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Review

Survey of research over the past fifty years in stochastic mathematics (S.M.) exploring interdisciplinary frontiers

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Abstract: This article provides an overview of the explorations over the past fifty years or more, as indicated in the title. The main new areas explored are the first three terms below. The fourth item is a related exploration result. The interaction of stochastic mathematics (S.M.) and statistical physics opens up new frontiers. Interaction of S.M. with other branches of mathematics: a nearly new theory for various stability and its speed estimation. Interaction of S.M. and economy: the first precise theory of economic optimization. Optimization method in deep mathematical research. It is hoped that these experiences would be helpful to future generations.

Keywords: stochastic mathematics; statistical physics; Markov processes; spectral gap/eigenvalues; economic optimization

1. Opening up new frontiers by interacting with statistical physics

After graduating from university in 1972, I was assigned to work at Guiyang Normal College. At that time, I was mainly engaged in promoting the optimization method. I soon realized that my theoretical knowledge was seriously lacking, making it difficult for me to delve into theoretical exploration. Therefore, I turned to my alma mater (Beijing Normal University), asking Professor Yan Shijian for help. Despite being in a difficult situation, Prof. Yan scoured several second-hand bookstores in Beijing and purchased over ten books for me: Classics in probability and statistics, including M. Loève's masterpiece [1]. He recommended that I carefully read this book. He also sent me the manuscript of his book co-authored with Wang Junxiang and Liu Xiufang, which was before the official publication of Fundamentals of Probability Theory [2]. Thanks to the latter's assistance, I was able to complete the former within two and a half years. As I entered the phase of specialized research, I happened to come across the famous paper by Professor Hou Zhenting [3]. In 1975, through the introduction of Prof. Yue Min-Yi from the Chinese Academy of Sciences, Prof. Hou accepted me as his disciple, and I embarked on the path of researching Markov chains. In 1976, I finally had the opportunity to seek his personal guidance in Changsha. Prof. Hou, let's delve into the classics of Kai-Lai Chung [4] together, not in the classroom or at a desk, but on the hillside, over the stones, reading the book page by page. Between 1976 and 1978, fortunately, Professor Qian Min from Peking University, leveraging his early background in physics, suggested researching equilibrium statistical physics models, which correspond to the reversible Markov processes. He studied reversible diffusion processes together with Qian Minping and Gong Guanglu; I studied reversible Markov chains with Professor Hou. What I find unforgettable is that in 1979, when I was still a graduate student, the three members of the Qian's family, after consultation with my supervisor, Prof. Yan, they decided to bring Professor M.L. Silverstein from the University of Washington, whom they invited, to Beijing Normal University, allow me to introduce to Professor Silverstein my research work on reversible Markov chains with Professor Hou. In that year, we co-published a research monograph [5] published by Hunan Science and Technology Press; see my homepage for details. The sixth chapter in the book is the paper co-authored by Professor Hou and me, titled "Markov Processes and Field Theory". In brief, we focus on the research of reversible Markov processes in terms of the mathematical tools for analyzing classical conservative field theory. This work later gained many applications, and now, we know that it can also be extended to Hermitian matrix theory and applied to quantum mechanics. As we all know, the core of classical conservative field theory is that the work done by the field along any closed path is zero (equivalently, we have the so-called path independence). In that year (1979), it was difficult for us to access foreign literature; hence only three years later, when I visited the United States, I learned: for finite Markov chains, the criterion we found "along any closed path, the work done by the chains equals zero" is equivalent to the "cycle condition" proposed by Kolmogorov (1936). Actually, there is no regret here, because in that year, we actually obtained a much simpler criterion. Thanks to the graph structure, it is not difficult to imagine: we need only to check the minimum closed loop; we do not need to check all closed paths. Therefore, in practice, only the two smallest closed loops, namely triangles and quadrilaterals, are commonly used.

