structure because they depend simultaneously on the spatial variable and the unknown function. Such a dependence allows nonautonomous models to capture spatially heterogeneous effects and external perturbations more accurately than their autonomous counterparts. Consequently, nonautonomous models are often more flexible and more accurate in capturing the influence of the surrounding environment on the behavior of the solution. Motivated by the above theoretical advantages and practical application requirements,

many scholars have studied the following elliptic equation with nonautonomous terms [12–16]:

$$\begin{cases} -\Delta u + \lambda u = \nu |u|^{p-2}u + f(x, u), & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 \, dx = a, & u \in H^1(\mathbb{R}^N). \end{cases} \quad (1)$$

When $\nu = 1$, $p = \frac{2N}{N-2}$, and $f(x, u) = h(x)|u|^{q-2}u$, constrained variational methods were used by Yang et al. [12] to examine normalized solutions pertaining to a class of elliptic equations possessing nonautonomous terms. By means of mass-preserving scaling, fiber maps, and the Pohozaev identity, they established a correspondence between critical points along scaling orbits and elements of the Pohozaev manifold. In the L^2 subcritical regime, the authors used the Gagliardo–Nirenberg inequality and the Sobolev inequality to derive quantitative estimates for the energy functional, thereby determining a mass threshold and an appropriate local minimization region. Under the assumptions that the infimum of the nonautonomous coefficient $h(x)$ coincides with its limit at infinity and that the prescribed mass is sufficiently small, they obtained a positive normalized solution through local minimization and concentration–compactness arguments. By imposing additional monotonicity and differentiability conditions on $h(x)$, they further investigated the property of natural constraint for the Pohozaev manifold and the existence of normalized ground states. In the L^2 critical and L^2 supercritical regimes, the authors adopted a minimization strategy on the Pohozaev manifold. By combining the monotonicity of the ground state energy level with respect to the prescribed mass, a comparison between the energy levels of the original problem and the autonomous limiting problem at infinity, the Brezis–Lieb splitting lemma, and concentration–compactness arguments, they proved that the minimum energy level is attained. They then applied the classical deformation lemma to obtain a positive normalized solution of mountain-pass type. Under the corresponding structural and monotonicity assumptions, if $h \in C^2(\mathbb{R}^N, \mathbb{R})$, the Pohozaev manifold provides a natural constraint for the problem, and the resulting solution is a normalized ground state. The corresponding conclusions are also subject to appropriate dimensional and mass restrictions. When $\nu = 1$ and $f(x, u) = h(x)$, Zou and Chen [13] established the existence of normalized solutions for Equation (1) by means of the mountain pass lemma and other techniques. In the case where $\nu = 0$ and $f(x, u) = g(x)h(u)$, $g(x)$ is assumed to satisfy the following assumptions:

Assumption 1. $g \in C(\mathbb{R}^N, [0, \infty))$, for all $x \in \mathbb{R}^N$, $0 < g_\infty := \lim_{|y| \rightarrow \infty} g(y) \leq g(x)$;

Assumption 2. $g \in C^1(\mathbb{R}^N, \mathbb{R})$, and for every $x \in \mathbb{R}^N$, the mapping $t \mapsto -2(tx) \cdot \nabla g(tx) + N(\sigma - 2)g(tx)$ is nonincreasing on $(0, \infty)$;

Assumption 3. For all $x \in \mathbb{R}^N$, the mapping $t \mapsto t^{N - \frac{\omega(N-2)}{2}} g(tx)$ is strictly increasing on $(0, \infty)$.

It follows from Assumption 1 that $g(x)$ is continuous and bounded on $\mathbb{R}^N$. Under the above assumptions, Tang and Chen [16] proved the existence of normalized

solutions to problem (1). In the mass supercritical case, instead of minimizing the energy functional directly, the problem was reformulated as a minimization problem on a suitable constraint manifold. The main difficulty in analyzing a minimizing sequence lies in recovering compactness. To overcome this difficulty, Tang and Chen exploited Assumption 3 to derive a monotonicity property of the energy with respect to mass. Together with the associated limit equation, this ruled out the splitting of both energy and mass. Finally, by means of a deformation lemma, the minimizer was shown to be a critical point of the energy functional under the prescribed mass constraint.

Over the past several decades, normalized solutions of elliptic equations with potential functions have also attracted extensive academic attention. Abundant theoretical achievements have continuously emerged in relevant literature [17–21]. Jin and Zhang [18] investigated normalized solutions to elliptic equations involving a Sobolev critical nonlinearity and potentials. They proved the existence of positive normalized solutions. Specifically, they established two types of Pohozaev identities under different conditions on the potential function. Using the augmented functional method, they constructed Palais–Smale sequences approximating the corresponding Pohozaev manifolds. With the help of a splitting lemma, they overcame the loss of compactness induced by the nonradial potential and the Sobolev critical term. Accordingly, they extended the relevant existence results from the potential-free case to the potential case under appropriate conditions. Using variational methods, the concentration-compactness principle, and other analytical tools, He and Suo [20] demonstrated that an equation with a singular potential admits radially symmetric normalized solutions. Molle et al. [21] studied an L^2 supercritical nonlinear Schrödinger equation with a negative potential. By employing constrained variational methods, they first estimated the potential term using the Hölder inequality, the Sobolev inequality, and related analytical tools. They then established the mountain-pass geometry of the associated energy functional and constructed a Palais–Smale sequence. Subsequently, a Pohozaev-type identity was used to prove the boundedness of this sequence. Finally, by combining the splitting lemma with an energy level comparison, the authors ruled out energy splitting caused by the lack of compactness and thereby established the existence of normalized solutions.

The Hardy potential is a typical and representative type of singular potential. Elliptic equations with Hardy potentials have direct or indirect applications in numerous physical models such as heat conduction and fluid mechanics. Due to the intrinsic singularity of the Hardy potential, the corresponding differential operators and energy functionals exhibit substantially more complicated analytical and variational structures than those arising in regular potential problems. This singular behavior often leads to difficulties related to coercivity, compactness, regularity, and the characterization of appropriate functional spaces. Therefore, the investigation of such equations is both complex and challenging. Over the past several decades, numerous researchers worldwide have systematically investigated whether solutions to elliptic equations with Hardy potentials exist, whether multiple solutions occur, and how these solutions behave asymptotically. The main analytical approaches include variational methods, the approach using upper and lower solutions, comparison principles and various

compactness techniques. These methods have played a fundamental role in overcoming the difficulties caused by the singular potential and in revealing the underlying structure of the associated nonlinear problems. In recent years, increasing attention has been devoted to the existence of normalized solutions for elliptic equations involving Hardy potentials [22–24]. This additional mass constraint further enriches the variational structure of the problem and introduces new analytical difficulties. Li and Zou [22] investigated an elliptic equation involving a Hardy potential and nonlinear terms. First, they reformulated the critical Schrödinger equation with a Hardy potential as a constrained variational problem under a prescribed L^2 mass condition. By introducing an L^2 -norm-preserving scaling, they constructed the associated fiber maps and used the Pohozaev identity to restrict the solutions to the Pohozaev manifold. They then analyzed the geometry of the energy functional by means of the Hardy inequality, the Sobolev–Hardy inequality, and the Gagliardo–Nirenberg inequality. In the L^2 subcritical, L^2 critical, and L^2 supercritical regimes, the authors constructed Palais–Smale sequences by employing local minimization, Ekeland’s variational principle, and minimax methods, respectively. By proving that the ground state energy lies below the critical Hardy–Sobolev threshold, they excluded concentration phenomena and energy splitting, recovered compactness, and consequently established the existence of normalized ground states. For the defocusing case, the authors obtained nonexistence results by showing that the infimum of the energy functional is not attained and by extending the Hadamard three-circles theorem to the Laplace operator with a Hardy potential. Finally, they investigated the asymptotic behavior of the normalized ground states as the relevant parameters tend to zero, using weak convergence, limiting equations, and energy convergence arguments. Fu [24] mainly employed variational methods, with the minimization of the energy functional under a mass constraint as the core approach. By combining the Hardy inequality, the Lagrange multiplier method, the concentration–compactness principle, the Pohozaev identity, the Mountain Pass Theorem, and the Palais–Smale condition, the author investigated the existence of ground states, global minimizers, and nontrivial solutions for Schrödinger equations with a Hardy term.

Motivated by the above research background and practical applications, we aim to investigate the existence of normalized solutions to the following nonautonomous elliptic equation with a Hardy potential posed in the whole space $\mathbb{R}^N$:

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} + \lambda u = \nu |u|^{q-2} u + |x|^\alpha h(u), & x \in \mathbb{R}^N \setminus \{0\}, \\ \|u\|_2^2 = \kappa, & u \in H^1(\mathbb{R}^N), \end{cases} \quad (2)$$

where $\kappa > 0$, $N \geq 3$, $\alpha \in (-\frac{4}{N}, 0)$, $\nu > 0$, $0 \leq \mu < \bar{\mu} = \frac{(N-2)^2}{4}$, $\bar{q} = 2 + \frac{4}{N} < q < 2^* = \frac{2N}{N-2}$, $\lambda \in \mathbb{R}$ denotes the Lagrange multiplier, and the function $h(u)$ satisfies the following hypotheses:

Hypothesis 1. Suppose h is a continuous odd mapping from $\mathbb{R}$ to $\mathbb{R}$ and not identically zero. For any $\tau \in \mathbb{R}$, define $H(\tau) = \int_0^\tau h(t) dt$.

Hypothesis 2. There exist two constants s, σ satisfy $s \in (-\alpha, \frac{4}{N})$, $\sigma \in (2 + \frac{4}{N}, 2^*(s))$

and $\sigma H(\tau) \leq h(\tau)\tau \leq 2^*(s)H(\tau)$ for any $\tau \in \mathbb{R}$. Here, $2^*(s) = \frac{2(N-s)}{N-2}$ denotes the Hardy-Sobolev critical exponent.

Hypothesis 3. The function $\frac{\tilde{H}(\tau)}{|\tau|^m}$ is strictly increasing on $(0, +\infty)$, where $\tilde{H}(\tau) = h(\tau)\tau - \bar{\theta}H(\tau)$, $\bar{\theta} = 2 + \frac{2\alpha}{N}$, $m = \frac{2N+2\alpha+4}{N}$.