The tools of field theory are not only applicable to Markov chains with countable states, but also to a class of infinitely dimensional particle systems, namely, equilibrium statistical physics models. Examples of the application of the "triangle condition" and "quadrangle condition" can be found in Section 11.1 of the author's monograph [6]. In 1977, when Kai-Lai Chung returned to China to give lectures, he mentioned that the Russian R.L. Dobrushin's school had created a new research field, the interaction between stochastic mathematics and equilibrium statistical physics, called "random field theory". This was a significant event in the history of the development of mathematics, marking a return to nature from the axiomatic movement. Therefore, when I was a graduate student in 1978, based on my previous foundation, Prof. Yan suggested that I pursue research in this direction. We spent a semester discussing C. Preston's book [7] and translated it into Chinese for publication in China. This was about static systems without time. Soon after, we came across T.M. Liggett's review paper [8], which was about dynamic systems with time parameters and was more relevant to our research background. So we focused on the study of equilibrium statistical physics theory as a class of infinite-dimensional interacting particle systems.

Also in 1977, Ilya Prigogine, a leading figure of the Brussels School, was awarded the Nobel Prize in Chemistry, for the first time in history, directly recognizing pioneering work in the field of non-equilibrium statistical physics. This and the equilibrium state mentioned in the previous part, can be regarded as two distinct worlds of statistical physics, but the former one has significantly greater difficulty than the latter. The author focuses on typical models, especially Schlögl model, and studies the construction, uniqueness, and stationary distribution of the infinite-dimensional reaction-diffusion process introduced by the author in 1985 [9]. The works are collected in the monograph [6], Part IV (102 pages), specifically addressing non-equilibrium systems, dealing with typical 15 models. Around 1985, following a visit by F. Spitzer from the United States, R. Durrett from the same university as Spitzer immediately requested a visit to China. Due to financial difficulties at my university back then, we arranged for him to visit Anhui Normal University, where domestic peers gathered for academic exchanges. On the night he arrived, I wrote him a three-page note, listing the models we were concerned about at that time. Nine years later, when I met him in the United States, he said he had been keeping those three pages of the note.

A. Perrut, R. Durrett, and six other people, in their articles, referred 4 times to the construction of the aforementioned process as "Chen's construction", Professor D.A. Dawson, former director of the Fields Institute in Canada, and others directly used my results and methods for the corresponding mean-field model in their research. A group of scholars led by R. Durrett in the United States have studied various simplified models of reaction-diffusion processes. A large number of articles constituted the content of his ten lectures at the French Probability Theory Summer School [10]. He began his lectures with the explicitly stated comment that "for a closely related model of Schlögl (1972) (See Part IV of Chen (1992) for a survey)." I deeply admire and appreciate his integrity and demeanor. After five years of exploration, we finally chose typical models of non-equilibrium statistical physics such as the Schlögl model. As the primary focus [see H. Haken (1977–1983), Synergetics for the physical background], we strive to explore the non-equilibrium mathematical foundation of statistical physics. R. Durrett and T.M. Liggett, as well as scholars from Italy, Germany, and other countries, have also contributed and participated in research in this direction. Mathematically speaking, these models are very difficult. Despite two decades of efforts, important issues remain unresolved.

From a local perspective, an infinite-dimensional reaction-diffusion process becomes a finite-dimensional jump process, and thus, the latter serves as the foundation for the former. For example, considering the uniqueness problem for the processes, it is non-trivial even in the two-dimensional case. However, here we need to use the uniqueness in any finite dimension, otherwise it cannot be considered as an infinite-dimensional model. Regarding this difficult problem, it took the author five years to find a solution, and another three years to complete its general form.

In that period, R.L. Dobrushin's school in Russia created a probability metric—Wasserstein distance, while the American school commonly used coupling method, it was only after many years that the author realized that the two are essentially sisters. Later, I repeatedly used "coupling and probability distance" as the title of talks, it gave rise to the "Trilogy of Couplings": "Markov Coupling, Optimal Markov Coupling, and Optimization of Distance with respect to Optimal Coupling" The "optimal Markov coupling" is classified based on a certain type of distance. Based on the aforementioned uniqueness criterion of the jump process, we obtain the output: both marginal processes are unique if and only if a certain (or equivalently, any) coupled jump process is unique. It should be noted that for given two marginal processes, the number of coupling processes of these margins may have an uncountable infinity. This step is also the primary tool for the next step of research, which focuses on "convergence rate estimation". In essence, as the cornerstone of research on reaction-diffusion processes, we have deepened and expanded a general theory of jump processes.

Due to our rapid entry into the international forefront, we quickly attracted the attention of the leaders of the two major schools of thought in Russia and the United States at that time. Russia: R.L. Dobrushin has proposed and applied for a Sino-Russian cooperative research project; the US: R. Durrett has proposed a Sino-US cooperative project, as detailed in my homepage features a "popular science work" [1] that includes touching materials.

Below are two overseas compliments regarding this achievement.

- S. Rachev (Zbl. Math.753 (1993), p. 301): "The author is an outstanding Chinese probabilist in probability and stochastic processes creating the Chinese school of Markov processes".
- T.M. Liggett, (Fellow of the American Academy of Sciences) (Math. Reviews 94a, (1994), p.439): "the school led by the author in Beijing has worked on the construction, and ergodic theory of the class of interacting particle systems, known as reaction diffusion processes."

2. Interaction with other branches of mathematics: A brand-new theory for various stability and its speed estimation

The core topic in statistical physics is phase transitions, the research on which requires infinite-dimensional mathematics since there is no phase transition in a finite-dimensional situation. Thus, a colorful landscape appeared. It reminds one of Leo Tolstoy's famous quote: "Happy families are all alike; every unhappy family is unhappy in its own way"

- The research methods for "unfortunate families" include: Peierls method, Sinai theory, Reflection Positivity theory, Durrett's coexistence phenomenon, and so on.
- For the "happy family" system, it should be in a stable (ergodic) region. We seek an exponential ergodic rate, equivalently, the spectral gap of the corresponding operator (the first non-trivial eigenvalue), as used for characterization. The more precise the speed is, the closer it is to the boundary of phase transition. The stability speed belongs to a common field in mathematical research. We choose this path, gradually forming new theories.

Three classical ergodicities $\|f\|_{\text{Var}} := \text{Var}(f) = \pi(f^2) - \pi(f)^2$

$$\textbf{Usual ergodicity}: \ \lim_{t\to\infty} \big\|P_t(x,\cdot) - \pi\big\|_{\text{Var}} = 0,$$

$$\textbf{Exponential ergodicity}: \ \exists \alpha > 0 \text{ such that } \big\|P_t(x,\cdot) - \pi\big\|_{\text{Var}} \leqslant C(x) e^{-\alpha t},$$

$$\textbf{Strong ergodicity}: \ \lim_{t\to\infty} \sup_x \big\|P_t(x,\cdot) - \pi\big\|_{\text{Var}} = 0$$

$$\Longleftrightarrow \exists \beta > 0 \text{ such that } \lim_{t\to\infty} e^{\beta t} \sup_x \big\|P_t(x,\cdot) - \pi\big\|_{\text{Var}} = 0,$$

where $P_t(x, dy)$ is the transition probability of the Markov process, π is its stationary distribution, and $\|\cdot\|_{\mathrm{Var}}$ is Total Variation Norm.