The energy functional corresponding to Equation (2) is given by

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx - \frac{\nu}{q} \int_{\mathbb{R}^N} |u|^q dx - \int_{\mathbb{R}^N} |x|^\alpha H(u) dx. \quad (3)$$

For $\bar{q} < q < 2^*$, it is not difficult to see that $E(u)$ is unbounded below on the set $\mathfrak{N}(\kappa)$. Therefore, we define the following manifold

$$\mathfrak{R}(\kappa) = \left\{ u \in \mathfrak{N}(\kappa) : J(u) = \frac{d}{dt} E\left(t^{\frac{N}{2}} u_t\right) \Big|_{t=1} = 0 \right\},$$

where for any $t > 0$ and $u \in H^1(\mathbb{R}^N)$, we define the scaling $u_t(x) = u(tx)$. It is readily seen that every critical point of the energy functional E restricted to $\mathfrak{N}(\kappa)$ satisfies $J(u) = 0$, and

$$\begin{aligned} J(u) = & \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx - \nu \gamma_q \int_{\mathbb{R}^N} |u|^q dx \\ & - \int_{\mathbb{R}^N} |x|^\alpha \left[\frac{N}{2} h(u)u - (N + \alpha)H(u) \right] dx. \end{aligned} \quad (4)$$

$$\begin{aligned} E\left(t^{\frac{N}{2}} u_t\right) = & \frac{t^2}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx - \frac{\nu}{q} t^{q\gamma_q} \int_{\mathbb{R}^N} |u|^q dx \\ & - t^{-(N+\alpha)} \int_{\mathbb{R}^N} |x|^\alpha H\left(t^{\frac{N}{2}} u\right) dx. \end{aligned} \quad (5)$$

Motivated by the ideas developed by Fan and Li [8], Chen and Tang [16], we investigate the existence of normalized solutions to Equation (2). The main result of this paper is stated as follows. where $H_r^1(\mathbb{R}^N)$ denotes the space of radial functions belonging to $H^1(\mathbb{R}^N)$.

Theorem 1. Assume that $\bar{q} < q < 2^*$, $0 \leq \mu < \bar{\mu}$, $\nu > 0$, and Hypotheses 1–3 hold. Then for any $\kappa > 0$, Equation (2) admits a normalized solution $(u_\kappa, \lambda_\kappa) \in H_r^1(\mathbb{R}^N) \times \mathbb{R}^+$ such that $r(\kappa) = \inf_{u \in \mathfrak{R}(\kappa)} E(u) = E(u_\kappa) > 0$.

Remark 1. One can readily verify that $h(v) = |v|^{q_1-2}v$ satisfies Hypotheses 1–3, where $q_1 = \frac{\sigma+2^*(s)}{2}$. Therefore, the assumptions imposed in this paper are consistent and nonempty.

2. Preliminaries

Lemma 1 (Hardy inequality [25]). Let $1 < p < N$ and $u \in W^{1,p}(\mathbb{R}^N)$. Then

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \leq \left(\frac{p}{N-p} \right)^p \int_{\mathbb{R}^N} |\nabla u|^p dx.$$

In particular, when $p = 2$, the inequality takes the form

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \leq \frac{1}{\mu} \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

Lemma 2 (Gagliardo–Nirenberg inequality [26]). *Let $N \geq 2$, $\eta \in [2, 2^*)$, and define $\gamma_\eta = \frac{N(\eta-2)}{2\eta}$. Then for any $u \in H^1(\mathbb{R}^N)$, the inequality*

$$\|u\|_\eta \leq C_{N,\eta} \|u\|_2^{(1-\gamma_\eta)} \|\nabla u\|_2^{\gamma_\eta},$$

holds with some positive constant $C_{N,\eta} > 0$.

Lemma 3 ([27]). *Suppose $N \geq 3$, $0 < \omega < 2$, and $2 < p < 2^*(\omega)$. There exists a constant $C_p > 0$ for which the following inequality holds for all $u \in H^1(\mathbb{R}^N)$:*

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^\omega} \leq C_p \|\nabla u\|_2^{pV_p} \|u\|_2^{p-pV_p},$$

where $V_p = \frac{N(p-2)+2\omega}{2p}$.

Lemma 4 ([28]). *Assume that $d_1 \geq 0$, $d_2 \geq 0$ and $a \geq 1$ are given, then*

$$(d_1 + d_2)^a \leq C_a (d_1^a + d_2^a),$$

where $C_a = 2^{a-1}$.

3. Proof of the main theorem

Lemma 5. *Assume that Hypothesis 1 and Hypothesis 2 are satisfied, then for any $m \in \mathbb{R}$,*

$$\begin{cases} |t|^{2^*(s)} H(m) \leq H(mt) \leq |t|^\sigma H(m), & t \in [0, 1]; \\ |t|^\sigma H(m) \leq H(mt) \leq |t|^{2^*(s)} H(m), & t \in [1, \infty). \end{cases}$$

Proof. Fix an arbitrary $m \in \mathbb{R}$. For $t > 0$, define $\phi_m(t) = \frac{H(mt)}{t^\sigma}$. Since the function H is differentiable and $H'(\tau) = h(\tau)$, differentiating ϕ_m with respect to t gives

$$\frac{d}{dt} [\phi_m(t)] = t^{-(\sigma+1)} [(mt)h(mt) - \sigma H(mt)].$$

By Hypothesis 2, for every $\tau \in \mathbb{R}$, $\tau h(\tau) \geq \sigma H(\tau)$. Applying this inequality with $\tau = mt$, we obtain $(mt)h(mt) - \sigma H(mt) \geq 0$. Thus, $\phi'_m(t) \geq 0$ for every $t > 0$, which shows that $t \mapsto \frac{H(mt)}{t^\sigma}$ is nondecreasing on $(0, \infty)$. Next, define $\psi_m(t) := \frac{H(mt)}{t^{2^*(s)}}$, $t > 0$. A similar computation yields

$$\frac{d}{dt} [\psi_m(t)] = t^{-(2^*(s)+1)} [(mt)h(mt) - 2^*(s)H(mt)].$$

Using the inequality in Hypothesis 2, we obtain $\psi'_m(t) \leq 0$. For this reason, the function $t \mapsto \frac{H(mt)}{t^{2^*(s)}}$ is nonincreasing on $(0, +\infty)$. Consequently, for $t \in [0, 1]$,

$$|t|^{2^*(s)} H(m) \leq H(mt) \leq |t|^\sigma H(m),$$

whereas for $t \in [1, \infty)$,

$$|t|^\sigma H(m) \leq H(mt) \leq |t|^{2^*(s)} H(m).$$

□

Lemma 6. *Suppose that Hypotheses 1–3 are true. Then for every $u \in \mathfrak{N}(\kappa)$, the following statements hold:*

$$\lim_{t \rightarrow 0^+} E\left(t^{\frac{N}{2}} u_t\right) = 0; \quad \lim_{t \rightarrow \infty} E\left(t^{\frac{N}{2}} u_t\right) = -\infty.$$

Proof. By Hypothesis 2 and Lemma 5, given any $\epsilon > 0$, we can find a constant $\delta > 0$ satisfying

$$H(\tau) \leq \epsilon |\tau|^{\bar{q}} + \delta |\tau|^{2^*(s)}. \quad (6)$$

We next prove that $H(\tau) > 0$ for every $\tau \in \mathbb{R} \setminus \{0\}$. We first show that H is nonnegative on $\tau \in \mathbb{R} \setminus \{0\}$. Suppose, to the contrary, that there exists some $\tau \neq 0$ such that $H(\tau) < 0$. Since $\sigma < 2^*(s)$, multiplying this inequality by the negative number $H(\tau)$ yields $\sigma H(\tau) > 2^*(s) H(\tau)$. On the other hand, Hypothesis 2 gives $\sigma H(\tau) \leq \tau h(\tau) \leq 2^*(s) H(\tau)$, which is a contradiction. Hence $H(\tau) \geq 0$ for every $\tau \in \mathbb{R} \setminus \{0\}$. Moreover, since h is not identically zero, there exists $\tau_1 \neq 0$ such that $h(\tau_1) \neq 0$. If $H(\tau_1) = 0$, then $\tau_1 h(\tau_1) = 0$, which leads to a contradiction. Therefore, $H(\tau_1) > 0$ holds at some nonzero point. By Lemma 5, if $H(\tau_1) > 0$, then given arbitrary $t > 0$, the estimates below are valid.

For $t \in (0, 1]$,

$$H(\tau_1 t) \geq t^{2^*(s)} H(\tau_1) > 0.$$

For $t \in [1, \infty)$,

$$H(\tau_1 t) \geq t^\sigma H(\tau_1) > 0.$$

Consequently, $H(\tau_1 t) > 0$ for every $t > 0$. If $\tau_1 > 0$, then $\tau_1 t > 0$; if $\tau_1 < 0$, then $\tau_1 t < 0$. Let $\tau_1 t = \tau$. It follows that $H(\tau) > 0$ for all $\tau \neq 0$. Combining Equations (5) and (6), and the Hardy inequality, we obtain

$$\begin{aligned} \frac{t^2}{2} \|\nabla u\|_2^2 &\geq E\left(t^{\frac{N}{2}} u_t\right) \\ &\geq \frac{t^2}{2} \left(1 - \frac{\mu}{\bar{\mu}}\right) \|\nabla u\|_2^2 - \frac{\nu}{q} t^{q\gamma_q} \int_{\mathbb{R}^N} |u|^q dx \\ &\quad - \left[\epsilon \int_{\mathbb{R}^N} |x|^\alpha \left|t^{\frac{N}{2}} u_t\right|^{\bar{q}} dx + \delta \int_{\mathbb{R}^N} |x|^\alpha \left|t^{\frac{N}{2}} u_t\right|^{2^*(s)} dx \right]. \end{aligned} \quad (7)$$

By a straightforward calculation, we obtain

$$\int_{\mathbb{R}^N} |x|^\alpha \left|t^{\frac{N}{2}} u_t\right|^{\bar{q}} dx = t^{2-\alpha} \int_{\mathbb{R}^N} |x|^\alpha |u|^{\bar{q}} dx, \quad (8)$$

$$\int_{\mathbb{R}^N} |x|^\alpha \left|t^{\frac{N}{2}} u_t\right|^{2^*(s)} dx = t^\theta \int_{\mathbb{R}^N} |x|^\alpha |u|^{2^*(s)} dx, \quad \theta = N \left(\frac{2^*(s)}{2} - 1\right) - \alpha > 2. \quad (9)$$

We next prove that, as $t \rightarrow 0$,

$$\epsilon t^{2-\alpha} \int_{\mathbb{R}^N} |x|^\alpha |u|^{\bar{q}} dx \rightarrow 0, \quad \delta t^\theta \int_{\mathbb{R}^N} |x|^\alpha |u|^{2^*(s)} dx \rightarrow 0.$$

Thanks to Lemma 3, we obtain that

$$\int_{\mathbb{R}^N} |x|^\alpha |u|^{\bar{q}} dx = \int_{\mathbb{R}^N} \frac{|u|^{\bar{q}}}{|x|^{-\alpha}} dx \leq C_{\bar{q}} \|\nabla u\|_2^{\bar{q}V_{\bar{q}}} \|u\|_2^{\bar{q}-\bar{q}V_{\bar{q}}}, \quad (10)$$

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^\alpha |u|^{2^*(s)} dx &= \int_{\mathbb{R}^N} \frac{|u|^{2^*(s)}}{|x|^{-\alpha}} dx \\ &\leq C_{2^*(s)} \|\nabla u\|_2^{2^*(s)V_{2^*(s)}} \|u\|_2^{2^*(s)-2^*(s)V_{2^*(s)}}. \end{aligned} \quad (11)$$

Since $u \in H^1(\mathbb{R}^N)$, we have $\|\nabla u\|_2 < \infty$. Consequently, as $t \rightarrow 0^+$,

$$\epsilon t^{2-\alpha} \int_{\mathbb{R}^N} |x|^\alpha |u|^{\bar{q}} dx \rightarrow 0, \quad \delta t^\theta \int_{\mathbb{R}^N} |x|^\alpha |u|^{2^*(s)} dx \rightarrow 0.$$

Hence,

$$\lim_{t \rightarrow 0^+} E\left(t^{\frac{N}{2}} u_t\right) = 0.$$

Since $H(\tau) > 0$ for all $\tau \neq 0$, we can derive the following estimate.

$$E\left(t^{\frac{N}{2}} u_t\right) \leq \frac{t^2}{2} \|\nabla u\|_2^2 - \frac{\nu}{q} t^{q\gamma_q} \int_{\mathbb{R}^N} |u|^q dx.$$

Because $\bar{q} < q < 2^*$, we can check that $q\gamma_q > 2$. Consequently, $E\left(t^{\frac{N}{2}} u_t\right) \rightarrow -\infty$ as $t \rightarrow +\infty$. $\square$

Lemma 7. Assume that Hypotheses 1–3 hold. Given any $u \in H^1(\mathbb{R}^N)$ and $t > 0$, it follows that

$$E(u) - E\left(t^{\frac{N}{2}} u_t\right) = \tilde{h}(t, u) + \frac{1-t^2}{2} J(u),$$

where $\tilde{h}(t, u) = \int_{\mathbb{R}^N} h_1(x, t, u) dx + \int_{\mathbb{R}^N} h_2(x, t, u) dx$,

$$\begin{aligned} h_1(x, t, u) &= t^{-N} |t^{-1}x|^\alpha H\left(t^{\frac{N}{2}} u\right) - |x|^\alpha H(u) + \frac{(1-t^2)N}{4} |x|^\alpha [h(u)u - 2H(u)] \\ &\quad - \frac{1-t^2}{2} (\nabla |x|^\alpha) x H(u). \end{aligned}$$

$$h_2(x, t, u) = \nu \left(\frac{1-t^2}{2} \gamma_q + \frac{t^{q\gamma_q} - 1}{q} \right) |u|^q.$$

Moreover, for $t \in (0, 1) \cup (1, \infty)$, $u \neq 0$, we have $\tilde{h}(t, u) > 0$.

Proof. It follows from Equations (3)–(5) that

$$\begin{aligned} E(u) - E\left(t^{\frac{N}{2}}u_t\right) &= \frac{1-t^2}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\right) dx \\ &\quad + \frac{\nu}{q} (t^{q\gamma_q} - 1) \int_{\mathbb{R}^N} |u|^q dx \\ &\quad + \int_{\mathbb{R}^N} \left[t^{-(N+\alpha)} |x|^\alpha H\left(t^{\frac{N}{2}}u\right) - |x|^\alpha H(u)\right] dx \\ &= \frac{1-t^2}{2} \left\{ \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\right) dx \right. \\ &\quad - \int_{\mathbb{R}^N} |x|^\alpha \left[\frac{N}{2} h(u)u - (N+\alpha)H(u)\right] dx \\ &\quad \left. - \nu\gamma_q \int_{\mathbb{R}^N} |u|^q dx \right\} \\ &\quad + \int_{\mathbb{R}^N} h_1(x, t, u) dx + \int_{\mathbb{R}^N} h_2(x, t, u) dx \\ &= \tilde{h}(t, u) + \frac{1-t^2}{2} J(u). \end{aligned}$$

We now verify $h_1(x, t, u) > 0$ for all $t \in (0, 1) \cup (1, \infty)$, $u \neq 0$,

$$\begin{aligned} h_1(x, t, u) &= t^{-(N+\alpha)} |x|^\alpha H\left(t^{\frac{N}{2}}u\right) - |x|^\alpha H(u) \\ &\quad + \frac{(1-t^2)N}{4} |x|^\alpha [h(u)u - 2H(u)] - \frac{\alpha(1-t^2)}{2} |x|^\alpha H(u) \\ &= |x|^\alpha h_3(x, t, u). \end{aligned}$$

where

$$\begin{aligned} h_3(x, t, u) &= t^{-(N+\alpha)} H\left(t^{\frac{N}{2}}u\right) - H(u) + \frac{N(1-t^2)}{4} [h(u)u - 2H(u)] \\ &\quad - \frac{\alpha(1-t^2)}{2} H(u). \\ \frac{dh_3(x, t, u)}{dt} &= -(N+\alpha)t^{-(N+\alpha)-1} H\left(t^{\frac{N}{2}}u\right) + t^{-(N+\alpha)} \frac{N}{2} h\left(t^{\frac{N}{2}}u\right) t^{\frac{N}{2}-1}u \\ &\quad - \frac{N}{2} th(u)u + NtH(u) + \alpha tH(u) \\ &= t^{-(N+\alpha)-1} \left[\frac{N}{2} h\left(t^{\frac{N}{2}}u\right) \left(t^{\frac{N}{2}}u\right) - (N+\alpha)H\left(t^{\frac{N}{2}}u\right) \right] \\ &\quad - \frac{N}{2} th(u)u + (N+\alpha)tH(u) \\ &= \frac{N}{2} t|u|^{2+\frac{4+2\alpha}{N}} \left[\frac{h\left(t^{\frac{N}{2}}u\right) \left(t^{\frac{N}{2}}u\right) - \left(2+\frac{2\alpha}{N}\right)H\left(t^{\frac{N}{2}}u\right)}{\left|t^{\frac{N}{2}}u\right|^{2+\frac{4+2\alpha}{N}}} \right. \\ &\quad \left. - \frac{h(u)u - \left(2+\frac{2\alpha}{N}\right)H(u)}{|u|^{2+\frac{4+2\alpha}{N}}} \right]. \end{aligned}$$

By Hypothesis 3, the function $F(s) = \frac{h(s)s - \left(2+\frac{2\alpha}{N}\right)H(s)}{|s|^{2+\frac{4+2\alpha}{N}}}$ is strictly increasing on

$(0, +\infty)$. Since h is odd, the definition $H(u) = \int_0^u h(\tau) d\tau$ implies that H is even, and hence the function F is even. Therefore

$$h'_3(x, t, u) \begin{cases} < 0, & t \in (0, 1), \\ > 0, & t \in (1, +\infty). \end{cases}$$

Recall that $h_1(x, t, u) = |x|^\alpha h_3(x, t, u)$. Since $x \neq 0$, the factor $|x|^\alpha$ is strictly positive and independent of t . Therefore, differentiating h_1 with respect to t and using the preceding result, we obtain

$$h'_1(x, t, u) = |x|^\alpha h'_3(x, t, u) \begin{cases} < 0, & t \in (0, 1), \\ > 0, & t \in (1, +\infty). \end{cases}$$

This yields $h_1(x, t, u) > h_1(x, 1, u) = 0$.

Moreover, $\frac{d}{dt} h_2(x, t, u) \equiv \nu |u|^{q\gamma_q} (t^{q\gamma_q-1} - t)$. Since $q\gamma_q > 2$, we have

$$\frac{d}{dt} h_2(x, t, u) \begin{cases} < 0, & t \in (0, 1), \\ > 0, & t \in (1, +\infty) \end{cases}$$

therefore, $h_2(x, t, u) > 0$ for all $t \in (0, 1) \cup (1, \infty)$, which completes the proof of $\tilde{h}(t, u) > 0$ for $t \in (0, 1) \cup (1, \infty)$, $u \neq 0$. $\square$

Lemma 8. Assume that Hypotheses 1–3 hold. For any $u \in \mathfrak{N}(\kappa)$, there exists a unique $t_u > 0$ such that $t_u^{\frac{N}{2}} u_{t_u} \in \mathfrak{R}(\kappa)$. Moreover, $E\left(t_u^{\frac{N}{2}} u_{t_u}\right) > E\left(t^{\frac{N}{2}} u_t\right)$ for all $t > 0$ with $t \neq t_u$.

Proof. For any $u \in \mathfrak{N}(\kappa)$, we have $t^{\frac{N}{2}} u_t \in \mathfrak{N}(\kappa)$. According to Equation (5),

$$\begin{aligned} E'\left(t^{\frac{N}{2}} u_t\right) &= t \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx - \nu \gamma_q t^{q\gamma_q-1} \int_{\mathbb{R}^N} |u|^q dx \\ &\quad + (N + \alpha) t^{-(N+\alpha)-1} \int_{\mathbb{R}^N} |x|^\alpha H\left(t^{\frac{N}{2}} u\right) dx \\ &\quad - \frac{N}{2} t^{-(N+\alpha)-1} \int_{\mathbb{R}^N} |x|^\alpha h\left(t^{\frac{N}{2}} u\right) t^{\frac{N}{2}} u dx \\ &= \frac{1}{t} J\left(t^{\frac{N}{2}} u_t\right). \end{aligned}$$

Therefore, $E'\left(t^{\frac{N}{2}} u_t\right) = 0 \iff J\left(t^{\frac{N}{2}} u_t\right) = 0 \iff t^{\frac{N}{2}} u_t \in \mathfrak{R}(\kappa)$. From Lemma 6, $\lim_{t \rightarrow 0^+} E\left(t^{\frac{N}{2}} u_t\right) = 0$ and $E\left(t^{\frac{N}{2}} u_t\right) < 0$ for sufficiently large t . In view of Equations (7)–(11), $E\left(t^{\frac{N}{2}} u_t\right) > 0$ for sufficiently small $t > 0$. Thus $\max_{t \in (0, \infty)} E\left(t^{\frac{N}{2}} u_t\right)$ is attained, so there exists $t_u > 0$ such that $E'\left(t_u^{\frac{N}{2}} u_{t_u}\right) = 0$ and $t_u^{\frac{N}{2}} u_{t_u} \in \mathfrak{R}(\kappa)$.