Theorem 1. *Let $(E,\mathscr{E})$ be a countably generated measurable space (i.e., $\mathscr{E}$ is countably generated). Then, for the reversible Markov process on it, as long as its transition probability regarding the reversible measure π has density, the following implication diagram holds.*

Figure 1 is complete, meaning that there are no new implication relationships between any two properties in the graph. In addition, only three classical ergodicities are known earlier in the figure, as can be seen, this diagram significantly enriches the ergodic theory [11].

$$
\begin{array}{ccc}
 & \textsf{Nash inequality} & \\
\swarrow\!\!\!\swarrow & & \searrow\!\!\!\searrow \\
\textsf{Logarithmic Sobolev inequality} & & L^1\textsf{-exponential convergence} \\
\Downarrow & & \| \\
\textsf{Exponential convergence in entropy} & & \pi\textsf{-a.s. Strong ergodicity} \\
\Downarrow & & \Downarrow \\
\textsf{Poincaré inequality} & \Longleftrightarrow & \pi\textsf{-a.s. Exp. ergodicity} \\
 & \Downarrow & \\
 & L^2\textsf{-algebraic convergence} & \\
 & \Downarrow & \\
 & \textsf{Ordinary ergodicity} &
\end{array}
$$

Figure 1. Diagram of Various Ergodicity Implication Relationships.

2.1. Various inequalities

The skip process may be described by the q-pair $(q(x),\ q(x, A))$ and, in the reversible case, with respect to its reversible measure π, it can be defined on the $L^2(\pi)$ space.

Dirichlet form

$$
\begin{aligned}
D(f,f) &= \frac{1}{2}\int_E \pi(dx)\int_E q(x,dy)(f(y)-f(x))^2,\\
f\in\mathscr{D}(D) &:= \{f\in L^2(\pi) : D(f,f)<\infty\} \subset L^2(\pi).
\end{aligned}
$$

Certainly, for every reversible Markov process, we have a Dirichlet form. Then, we can define the three inequalities on the left side of the diagram above:

Poincaré inequality : $\mathrm{Var}(f) \leqslant CD(f), \quad f \in L^2(\pi),$

Logarithmic Sobolev inequality : $\displaystyle\int f^2 \log \frac{f^2}{\|f\|_2^2} d\pi \leqslant CD(f), \quad f \in L^2(\pi),$

Nash inequality : $\mathrm{Var}(f) \leqslant CD(f)^{1/p}\|f\|_1^{2/p^*}, \qquad f \in L^2(\pi),$

Here, $1/p^* + 1/p = 1.$

2.2. Corresponding stability of various inequalities

$$\textbf{Poincaré inequality} \Longleftrightarrow L^2\text{-Exp-convergence}$$
$$\mathrm{Var}(P_t f) \leqslant \mathrm{Var}(f)\exp[-2\lambda_1 t],$$
$$\textbf{Log Sobolev inequality} \Longrightarrow \text{Exp-convergence in Entropy :}$$
$$\mathrm{Ent}(P_t f) \leqslant \mathrm{Ent}(f)\exp[-2\sigma t],$$
$$\text{where } \mathrm{Ent}(f) = \pi(f\log f) - \pi(f)\log\|f\|_1.$$
$$\textbf{Nash inequality} \Longleftrightarrow \mathrm{Var}(\mathbf{P_t f}) \leqslant \mathbf{C}\|\mathbf{f}\|_{\mathbf{1}}^{\mathbf{2}}/\mathbf{t}^{\mathbf{p^*-1}}.$$

The reason one needs different convergences is as stated by Kolmogorov: "every approximation or convergence there must be a most suitable metric". As we all know, convergence problems are common research topics in most branches of mathematics; however, research on its speed is quite limited, which stems from the difficulty of the problem, as quantitative research is much more challenging than qualitative research. Therefore, a mathematical theory that only has qualitative results of stability without an estimate of the rate of stability is not sufficiently mature, whether viewed from a theoretical or an applied perspective. Now consider a one-dimensional diffusion process on the interval $[0,\infty)$ with the left endpoint 0 as a reflecting boundary, i.e., $f'(0)=0$. The operator is