We now prove the uniqueness of t_u for any $u \in \mathfrak{N}(\kappa)$. Let $t_1, t_2 > 0$ satisfy

$t_1^{\frac{N}{2}} u_{t_1} \in \mathfrak{R}(\kappa)$, $t_2^{\frac{N}{2}} u_{t_2} \in \mathfrak{R}(\kappa)$. By Lemma 7,

$$\begin{aligned} E\left(t_1^{\frac{N}{2}} u_{t_1}\right) &= E\left(\left(\frac{t_2}{t_1}\right)^{\frac{N}{2}} \left(t_1^{\frac{N}{2}} u_{t_1}\right)_{\frac{t_2}{t_1}}\right) + \frac{1 - \left(\frac{t_2}{t_1}\right)^2}{2} J\left(t_1^{\frac{N}{2}} u_{t_1}\right) + \tilde{h}\left(\frac{t_2}{t_1}, t_1^{\frac{N}{2}} u_{t_1}\right) \\ &= E\left(t_2^{\frac{N}{2}} u_{t_2}\right) + \tilde{h}\left(\frac{t_2}{t_1}, t_1^{\frac{N}{2}} u_{t_1}\right). \end{aligned} \quad (12)$$

Similarly,

$$\begin{aligned} E\left(t_2^{\frac{N}{2}} u_{t_2}\right) &= E\left(\left(\frac{t_1}{t_2}\right)^{\frac{N}{2}} \left(t_2^{\frac{N}{2}} u_{t_2}\right)_{\frac{t_1}{t_2}}\right) + \frac{1 - \left(\frac{t_1}{t_2}\right)^2}{2} J\left(t_2^{\frac{N}{2}} u_{t_2}\right) + \tilde{h}\left(\frac{t_1}{t_2}, t_2^{\frac{N}{2}} u_{t_2}\right) \\ &= E\left(t_1^{\frac{N}{2}} u_{t_1}\right) + \tilde{h}\left(\frac{t_1}{t_2}, t_2^{\frac{N}{2}} u_{t_2}\right). \end{aligned} \quad (13)$$

Combining Equations (12), (13) and Lemma 7, we deduce $t_1 = t_2$, this establishes the result. $\square$

Lemma 9. Assume that Hypotheses 1–3 hold. Then

(1) There exists $\rho > 0$ such that $\inf_{u \in \mathfrak{R}(\kappa)} \|\nabla u\|_2 > \rho > 0$;

(2) $r(\kappa) = \inf_{u \in \mathfrak{R}(\kappa)} E(u) > 0$.

Proof. Let $u \in \mathfrak{R}(\kappa)$ be arbitrary. Then

$$\begin{aligned} \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx &= \nu \gamma_q \int_{\mathbb{R}^N} |u|^q dx \\ &\quad + \int_{\mathbb{R}^N} |x|^\alpha \left[\frac{N}{2} h(u) u - (N + \alpha) H(u) \right] dx. \end{aligned} \quad (14)$$

From Hypothesis 2 and Lemma 5, we have

$$|h(v)v| \leq \epsilon |v|^{\bar{q}} + \delta |v|^{2^*(s)}. \quad (15)$$

It follows from Lemma 5 and Hypothesis 2 that $H(u)$ is positive for all $u \neq 0$. Since $N + \alpha > 0$, it follows from Equation (14) and the Hardy inequality that

$$\left(1 - \frac{\mu}{\bar{\mu}}\right) \int_{\mathbb{R}^N} |\nabla u|^2 dx \leq \nu \gamma_q \int_{\mathbb{R}^N} |u|^q dx + \frac{N}{2} \int_{\mathbb{R}^N} |x|^\alpha h(u) u dx. \quad (16)$$

Together with Equations (10), (11), and (15), we get

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^\alpha h(u) u dx &\leq \epsilon \int_{\mathbb{R}^N} |x|^\alpha |u|^{\bar{q}} dx + \delta \int_{\mathbb{R}^N} |x|^\alpha |u|^{2^*(s)} dx \\ &\leq \epsilon C_{\bar{q}} \kappa^{\frac{\bar{q}-qV_{\bar{q}}}{2}} \|\nabla u\|_2^{\bar{q}V_{\bar{q}}} \\ &\quad + \delta C_{2^*(s)} \kappa^{\frac{2^*(s)-2^*(s)V_{2^*(s)}}{2}} \|\nabla u\|_2^{2^*(s)V_{2^*(s)}}. \end{aligned}$$

Combining Equation (16) and the Gagliardo–Nirenberg inequality with the above

estimate, we obtain

$$\begin{aligned} \left(1 - \frac{\mu}{\bar{\mu}}\right) \int_{\mathbb{R}^N} |\nabla u|^2 dx &\leq \nu \gamma_q C_{N,q} \kappa^{\frac{q(1-\gamma_q)}{2}} \|\nabla u\|_2^{q\gamma_q} \\ &\quad + \frac{N}{2} \epsilon C_{\bar{q}} \kappa^{\frac{\bar{q}-\bar{q}V_{\bar{q}}}{2}} \|\nabla u\|_2^{\bar{q}V_{\bar{q}}} \\ &\quad + \frac{N}{2} \delta C_{2^*(s)} \kappa^{\frac{2^*(s)-2^*(s)V_{2^*(s)}}{2}} \|\nabla u\|_2^{2^*(s)V_{2^*(s)}}. \end{aligned}$$

Since $q\gamma_q > 2$, $\bar{q}V_{\bar{q}} > 2$, $2^*(s)V_{2^*(s)} > 2$, we deduce that $\|\nabla u\|_2$ cannot converge to 0. Based on the foregoing results, we can select a constant $\rho > 0$ that fulfills

$$\inf_{u \in \mathfrak{R}(\kappa)} \|\nabla u\|_2 > \rho > 0,$$

which completes the proof of assertion (1) in Lemma 9.

We now verify assertion (2). For all u belonging to $\mathfrak{R}(\kappa)$, the equality $J(u) = 0$ is satisfied. Set $b_1 = \min\{\sigma, q\}$, then

$$\begin{aligned} E(u) &= E(u) - \frac{2}{(b_1 - 2)N} J(u) \\ &= \left(\frac{1}{2} - \frac{2}{(b_1 - 2)N}\right) \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\right) dx \\ &\quad + \frac{1}{b_1 - 2} \int_{\mathbb{R}^N} |x|^\alpha [h(u)u - b_1 H(u)] dx - \frac{2\alpha}{(b_1 - 2)N} \int_{\mathbb{R}^N} |x|^\alpha H(u) dx \\ &\quad + \nu \left(\frac{2\gamma_q}{(b_1 - 2)N} - \frac{1}{q}\right) \int_{\mathbb{R}^N} |u|^q dx \\ &\geq \left(\frac{1}{2} - \frac{2}{(b_1 - 2)N}\right) \left(1 - \frac{\mu}{\bar{\mu}}\right) \|\nabla u\|_2^2 + \nu \left(\frac{2\gamma_q}{(b_1 - 2)N} - \frac{1}{q}\right) \int_{\mathbb{R}^N} |u|^q dx \\ &\quad + \frac{1}{b_1 - 2} \int_{\mathbb{R}^N} |x|^\alpha [h(u)u - b_1 H(u)] dx \\ &\geq \left(\frac{1}{2} - \frac{2}{(b_1 - 2)N}\right) \left(1 - \frac{\mu}{\bar{\mu}}\right) \|\nabla u\|_2^2 \\ &\geq \left(\frac{1}{2} - \frac{2}{(b_1 - 2)N}\right) \left(1 - \frac{\mu}{\bar{\mu}}\right) \rho^2. \end{aligned} \tag{17}$$

As a consequence, $\inf_{u \in \mathfrak{R}(\kappa)} E(u) > 0$. $\square$

Lemma 10. Suppose that Hypotheses 1–3 are satisfied. Then $r(\kappa)$ is non-increasing with respect to κ on $(0, \infty)$. Moreover, whenever $r(\kappa)$ is attained, it follows that $r(\kappa) > r(\kappa')$ for every $\kappa' > \kappa$.

Proof. Given any $0 < \kappa_1 < \kappa_2$, we can find a sequence $\{u_n\} \subset \mathfrak{R}(\kappa_1)$ satisfying $E(u_n) < r(\kappa_1) + 1/n$. Let

$$\varsigma = \sqrt{\frac{\kappa_2}{\kappa_1}} \in (1, \infty), \quad v_n(x) = \varsigma^{(2-N)/2} u_n(\varsigma^{-1}x).$$

Then

$$\|v_n\|_2^2 = \kappa_2, \quad \int_{\mathbb{R}^N} \frac{|v_n|^2}{|x|^2} dx = \int_{\mathbb{R}^N} \frac{|u_n|^2}{|x|^2} dx, \quad \|\nabla v_n\|_2 = \|\nabla u_n\|_2, \quad \|v_n\|_q = \varsigma^{1-\gamma_q} \|u_n\|_q.$$

Since $s + \alpha > 0$, by Lemma 5 we have

$$\begin{aligned} & t_n^{-N} \int_{\mathbb{R}^N} |t_n^{-1}x|^\alpha H(t_n^{\frac{N}{2}} u_n) dx - t_n^{-N} \int_{\mathbb{R}^N} |t_n^{-1}x|^\alpha H(t_n^{\frac{N}{2}} v_n) dx \\ &= t_n^{-N} \int_{\mathbb{R}^N} |t_n^{-1}x|^\alpha H(t_n^{\frac{N}{2}} u_n) dx - t_n^{-N} \int_{\mathbb{R}^N} |\varsigma t_n^{-1}x|^\alpha \varsigma^N H(\varsigma^{\frac{2-N}{2}} t_n^{\frac{N}{2}} u_n) dx \quad (18) \\ &\leq t_n^{-N} \int_{\mathbb{R}^N} \left[|t_n^{-1}x|^\alpha - \varsigma^s |\varsigma t_n^{-1}x|^\alpha \right] H(t_n^{\frac{N}{2}} u_n) dx < 0. \end{aligned}$$

Moreover, since $\bar{q} < q < 2^*$, it follows that $\gamma_q < 1$, and hence

$$-\frac{\nu}{q} t_n^{q\gamma_q} \int_{\mathbb{R}^N} |v_n|^q dx + \frac{\nu}{q} t_n^{q\gamma_q} \int_{\mathbb{R}^N} |u_n|^q dx \leq 0. \quad (19)$$

From Lemma 8, we can choose $t_n > 0$ satisfying $t_n^{\frac{N}{2}} (v_n)_{t_n} \in \mathfrak{R}(\kappa_2)$. Then, by Lemma 9, Equations (5), (18), and (19), we have

$$\begin{aligned} r(\kappa_2) &\leq E\left(t_n^{\frac{N}{2}} (v_n)_{t_n}\right) \\ &\leq E\left(t_n^{\frac{N}{2}} (u_n)_{t_n}\right) + t_n^{-N} \int_{\mathbb{R}^N} \left[|t_n^{-1}x|^\alpha - \varsigma^s |t_n^{-1}x\varsigma|^\alpha \right] H\left(t_n^{\frac{N}{2}} u_n\right) dx \\ &\leq E\left(t_n^{\frac{N}{2}} (u_n)_{t_n}\right) \leq E(u_n) < r(\kappa_1) + \frac{1}{n}. \end{aligned}$$

Taking $n \rightarrow \infty$, we get $r(\kappa_2) \leq r(\kappa_1)$. This completes the proof of the non-increasing property.