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}.$$

Definition
$$C(x) = \int_0^x b/a, \qquad \mu[x,y] = \int_x^y e^C/a$$

We often omit writing the Lebesgue measure dx. So, we have 11 explicit criteria. The rows 4, 6, and 7 in the property column of the **Table 1**, each contain two entries, with the same criteria. This indicates that there are two equivalent properties. Of course, the proof of this result is not simple. For example, for the two properties in the fourth line, we have introduced the single integral operator I and the double integral operator II:

$$I(f)(x) = \frac{e^{-C(x)}}{f'(x)}\int_x^D \left[fe^C/a\right](u), \qquad f\in\mathscr{F},$$
$$II(f)(x) = \frac{1}{f(x)}\int_0^x e^{-C(y)}\int_y^D \left[fe^C/a\right](u), \quad f\in\mathscr{F}'.$$

Among which

$$\mathscr{F} = \left\{f\in C[0,D]\cap C^1(0,D) : f(0)=0,\ f'|_{(0,D)}>0\right\},$$
$$\mathscr{F}' = \left\{f\in C[0,D] : f(0)=0, f|_{(0,D)}>0\right\}.$$

Table 1. Eleven criteria for one-dimensional diffusions.

Property	Criterion
Uniqueness	$\int_0^\infty \mu[0,x]e^{-C(x)} = \infty \quad (*)$

Table 1. *Cont.*

Property	Criterion
Recurrence	$\int_0^\infty e^{-C(x)} = \infty$
Ergodicity	$(*)$ & $\mu[0,\infty) < \infty$
Exponential ergodicity L^2-exp. convergence	$(*)$ & $\sup_{x>0} \mu[x,\infty) \int_0^x e^{-C} < \infty$
Discrete spectrum	$(*)$ & $\lim_{x\to\infty} \mu[x,\infty) \int_0^x e^{-C} = 0$
Log. Sobolev inequality Exp. convergence in Entropy	$(*)$ & $\sup_{x>0} \mu[x,\infty)\log[\mu[x,\infty)^{-1}] \int_0^x e^{-C} < \infty$
Strong ergodicity L^1-exp. convergence	$(*)$ & $\int_0^\infty \mu[x,\infty) e^{-C(x)} < \infty$
Nash inequality	$(*)$ & $\sup_{x>0} \mu[x,\infty)^{(\nu-2)/\nu} \int_0^x e^{-C} < \infty$

Note: Here (*) means that the uniqueness condition in the first line is required in each of the later lines.

So far, we have only dealt with the one-dimensional case. For the high-dimensional case, Wang Feng-Yu and I have published a paper [12]. I remember when we submitted this paper, the reviewer criticized us for not carefully reviewing the literature, as if this important issue was not taken seriously. Previous researchers may have achieved a lot. At that time, I could only find two simple examples from the literature available to me. There was no systematic survey. So Wang and I went to the college archive and brought over the "Journal of Mathematical Reviews", we reviewed all the papers related to eigenvalue estimation. It was found that all the articles were limited to Dirichlet boundaries. We have not seen any articles dealing with ergodic cases. We informed the journal of this fact, and as a result, this article was published. The fundamental reason lies in the fact that the only mathematical tool available at that time was the maximum principle (which was later proven to be equivalent to the positivity of the Dirichlet eigenvalues) [13]. However, in the ergodic case, the hypersurface where the eigen-function being zero is certainly within the inner region, it cannot be explicitly characterized! Therefore, the maximum principle is basically not applicable. By the way, we mention that we also use the splitting technique, that is, using the flowing zero-surface serves as the boundary, splitting the region into two parts, where the maximum principle is applied separately. With Wang together, we used this to improve the famous Cheeger inequality in geometry. However, the latter is not an explicit estimate, and the result is relatively coarse. Our genuine breakthrough lies in replacing the maximum principle with probabilistic methods. The idea is to elevate the known process from a single level to two: consider two stochastic processes starting from two different points (the essential process remains unchanged, only the starting points change). One can be easily imagined, for given arbitrarily any two starting points, if the two processes can always converge, then the original process should be ergodic. But be careful, if it's not awesome, then everyone will go their own way, and it will be hard to meet again. So intervention is needed. One approach is as follows: construct a vertical hyperplane at the midpoint between the two starting points. When the process on the left side moves, then the process on the right is commanded to follow the reflection trajectory of the process on the left with respect to the hyperplane. In this way, the two processes are easy to meet. We refer to the process on the right side constructed in this way as the reflection of the process on the left side. Let's combine these two processes into the joint process on a product space; we only care about when its two components (referred to as marginal processes) meet. The distance between the two margins is a stochastic process on the half-space $[0, \infty)$. We are interested in when the distance process hits zero. This goes back to the one-dimensional case we have treated before.