Now suppose that $r(\kappa)$ is attainable, that is, there is an element $\tilde{u} \in \mathfrak{R}(\kappa)$ satisfying $E(\tilde{u}) = r(\kappa)$. For any $\kappa' > \kappa$, set $\xi = \sqrt{\frac{\kappa'}{\kappa}} \in (1, \infty)$, $\tilde{v}(x) = \xi^{(2-N)/2} \tilde{u}(\xi^{-1}x)$. By Lemma 8, there exists $t_0 > 0$ such that $t_0^{\frac{N}{2}} \tilde{v}_{t_0} \in \mathfrak{R}(\kappa')$. Combining Lemma 9, Equations (5), (18), and (19), we deduce

$$r(\kappa') \leq E(t_0^{\frac{N}{2}} \tilde{v}_{t_0}) < E(t_0^{\frac{N}{2}} \tilde{u}_{t_0}) \leq E(\tilde{u}) = r(\kappa),$$

which finishes the whole proof. $\square$

Lemma 11. Assume that Hypotheses 1–3 hold. Let $\{u_n\} \subset \mathfrak{R}(\kappa)$ be a minimizing sequence for $r(\kappa)$. Then the sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$.

Proof. Set $b_1 = \min\{\sigma, q\}$. By Equation (17), we deduce

$$\begin{aligned} r(\kappa) + o(1) &= E(u_n) \\ &= \left(\frac{1}{2} - \frac{2}{(b_1 - 2)N} \right) \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 - \mu \frac{u_n^2}{|x|^2} \right) dx \\ &\quad + \frac{1}{b_1 - 2} \int_{\mathbb{R}^N} |x|^\alpha [h(u_n)u_n - b_1 H(u_n)] dx \\ &\quad - \frac{2\alpha}{(b_1 - 2)N} \int_{\mathbb{R}^N} |x|^\alpha H(u_n) dx + \left(\frac{2\nu\gamma_q}{(b_1 - 2)N} - \frac{\nu}{q} \right) \int_{\mathbb{R}^N} |u_n|^q dx. \end{aligned}$$

Since $b_1 = \min\{\sigma, q\}$, we have $\frac{1}{2} - \frac{2}{(b_1 - 2)N} > 0$. Hypothesis 2 implies $h(u_n)u_n -$

$b_1 H(u_n) \geq 0$. In addition, $\frac{2\gamma_q}{(b_1-2)N} - \frac{1}{q} \geq 0$, so that

$$r(\kappa) + o(1) \geq \left(\frac{1}{2} - \frac{2}{(b_1-2)N} \right) \left(1 - \frac{\mu}{\bar{\mu}} \right) \|\nabla u_n\|_2^2.$$

Hence, the boundedness of $\|\nabla u_n\|_2$ implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. $\square$

Lemma 12. Assume that Hypotheses 1–3 hold. For any $t \in (2, 2^*)$, $u_n \rightarrow u$ in $L^t(\mathbb{R}^N)$. Then

$$\int_{\mathbb{R}^N} |x|^\alpha [H(u_n) - H(u)] dx \rightarrow 0.$$

Proof. In view of Hypothesis 2 and Lemma 5, corresponding to any $\epsilon > 0$, there exists a positive constant δ satisfying

$$|h(u)| \leq \epsilon |u|^{\bar{q}-1} + \delta |u|^{2^*(s)-1}. \quad (20)$$

According to Hypothesis 1, Equation (20), and Lemma 4, we have

$$\begin{aligned} |H(u_n) - H(u)| &= \left| \int_0^1 \frac{d}{dt} H(u + t(u_n - u)) dt \right| \\ &\leq |u_n - u| \max_{t \in [0,1]} |h(u + t(u_n - u))| \\ &\leq \left[C_1 (|u_n|^{\bar{q}-1} + |u|^{\bar{q}-1}) + C_2 (|u_n|^{2^*(s)-1} + |u|^{2^*(s)-1}) \right] |u_n - u|. \end{aligned}$$

It follows that

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^\alpha |H(u_n) - H(u)| dx &\leq C_1 \int_{\mathbb{R}^N} |x|^\alpha (|u_n|^{\bar{q}-1} + |u|^{\bar{q}-1}) |u_n - u| dx \\ &\quad + C_2 \int_{\mathbb{R}^N} |x|^\alpha (|u_n|^{2^*(s)-1} + |u|^{2^*(s)-1}) |u_n - u| dx. \end{aligned}$$

For $x \in \mathbb{R}^N \setminus B_1(0)$, by the Hölder inequality,

$$\begin{aligned} &\int_{\mathbb{R}^N \setminus B_1(0)} |x|^\alpha |u_n|^{\bar{q}-1} |u_n - u| dx \\ &\leq \int_{\mathbb{R}^N \setminus B_1(0)} |u_n|^{\bar{q}-1} |u_n - u| dx \\ &\leq \left(\int_{\mathbb{R}^N \setminus B_1(0)} |u_n|^{\bar{q}} dx \right)^{\frac{\bar{q}-1}{\bar{q}}} \left(\int_{\mathbb{R}^N \setminus B_1(0)} |u_n - u|^{\bar{q}} dx \right)^{\frac{1}{\bar{q}}} \rightarrow 0. \end{aligned} \quad (21)$$

For $x \in B_1(0)$, there exist constants $e > 1$, $a > 1$, $b > 1$, such that $\frac{1}{e} + \frac{1}{a} + \frac{1}{b} = 1$, $2 < a(\bar{q} - 1) < 2^*$, $2 < b < 2^*$, $\alpha e > -N$. Applying the Hölder inequality again, we obtain

$$\begin{aligned} &\int_{B_1(0)} |x|^\alpha |u_n|^{\bar{q}-1} |u_n - u| dx \\ &\leq \left(\int_{B_1(0)} |x|^{\alpha e} dx \right)^{\frac{1}{e}} \left(\int_{B_1(0)} |u_n|^{a(\bar{q}-1)} dx \right)^{\frac{1}{a}} \left(\int_{B_1(0)} |u_n - u|^b dx \right)^{\frac{1}{b}} \rightarrow 0. \end{aligned} \quad (22)$$

From Equations (21) and (22), we have $\int_{\mathbb{R}^N} |x|^\alpha |u_n|^{\bar{q}-1} |u_n - u| dx \rightarrow 0$.

Similarly, $\int_{\mathbb{R}^N} |x|^\alpha |u|^{\bar{q}-1} |u_n - u| dx \rightarrow 0$, $\int_{\mathbb{R}^N} |x|^\alpha |u_n|^{2^*(s)-1} |u_n - u| dx \rightarrow 0$, $\int_{\mathbb{R}^N} |x|^\alpha |u|^{2^*(s)-1} |u_n - u| dx \rightarrow 0$. Hence, $\int_{\mathbb{R}^N} |x|^\alpha |H(u_n) - H(u)| dx \rightarrow 0$, which yields

$$\int_{\mathbb{R}^N} |x|^\alpha [H(u_n) - H(u)] dx \rightarrow 0.$$

This yields the desired conclusion. $\square$

Lemma 13. Assume that Hypotheses 1–3 hold. For any $t \in (2, 2^*)$, $u_n \rightarrow u$ in $L^t(\mathbb{R}^N)$. Then

$$\int_{\mathbb{R}^N} |x|^\alpha [h(u_n)u_n - h(u)u] dx \rightarrow 0.$$

Proof. From the proofs of Lemma 6 and Lemma 7 in Wang and Fan's study [29], we see that

$$\int_{B_1(0)} [|x|^\alpha h(u_n)u_n - |x|^\alpha h(u)u] dx \rightarrow 0. \quad (23)$$

By Lemma 2.3 in the study by ZX Zhang and ZT Zhang [30], we have $h(u_n)u_n \rightarrow h(u)u$ in $L^1(\mathbb{R}^N)$. It follows that

$$\begin{aligned} 0 &\leq \left| \int_{\mathbb{R}^N \setminus B_1(0)} |x|^\alpha [h(u_n)u_n - h(u)u] dx \right| \leq \int_{\mathbb{R}^N \setminus B_1(0)} |h(u_n)u_n - h(u)u| dx \\ &\leq \int_{\mathbb{R}^N} |h(u_n)u_n - h(u)u| dx \rightarrow 0. \end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^N \setminus B_1(0)} |x|^\alpha [h(u_n)u_n - h(u)u] dx \rightarrow 0. \quad (24)$$

By the additivity of the integral over the domain of integration, together with Equations (23) and (24), we obtain

$$\int_{\mathbb{R}^N} |x|^\alpha [h(u_n)u_n - h(u)u] dx \rightarrow 0.$$

which completes the proof. $\square$

Lemma 14. Assume that Hypotheses 1–3 hold. Then $r(\kappa)$ is attainable.

Proof. By Hypothesis 1, h is an odd function. Hence $\{|u_n|\} \subset \mathfrak{R}(\kappa)$ and $E(|u_n|) \rightarrow r(\kappa)$. Let $\{v_n\} = \{|u_n|^*\}$ be the Schwarz symmetric rearrangement of $\{|u_n|\}$. Based on the properties of Schwarz symmetrization [31], we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx &\leq \int_{\mathbb{R}^N} |\nabla |u_n||^2 dx, & \int_{\mathbb{R}^N} \frac{v_n^2}{|x|^2} dx &\geq \int_{\mathbb{R}^N} \frac{|u_n|^2}{|x|^2} dx, \\ \int_{\mathbb{R}^N} |v_n|^q dx &= \int_{\mathbb{R}^N} |u_n|^q dx. \end{aligned}$$

Next, we verify that $\int_{\mathbb{R}^N} |x|^\alpha H(v_n) dx \geq \int_{\mathbb{R}^N} |x|^\alpha H(|u_n|) dx$. Let $L = \lim_{s \rightarrow +\infty} H(s) \in (0, +\infty]$. Since H is continuous and strictly increasing on $[0, \infty)$, with $H(0) = 0$, for every $0 < t < L$, there exists a unique $s_t > 0$ such that $H(s_t) = t$, equivalently, $s_t = H^{-1}(t)$. Therefore, $\{H(|u_n|) > t\} = \{|u_n| > s_t\}$. Similarly,

$\{H(v_n) > t\} = \{v_n > s_t\}$. Since $v_n = |u_n|^*$, by the properties of the Schwarz rearrangement, we have $\{v_n > s_t\} = \{|u_n| > s_t\}^*$. Therefore,

$$\{H(v_n) > t\} = \{v_n > s_t\} = \{|u_n| > s_t\}^* = \{H(|u_n|) > t\}^*. \quad (25)$$

We now show that these superlevel sets have finite Lebesgue measure. Since $\{u_n\} \subset H^1(\mathbb{R}^N)$, by Chebyshev inequality, we obtain

$$|\{|u_n| > s_t\}| \leq \frac{1}{s_t^2} \int_{\mathbb{R}^N} |u_n(x)|^2 dx < +\infty.$$

Therefore, $|\{H(|u_n|) > t\}| = |\{|u_n| > s_t\}| < +\infty$. According to the characterization of superlevel sets under the Schwarz rearrangement,

$$\{(H(|u_n|))^* > t\} = \{H(|u_n|) > t\}^*. \quad (26)$$

From Equations (25) and (26), for any $0 < t < L$, we obtain $\{H(v_n) > t\} = \{(H(|u_n|))^* > t\}$. If $L < +\infty$, we further consider the case $t \geq L$. Since for every $s \geq 0$, $H(s) < L \leq t$, it follows that $\{H(|u_n|) > t\} = \emptyset$, and $\{H(v_n) > t\} = \emptyset$. Therefore, for t exceeding the range of H , the corresponding superlevel sets are both empty and hence have Lebesgue measure zero. Thus, the conclusion still holds. In summary, for every $t > 0$, we have $\{H(v_n) > t\} = \{(H(|u_n|))^* > t\}$. Therefore, $H(v_n) = (H(|u_n|))^*$ almost everywhere. By Lieb and Loss [31], if f and g are nonnegative functions on $\mathbb{R}^n$ and $|\{x : |f(x)| > t\}| < \infty$, $|\{x : |g(x)| > t\}| < \infty$, then $\int_{\mathbb{R}^n} f(x)g(x) dx \leq \int_{\mathbb{R}^n} f^*(x)g^*(x) dx$. Let $f(x) = |x|^\alpha$, $g(x) = H(u_n(x))$, since $\{x : |x|^\alpha > t\} = B_{t^{\frac{1}{\alpha}}}(0)$, we have $|\{x : |x|^\alpha > t\}| < \infty$. It follows from the above analysis that $\int_{\mathbb{R}^N} |x|^\alpha H(v_n) dx \geq \int_{\mathbb{R}^N} |x|^\alpha H(|u_n|) dx$.

Thus $\{v_n\} \subset \mathfrak{N}(\kappa) \cap H_r^1(\mathbb{R}^N)$. By Lemma 8, there exists a unique $\tau_n > 0$ such that $J(\tau_n^{\frac{N}{2}}(v_n)_{\tau_n}) = 0$. Therefore

$$\begin{aligned} r(\kappa) &\leq E(\tau_n^{\frac{N}{2}}(v_n)_{\tau_n}) \\ &= \frac{\tau_n^2}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \frac{\mu\tau_n^2}{2} \int_{\mathbb{R}^N} \frac{v_n^2}{|x|^2} dx - \frac{\nu\tau_n^{q\gamma_q}}{q} \int_{\mathbb{R}^N} |v_n|^q dx \\ &\quad - \tau_n^{-(N+\alpha)} \int_{\mathbb{R}^N} |x|^\alpha H(\tau_n^{\frac{N}{2}} v_n) dx \\ &\leq \frac{\tau_n^2}{2} \int_{\mathbb{R}^N} |\nabla |u_n||^2 dx - \frac{\mu\tau_n^2}{2} \int_{\mathbb{R}^N} \frac{|u_n|^2}{|x|^2} dx - \frac{\nu\tau_n^{q\gamma_q}}{q} \int_{\mathbb{R}^N} |u_n|^q dx \\ &\quad - \tau_n^{-(N+\alpha)} \int_{\mathbb{R}^N} |x|^\alpha H(\tau_n^{\frac{N}{2}} |u_n|) dx \\ &= E(\tau_n^{\frac{N}{2}}(|u_n|)_{\tau_n}) \leq E(|u_n|) = r(k) + o(1). \end{aligned}$$

Hence $\tau_n^{\frac{N}{2}}(v_n)_{\tau_n} \subset \mathfrak{R}(\kappa) \cap H_r^1(\mathbb{R}^N)$ is a minimizing sequence for $r(\kappa)$. By Lemma 11, the sequence $\tau_n^{\frac{N}{2}}(v_n)_{\tau_n}$ remains bounded in $H_r^1(\mathbb{R}^N)$. Consequently, there exists some $u \in H_r^1(\mathbb{R}^N)$ such that $\tau_n^{\frac{N}{2}}(v_n)_{\tau_n}$ converges weakly to u in $H^1(\mathbb{R}^N)$, converges strongly to u in $L^t(\mathbb{R}^N)$ for all $t \in (2, 2^*)$, and converges to u almost everywhere on

$\mathbb{R}^N$. Consequently, we have

$$\int_{\mathbb{R}^N} |u|^2 dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left| \tau_n^{\frac{N}{2}}(v_n)_{\tau_n} \right|^2 dx = \kappa, \quad (27)$$

$$\begin{aligned} & \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx \\ & \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left[\left| \nabla \left(\tau_n^{\frac{N}{2}}(v_n)_{\tau_n} \right) \right|^2 - \mu \frac{\left| \tau_n^{\frac{N}{2}}(v_n)_{\tau_n} \right|^2}{|x|^2} \right] dx. \end{aligned} \quad (28)$$

From Lemmas 12 and 13, we obtain

$$\int_{\mathbb{R}^N} |x|^\alpha H \left(\tau_n^{\frac{N}{2}}(v_n)_{\tau_n} \right) dx \rightarrow \int_{\mathbb{R}^N} |x|^\alpha H(u) dx, \quad (29)$$

$$\int_{\mathbb{R}^N} |x|^\alpha \left[h \left(\tau_n^{\frac{N}{2}}(v_n)_{\tau_n} \right) \tau_n^{\frac{N}{2}}(v_n)_{\tau_n} - h(u)u \right] dx \rightarrow 0. \quad (30)$$

Since $\tau_n^{\frac{N}{2}}(v_n)_{\tau_n} \subset \mathfrak{R}(\kappa) \cap H_r^1(\mathbb{R}^N)$ forms a minimizing sequence for the value $r(\kappa)$, and $\bar{q} < q < 2^*$, together with $h(u)u \leq 2^*(s)H(u)$ given in Hypothesis 2, we deduce

$$\begin{aligned} r(\kappa) + o(1) &= E \left(\tau_n^{\frac{N}{2}}(v_n)_{\tau_n} \right) - \frac{1}{2} J \left(\tau_n^{\frac{N}{2}}(v_n)_{\tau_n} \right) \\ &= \left(\frac{\nu \gamma q}{2} - \frac{\nu}{q} \right) \int_{\mathbb{R}^N} \left| \tau_n^{\frac{N}{2}}(v_n)_{\tau_n} \right|^q dx \\ &\quad + \int_{\mathbb{R}^N} |x|^\alpha \left[\frac{N}{4} h \left(\tau_n^{\frac{N}{2}}(v_n)_{\tau_n} \right) \tau_n^{\frac{N}{2}}(v_n)_{\tau_n} \right. \\ &\quad \left. - \frac{N + \alpha + 2}{2} H \left(\tau_n^{\frac{N}{2}}(v_n)_{\tau_n} \right) \right] dx \\ &\leq \nu \left(\frac{\gamma q}{2} - \frac{1}{q} \right) \int_{\mathbb{R}^N} \left| \tau_n^{\frac{N}{2}}(v_n)_{\tau_n} \right|^q dx \\ &\quad + \left(\frac{N(N-s)}{2(N-2)} - \frac{N + \alpha + 2}{2} \right) \int_{\mathbb{R}^N} |x|^\alpha H \left(\tau_n^{\frac{N}{2}}(v_n)_{\tau_n} \right) dx. \end{aligned} \quad (31)$$

From Equations (29) and (31), we have

$$r(\kappa) \leq \left(\frac{\nu \gamma q}{2} - \frac{\nu}{q} \right) \int_{\mathbb{R}^N} |u|^q dx + \left(\frac{N(N-s)}{2(N-2)} - \frac{N + \alpha + 2}{2} \right) \int_{\mathbb{R}^N} |x|^\alpha H(u) dx.$$

By Lemma 9, we get $u \neq 0$. By taking into account the conclusions obtained in Equations (28)–(30), we can readily deduce that the functional J satisfies $J(u) \leq 0$.

Let $\kappa_0 = \|u\|_2^2$. From Equation (27), $\kappa_0 \leq \kappa$. By Lemma 8, there exists $t_m \in (0, 1]$ such that $t_m^{\frac{N}{2}} u_{t_m} \in \mathfrak{R}(\kappa_0)$. Lemma 7 implies $E(u) - E(t_m^{\frac{N}{2}} u_{t_m}) \geq \frac{1-t_m^2}{2} J(u)$, which yields

$$\begin{aligned}
 r(\kappa_0) &\leq E\left(t_m^{\frac{N}{2}} u_{t_m}\right) \leq E(u) - \frac{1-t_m^2}{2} J(u) \\
 &\leq E(u) - \frac{1}{2} J(u) \\
 &= \left(\frac{\nu\gamma_q}{2} - \frac{\nu}{q}\right) \int_{\mathbb{R}^N} |u|^q dx \\
 &\quad + \int_{\mathbb{R}^N} |x|^\alpha \left[\frac{N}{4} h(u) u - \frac{N+\alpha+2}{2} H(u) \right] dx \\
 &= \lim_{n \rightarrow \infty} \left\{ \left(\frac{\nu\gamma_q}{2} - \frac{\nu}{q}\right) \int_{\mathbb{R}^N} \left| \tau_n^{\frac{N}{2}}(v_n) \right|^q dx \right. \\
 &\quad + \int_{\mathbb{R}^N} |x|^\alpha \left[\frac{N}{4} h\left(\tau_n^{\frac{N}{2}}(v_n)\right) \tau_n^{\frac{N}{2}}(v_n) \right. \\
 &\quad \left. \left. - \frac{N+\alpha+2}{2} H\left(\tau_n^{\frac{N}{2}}(v_n)\right) \right] dx \right\} \\
 &= \lim_{n \rightarrow \infty} \left[E\left(\tau_n^{\frac{N}{2}}(v_n)\right) - \frac{1}{2} J\left(\tau_n^{\frac{N}{2}}(v_n)\right) \right] \\
 &= r(\kappa) \leq r(\kappa_0).
 \end{aligned}$$

Since $r(\kappa_0)$ is attainable, Lemma 10 implies that $\kappa_0 = \kappa$, $t_m = 1$, $u \in \mathfrak{R}(\kappa)$ and $E(u) = r(\kappa)$. The proof of the conclusion is complete. $\square$

Lemma 15. Suppose that Hypotheses 1–3 are satisfied. If $\bar{u} \in \mathfrak{R}(\kappa)$ attains the value $r(\kappa)$, that is, $E(\bar{u}) = r(\kappa)$, thus, $\bar{u}$ can be regarded as a critical point of the restricted functional $E|_{\mathfrak{N}(\kappa)}$.