Based on the above ideas, Chen and Wang (1997) [14] obtained the following result.

Theorem 2. *For a compact Riemannian manifold, let d, D, and K denote its dimension, diameter, and lower bound of the Ricci curvature of the manifold, respectively, then there is a formula:*

$$\lambda_1 \geqslant 4 \sup_{f \in \mathscr{F}} \inf_{r \in (0,D)} \frac{f(r)}{\int_0^r C(s)^{-1} ds \int_s^D (fC)(u) du}.$$

*Here, only two notations are used: $C(r) = \left(\cosh\left[\frac{r}{2}\sqrt{\frac{-K}{d-1}}\right]\right)^{d-1}$, and $\mathscr{F}$ is the set of all positive continuous functions on the interval $[0, D]$. This formula not only unifies but also improves all ten famous estimates obtained by geometers (including A. Lichnerowicz, Shiing-Tung Yau, and others) over the past four decades, as shown below in **Figure 2**.*

Author(s)	Lower bound	
A. Lichnerowicz(1958)	$\boldsymbol{\frac{d}{d-1}K, \quad K \geqslant 0}$	(1.1)
P.H. Bérard, G. Besson, & S. Gallot(1985)	$\boldsymbol{d\left\{\frac{\int_0^{\pi/2}\cos^{d-1}t\,dt}{\int_0^{D/2}\cos^{d-1}t\,dt}\right\}^{2/d}, \quad K = d-1 > 0}$	(1.2)
P. Li & S.T. Yau(1980)	$\frac{\pi^2}{2\,D^2}, \quad K \geqslant 0$	(1.3)
J.Q. Zhong & H.C. Yang(1984)	$\boldsymbol{\frac{\pi^2}{D^2}, \quad K \geqslant 0}$	(1.4)
D.G. Yang(1999)	$\boldsymbol{\frac{\pi^2}{D^2} + \frac{K}{4}, \quad K \geqslant 0}$	(1.5)
P. Li & S.T. Yau (1980)	$\frac{1}{D^2(d-1)\exp\left[1+\sqrt{1+16\alpha^2}\right]}, \quad K \leqslant 0$	(1.6)
K.R. Cai(1991)	$\boldsymbol{\frac{\pi^2}{D^2} + K, \quad K \leqslant 0}$	(1.7)
D. Zhao(1999)	$\boldsymbol{\frac{\pi^2}{D^2} + 0.52K, \quad K \leqslant 0.}$	(1.8)
H.C. Yang(1990) & F. Jia(1991)	$\boldsymbol{\frac{\pi^2}{D^2}e^{-\alpha}, \quad \textbf{if } d \geqslant 5, \quad K \leqslant 0}$	(1.9)
H.C. Yang(1990) & F. Jia(1991)	$\frac{\pi^2}{2\,D^2}e^{-\alpha'}, \quad \text{if } 2 \leqslant d \leqslant 4, \quad K \leqslant 0$	(1.10)

Figure 2. Ten Famous Estimates (Seven of Which with Bold Letters Are Optimal).