Proof. First, we prove that $E \in C^1$. Let $E_1 = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 - \mu \frac{u^2}{|x|^2}) dx - \frac{\nu}{q} \int_{\mathbb{R}^N} |u|^q dx$, $E_2 = \int_{\mathbb{R}^N} |x|^\alpha H(u) dx$. It is easy to see that $E_1 \in C^1$. Therefore, it remains only to prove that $E_2 \in C^1$. For any $v \in H^1(\mathbb{R}^N)$, we have $\langle E'_2(u), v \rangle = \int_{\mathbb{R}^N} |x|^\alpha h(u) v dx$. We now prove that $\int_{\mathbb{R}^N} |x|^\alpha h(u) v dx$ defines a bounded linear functional on $H^1(\mathbb{R}^N)$. By Equation (20), we have

$$\left| \int_{\mathbb{R}^N} |x|^\alpha h(u) v dx \right| \leq \epsilon \int_{\mathbb{R}^N} |x|^\alpha |u|^{\bar{q}-1} |v| dx + \delta \int_{\mathbb{R}^N} |x|^\alpha |u|^{2^*(s)-1} |v| dx.$$

By Hölder inequality, we obtain

$$\begin{aligned}
 \int_{\mathbb{R}^N} |x|^\alpha |u|^{\bar{q}-1} |v| dx &= \int_{\mathbb{R}^N} \left(|x|^{\frac{\alpha(\bar{q}-1)}{\bar{q}}} |u|^{\bar{q}-1} \right) \left(|x|^{\frac{\alpha}{\bar{q}}} |v| \right) dx \\
 &\leq \left[\int_{\mathbb{R}^N} \left(|x|^{\frac{\alpha(\bar{q}-1)}{\bar{q}}} |u|^{\bar{q}-1} \right)^{\frac{\bar{q}}{\bar{q}-1}} dx \right]^{\frac{\bar{q}-1}{\bar{q}}} \left[\int_{\mathbb{R}^N} \left(|x|^{\frac{\alpha}{\bar{q}}} |v| \right)^{\bar{q}} dx \right]^{\frac{1}{\bar{q}}} \\
 &= \left(\int_{\mathbb{R}^N} |x|^\alpha |u|^{\bar{q}} dx \right)^{\frac{\bar{q}-1}{\bar{q}}} \left(\int_{\mathbb{R}^N} |x|^\alpha |v|^{\bar{q}} dx \right)^{\frac{1}{\bar{q}}} \\
 &\leq C_3 \|u\|_{H^1}^{\bar{q}-1} \|v\|_{H^1}.
 \end{aligned}$$

Similarly, we have $\int_{\mathbb{R}^N} |x|^\alpha |u|^{2^*(s)-1} |v| dx \leq C \|u\|_{H^1}^{2^*(s)-1} \|v\|_{H^1}$. Hence $\left| \int_{\mathbb{R}^N} |x|^\alpha h(u) v dx \right| \leq C (\|u\|_{H^1}^{\bar{q}-1} + \|u\|_{H^1}^{2^*(s)-1}) \|v\|_{H^1}$. Thus the mapping

$v \mapsto \int_{\mathbb{R}^N} |x|^\alpha h(u) v$ belongs to $(H^1(\mathbb{R}^N))^*$. Hence $\int_{\mathbb{R}^N} |x|^\alpha h(u) v dx$ defines a bounded linear functional. Next, we prove that the mapping $u \mapsto E'_2(u)$ is continuous. Let $u_n \rightarrow u$ in $H^1(\mathbb{R}^N)$. We aim to prove that

$$\|E'_2(u_n) - E'_2(u)\|_{(H^1)^*} = \sup_{\|v\|_{H^1} \leq 1} \left| \int_{\mathbb{R}^N} |x|^\alpha [h(u_n) - h(u)] v dx \right| \rightarrow 0, \quad n \rightarrow \infty.$$

By Hölder inequality,

$$\begin{aligned} & \int_{\mathbb{R}^N} |x|^\alpha |h(u_n) - h(u)| |v| dx \\ &= \int_{\mathbb{R}^N} |x|^{\alpha \frac{2^*(-\alpha)-1}{2^*(-\alpha)}} |h(u_n) - h(u)| \cdot |x|^{\frac{\alpha}{2^*(-\alpha)}} |v| dx \\ &\leq \left(\int_{\mathbb{R}^N} \left(|x|^{\alpha \frac{2^*(-\alpha)-1}{2^*(-\alpha)}} |h(u_n) - h(u)| \right)^{\frac{2^*(-\alpha)}{2^*(-\alpha)-1}} dx \right)^{\frac{2^*(-\alpha)-1}{2^*(-\alpha)}} \\ &\quad \times \left(\int_{\mathbb{R}^N} \left(|x|^{\frac{\alpha}{2^*(-\alpha)}} |v| \right)^{2^*(-\alpha)} dx \right)^{\frac{1}{2^*(-\alpha)}} \\ &= \left(\int_{\mathbb{R}^N} |x|^\alpha |h(u_n) - h(u)|^{\frac{2^*(-\alpha)}{2^*(-\alpha)-1}} dx \right)^{\frac{2^*(-\alpha)-1}{2^*(-\alpha)}} \\ &\quad \times \left(\int_{\mathbb{R}^N} |x|^\alpha |v|^{2^*(-\alpha)} dx \right)^{\frac{1}{2^*(-\alpha)}}. \end{aligned} \tag{32}$$

Notice that

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^\alpha |h(u_n) - h(u)|^{\frac{2^*(-\alpha)}{2^*(-\alpha)-1}} dx &= \int_{0 < |x| \leq R} |x|^\alpha |h(u_n) - h(u)|^{\frac{2^*(-\alpha)}{2^*(-\alpha)-1}} dx \\ &\quad + \int_{|x| \geq R} |x|^\alpha |h(u_n) - h(u)|^{\frac{2^*(-\alpha)}{2^*(-\alpha)-1}} dx. \end{aligned} \tag{33}$$

It follows from Lemma 4 and Equation (20) that

$$|h(u_n) - h(u)|^{\frac{2^*(-\alpha)}{2^*(-\alpha)-1}} \leq C_4 \left(\sum_{i=1}^2 |u_n|^{b_i} + \sum_{i=1}^2 |u|^{b_i} \right), \tag{34}$$

where $b_1 = \frac{2^*(-\alpha)(\bar{q}-1)}{2^*(-\alpha)-1}$, $b_2 = \frac{2^*(-\alpha)(2^*(s)-1)}{2^*(-\alpha)-1}$, obviously, $2 < b_1 < b_2 < 2^*$. Since $u_n \rightarrow u$ in $H^1(\mathbb{R}^N)$, from the embedding theorem, we know that there exists $N_1 > 0$ such that, whenever $n > N_1$, $\|u_n - u\|_{b_i} \leq 1$. Note that $|u_n| = |u_n - u + u| \leq |u_n - u| + |u|$, combining with Equation (34), we have, for any $\varepsilon > 0$, there exists a

sufficiently large R such that

$$\begin{aligned} & \int_{|x| \geq R} |x|^\alpha |h(u_n) - h(u)|^{\frac{2^*(-\alpha)}{2^*(-\alpha)-1}} dx \\ & \leq C_5 R^\alpha \int_{|x| \geq R} \left(\sum_{i=1}^2 |u_n|^{b_i} + \sum_{i=1}^2 |u|^{b_i} \right) dx \\ & \leq C_6 R^\alpha \left[\int_{|x| \geq R} \sum_{i=1}^2 |u_n - u|^{b_i} dx + 2 \int_{|x| \geq R} \sum_{i=1}^2 |u|^{b_i} dx \right] \\ & < \frac{\varepsilon}{2}, \end{aligned} \quad (35)$$

for any $n > N_1$.

On the other hand, since $b_2 < 2^*$ and $-4/N < \alpha < 0$, we can choose $r > 1$ and $r' > 1$ such that $\frac{1}{r} + \frac{1}{r'} = 1$, $b_2 r < 2^*$, $\alpha r' > -N$. By Hölder inequality, we have

$$\begin{aligned} & \int_{|x| \leq R} |x|^\alpha |h(u_n) - h(u)|^{\frac{2^*(-\alpha)}{2^*(-\alpha)-1}} dx \\ & \leq \left(\int_{|x| \leq R} |x|^{\alpha r'} dx \right)^{\frac{1}{r'}} \left(\int_{|x| \leq R} |h(u_n) - h(u)|^{\frac{2^*(-\alpha)}{2^*(-\alpha)-1} r} dx \right)^{\frac{1}{r}}. \end{aligned} \quad (36)$$

Let $b = \frac{2^*(-\alpha)r}{2^*(-\alpha)-1}$ and $p = (2^*(s) - 1)b$. Notice that $|h(u)| \leq \epsilon |u|^{\bar{q}-1} + \delta |u|^{2^*(s)-1} \leq a + C_7 |u|^{2^*(s)-1}$, where $a > 0$, $C_7 > 0$ is a constant. Since $1 \leq p = b_2 r < 2^*$, $a \in L^b(B_R(0))$, thus, by the properties of the Nemytskii map, the map $h : L^p(B_R(0)) \rightarrow L^b(B_R(0))$ is continuous. This implies that $\int_{B_R(0)} |h(u_n) - h(u)|^b dx \rightarrow 0$. Therefore, there exists a sufficiently large N_2 such that for any $n > N_2$

$$\int_{B_R(0)} |x|^\alpha |h(u_n) - h(u)|^{\frac{2^*(-\alpha)}{2^*(-\alpha)-1}} dx < \frac{\varepsilon}{2}. \quad (37)$$

Choosing $N = \max\{N_1, N_2\}$, when $n > N$, from Equations (35) and (37), we have

$$\int_{R^N} |x|^\alpha |h(u_n) - h(u)|^{\frac{2^*(-\alpha)}{2^*(-\alpha)-1}} dx < \varepsilon.$$

By Equation (32), we can obtain $\int_{R^N} |x|^\alpha |h(u_n) - h(u)| |v| dx \rightarrow 0$ ($n \rightarrow \infty$), thus, $\sup_{\|v\| \leq 1} \left| \int_{R^N} |x|^\alpha [h(u_n) - h(u)] v dx \right| \rightarrow 0$ ($n \rightarrow \infty$).

Suppose $\bar{u}$ fails to be a critical point for the functional $E|_{\aleph(\kappa)}$. There exist $\varepsilon > 0$, $\wp > 0$ such that

$$u \in \aleph(\kappa), \|u - \bar{u}\| \leq 3\varepsilon \text{ and } \|E'(u)\| \geq \wp.$$

Clearly,

$$\lim_{t \rightarrow 1} \|t^{N/2} \bar{u}_t - \bar{u}\| = 0. \quad (38)$$

This yields the existence of a constant $\varepsilon_1 \in (0, 1/4)$ satisfying

$$|t - 1| < \varepsilon_1 \text{ and } \|t^{N/2} \bar{u}_t - \bar{u}\| < \varepsilon. \quad (39)$$

By Lemma 7, we know that

$$E\left(t^{\frac{N}{2}}\bar{u}_t\right) = E(\bar{u}) - \tilde{h}(t, \bar{u}) = r(\kappa) - \tilde{h}(t, \bar{u}), \quad \forall t > 0. \quad (40)$$

According to Lemma 8, one can find $T_1 \in (0, 1)$ and $T_2 \in (1, \infty)$ such that

$$J\left(T_1^{\frac{N}{2}}\bar{u}_{T_1}\right) > 0, \quad J\left(T_2^{\frac{N}{2}}\bar{u}_{T_2}\right) < 0. \quad (41)$$

Set

$$\varrho := \min \left\{ \frac{\tilde{h}(T_1, \bar{u})}{4}, \frac{\tilde{h}(T_2, \bar{u})}{4}, 1, \frac{\varepsilon_{\varrho}}{8} \right\}, \quad S := B(\bar{u}, \varepsilon) \cap \mathfrak{N}(\kappa).$$

Applying Lemma 5.15 in Willem's study in 1996 [32], we obtain a continuous deformation

$$\beta : [0, 1] \times \mathfrak{N}(\kappa) \rightarrow \mathfrak{N}(\kappa)$$

with the following properties:

- (i) $\beta(1, u) = u$ if $E(u) < r(\kappa) - 2\varrho$ or $E(u) > r(\kappa) + 2\varrho$;
- (ii) $\beta(1, \mathfrak{N}(\kappa)) \cap B(\bar{u}, \varepsilon) \cap E^{r(\kappa)+\varrho} \subset E^{r(\kappa)-\varrho}$;
- (iii) $E(\beta(1, u)) \leq E(u)$, $\forall u \in \mathfrak{N}(\kappa)$.

By Lemma 8, we have $E\left(t^{\frac{N}{2}}\bar{u}_t\right) \leq E(\bar{u}) = r(\kappa)$, for $t > 0$. Combined with Equation (39) and property (ii), we obtain

$$E\left(\beta(1, t^{\frac{N}{2}}\bar{u}_t)\right) \leq r(\kappa) - \varrho, \quad \forall t > 0, |t - 1| < \varepsilon_1. \quad (42)$$

From Equation (40) and property (iii), we have

$$\begin{aligned} E\left(\beta(1, t^{\frac{N}{2}}\bar{u}_t)\right) &\leq E\left(t^{\frac{N}{2}}\bar{u}_t\right) \\ &= r(\kappa) - \tilde{h}(t, \bar{u}) \leq r(\kappa) - \varepsilon_2, \quad \forall t > 0, |t - 1| \geq \varepsilon_1, \end{aligned} \quad (43)$$

where $\varepsilon_2 = \min \{\tilde{h}(1 - \varepsilon_1, \bar{u}), \tilde{h}(1 + \varepsilon_1, \bar{u})\} > 0$. Combining Equations (42) and (43), we obtain $\max_{t \in [T_1, T_2]} E\left(\beta(1, t^{\frac{N}{2}}\bar{u}_t)\right) < r(\kappa)$. Let $\Phi(t) = J\left(\beta(1, t^{\frac{N}{2}}\bar{u}_t)\right)$, $t > 0$. From Equation (40) and property (i), we have $\beta(1, t^{\frac{N}{2}}\bar{u}_t) = \bar{u}_t$, for $t = T_1$ and $t = T_2$. Combined with Equation (41), we obtain

$$\Phi(T_1) = J\left(T_1^{\frac{N}{2}}\bar{u}_{T_1}\right) > 0, \quad \Phi(T_2) = J\left(T_2^{\frac{N}{2}}\bar{u}_{T_2}\right) < 0.$$

Since $\Phi(t)$ is continuous on $(0, +\infty)$, there exists $\tilde{t} \in [T_1, T_2]$ such that $\Phi(\tilde{t}) = 0$. Hence $\mathfrak{R}(\kappa) \cap \beta(1, \tilde{t}^{\frac{N}{2}}\bar{u}_{\tilde{t}}) \neq \emptyset$, which contradicts the definition of $r(\kappa)$ as the minimal value. The proof is complete. $\square$

Proof of Theorem 1. From Lemmas 14 and 15, there exists $u_\kappa \in \mathfrak{R}(\kappa)$ such that $E(u_\kappa) = r(\kappa)$ and $E'|_{\mathfrak{N}(\kappa)}(u_\kappa) = 0$. An application of the Lagrange multiplier principle yields a Lagrange multiplier $\lambda_\kappa \in \mathbb{R}$ such that the pair $(u_\kappa, \lambda_\kappa)$ solves the equation given below

$$-\Delta u - \mu \frac{u}{|x|^2} + \lambda u = |x|^\alpha h(u) + \nu |u|^{q-2} u.$$

It follows that

$$\begin{aligned} & \int_{\mathbb{R}^N} \left(|\nabla u_\kappa|^2 - \mu \frac{|u_\kappa|^2}{|x|^2} \right) dx + \lambda_\kappa \int_{\mathbb{R}^N} |u_\kappa|^2 dx \\ & - \int_{\mathbb{R}^N} |x|^\alpha h(u_\kappa) u_\kappa dx - \nu \int_{\mathbb{R}^N} |u_\kappa|^q dx = 0. \end{aligned} \quad (44)$$

Since $J(u_\kappa) = 0$, $1 - \gamma_q > 0$ and $s + \alpha > 0$, combining Equation (44) and the inequality $h(\tau)\tau \leq 2^*(s)H(\tau)$ from Hypothesis 2, we obtain

$$\begin{aligned} \lambda_\kappa &= \frac{1}{\kappa} \left[\nu \int_{\mathbb{R}^N} |u_\kappa|^q dx + \int_{\mathbb{R}^N} |x|^\alpha h(u_\kappa) u_\kappa dx - \int_{\mathbb{R}^N} \left(|\nabla u_\kappa|^2 - \mu \frac{|u_\kappa|^2}{|x|^2} \right) dx \right] \\ &= \frac{1}{\kappa} \left[\nu(1 - \gamma_q) \int_{\mathbb{R}^N} |u_\kappa|^q dx + \frac{2 - N}{2} \int_{\mathbb{R}^N} |x|^\alpha h(u_\kappa) u_\kappa dx \right. \\ &\quad \left. + (N + \alpha) \int_{\mathbb{R}^N} |x|^\alpha H(u_\kappa) dx \right] \\ &\geq \frac{1}{\kappa} \left[\nu(1 - \gamma_q) \int_{\mathbb{R}^N} |u_\kappa|^q dx + (s + \alpha) \int_{\mathbb{R}^N} |x|^\alpha H(u_\kappa) dx \right] > 0. \end{aligned}$$

Therefore, $(u_\kappa, \lambda_\kappa) \in H_r^1(\mathbb{R}^N) \times \mathbb{R}^+$ is a normalized solution of the equation. The proof of the theorem is now complete. $\square$

4. Conclusion

Most existing results are concerned only with elliptic equations whose nonautonomous terms are bounded. In contrast, in this paper we focus on elliptic equations involving both a Hardy potential and a singular unbounded nonautonomous term, and we investigate the existence of normalized solutions. Since the singular nonautonomous term is unbounded near the origin, standard nonlinear estimates cannot be applied directly. Therefore, our analysis relies on the weighted interpolation inequality established in the study by Wang and Yang [27]. Moreover, the introduction of the singular weight changes the scaling exponent of the nonlinear energy integral, making the competition among different powers of the L^2 norm of the gradient more intricate when deriving scaling estimates for the energy functional. To cope with analytical obstacles produced by multiple singularities arising from coexisting Hardy potential and singular nonautonomous term, we combine the Hardy inequality with Lemma 3 to obtain the key estimates, thereby effectively overcoming the aforementioned obstacles.

In Chen and Tang's study [16], the nonautonomous term $g(x)$ is continuous and bounded, and converges at infinity to a nonzero constant g_∞ . When the minimizing sequence $v_n \rightharpoonup 0$ and vanishing does not occur, the concentration-compactness principle yields a sequence of translations y_n such that $|y_n| \rightarrow \infty$. Let $\hat{v}_n(x) = v_n(y_n + x)$. Since $g(x) \rightarrow g_\infty$ as $|x| \rightarrow \infty$, we have $g(x + y_n) \rightarrow g_\infty$. Together with the fact that the remainder integral tends to 0, it follows that the energy of the original problem is asymptotic to that of the limiting problem. In contrast, the Hardy potential and the nonautonomous term considered in the present paper are singular, therefore the

method introduced above cannot be directly utilized. In order to resolve this challenge, we employ Schwarz symmetric rearrangement to restrict the problem to the space of radial functions, and then we use the compactness of the radial embedding to rule out the possibility that the weak limit is zero.

Furthermore, future studies may explore the existence of normalized solutions for p -Laplacian equations with Hardy potentials. Since the p -Laplacian operator is nonlinear and, for $p \neq 2$, the natural working space $W^{1,p}(\mathbb{R}^N)$ is no longer a Hilbert space, many analytical tools relying on the Hilbert space framework available in the case $p = 2$ are no longer directly applicable. This creates additional difficulties in analyzing such problems. On the other hand, the present work focuses only on the existence of normalized solutions in the L^2 -supercritical regime. By imposing suitable growth and structural assumptions on the nonlinearity, it would be interesting to investigate further the existence and related properties of normalized solutions in the L^2 -subcritical regime. Moreover, the above problems could be extended to systems of elliptic equations, where the existence and variational structure of normalized solutions deserve further investigation. Meanwhile, with the development of intelligent solution methods, we may also attempt to perform numerical verification for the existence of normalized solutions.

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