Corresponding to the 11 criteria for diffusion processes, we have almost parallel 10 explicit criteria for birth and death processes, with only one exception. The LogSobolev inequality is no longer equivalent to, but stronger than, the exponential convergence with respect to the relative entropy. Unfortunately, we have not yet seen an explicit discriminant criterion for the latter. After 20 years of exploration and a set of theories, we return to the starting point: study the phenomenon of phase transition and review the effectiveness of the developed tools. We investigate Euclidean quantum field model on lattice: φ^4 model. The lattice is $\mathbb{Z}^d$. At each lattice point, $i \in \mathbb{Z}^d$ above, the potential function is 4 order:

$$u(x_i) = x_i^4 - \beta x_i^2, \qquad x_i \in \mathbb{R},$$

where β indicating anti-temperature. The interaction of the system is adjacent:

$$H(x) = -J \sum_{\langle ij \rangle} x_i x_j, \qquad J \geqslant 0, \ \ i, j \in \mathbb{Z}^d,$$

where $\langle ij \rangle$ representing the adjacent edge in $\mathbb{Z}^d$, $x_i, x_j \in \mathbb{R}$. For fixed boundaries ω and finite box, $\Lambda \subset \mathbb{Z}^d$, the local operators of the system can be written as

$$L_\Lambda^\omega = \sum_{i \in \Lambda} \left[\partial_{ii} - \partial_i (u + H_\Lambda^\omega) \partial_i \right].$$

The goal is to find the spectral gap of this operator $\lambda_1^\beta(\lambda, \omega)$ and best Logarithm Sobolev constant $\sigma^\beta(\Lambda, \omega)$ about their uniform estimation with respect to the boundary conditions ω and finite boces Λ.

2.3. Euclidean lattice quantum field φ^4 model

Theorem 3 (Chen 2008 [15])**.** *We have*

$$\inf_{\Lambda \Subset \mathbb{Z}^d} \inf_{\omega \in \mathbb{R}^{\mathbb{Z}^d}} \lambda_1^\beta(\Lambda, \omega) \approx \inf_{\Lambda \Subset \mathbb{Z}^d} \inf_{\omega \in \mathbb{R}^{\mathbb{Z}^d}} \sigma^\beta(\Lambda, \omega)$$
$$\approx \exp\left[-\beta^2/4 - c \log \beta \right] - 4dJ.$$

*Among them, $c := c(\beta) \in [1, 2]$. Here $f(\beta) \approx g(\beta)$ means that when $\beta \to \infty$, $f(\beta)$ has the same convergence principal order as $g(\beta)$. The shaded area in the image on **Figure 3** is the exponentially ergodic area, and the curve slightly outside the upper left corner is the non-ergodic area. This model will undergo a phase transition because, as shown in the subsequent graph of u in **Figure 4**, the potential function has a double well, and when in one pit, it is difficult to cross over to another pit in the low-temperature region. This is a phase separation. **Figure 5** shows the comparison of the variance of the random variable.*

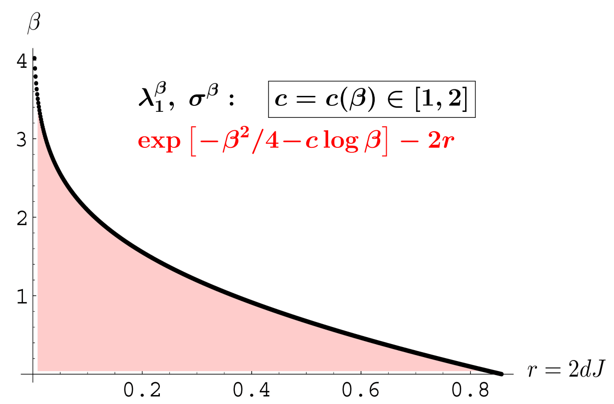
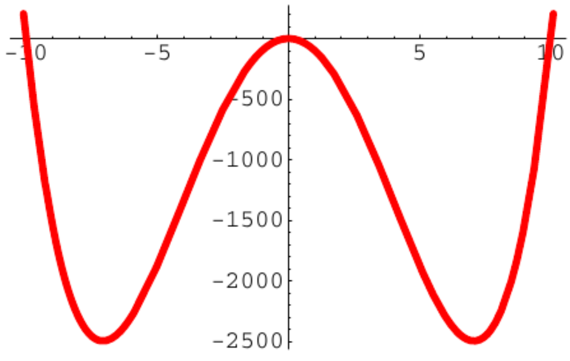


Figure 3. Infinite-dimensional phase transition.



$u(x) = x^4 - \beta x^2, \;\; \beta = 100$

Figure 4. Infinite-dimensional phase transition where the potential function $\boldsymbol{u}$ has double wells.

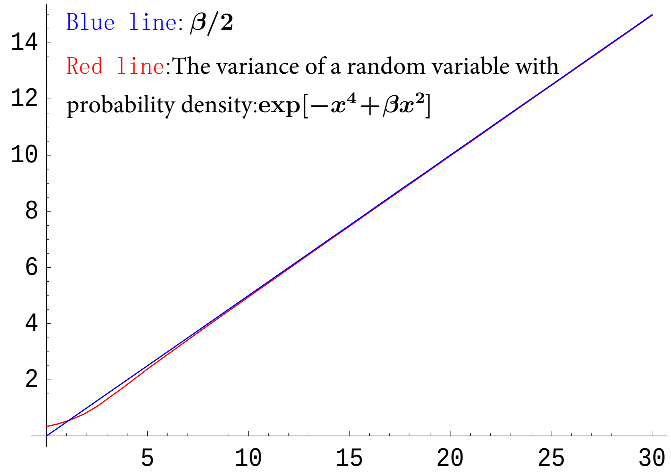


Figure 5. Asymptotic comparison between Var(X) and $\beta/2$ for probability density exp$[-x^4 + \beta x^2]$.

The encouragement from overseas colleagues regarding the second achievement:

> M. Znojil (2003), Zbl. Math., 01782097.
> "The work is a continuation of its seven (all self-) references...but offers very nice results (their essence being well characterized by the title, and they look 'final'. Amazingly enough, their explicit bounds are complete (i.e., both-sided)! Three illustrative examples demonstrate their power in applications."

Book reviews on the second achievement and monograph [1]:

- L. Miclo's review (Math Review, MR2091955, 2005): "these fields have been very active recently, in particular due to the contributions of the Beijing school led by Mr. Chen."
- Nine specialists, including D.G. Aldous who is a member of the Royal Society, have repeatedly listed the monograph [1] as a "standard reference."
- R. Durrett (Fellow of the American Academy of Sciences) (SIAM Review, Book Reviews 48:1 (2006), p.164): "Inequalities for the spectral gap and their

applications are a huge topic. Chen's 228-page book takes the reader on a tour of some of this material. The emphasis is on topics that Chen has contributed to (and I think this is appropriate), but this includes a wide variety of interesting topics."

"Chen has carried out an active research program and traveled extensively (as indicated by the long list in the acknowledgments).

Chen is an experienced researcher who has won prizes in China and spoken at the International Congress of Math, and he is an extensive writer of lecture notes."

For more book reviews, please refer to the 'Book Reviews' section on the author's homepage.

2.4. National innovation research team

Before 1978, I was single-handedly for several decades without any communication environment or opportunities. I went to graduate school in 1978. At the beginning, with the formation of a small collective, a dream arose: hoping that in the future we could have a good team so that the overseas know that there is also a small team in China doing well. Here we list some informations afterwise.

- In 2001, the National Natural Science Foundation of China approved the probability theory innovation research team at Beijing Normal University, which is the first team in mathematics in China.
- During the ten years from 2000 to 2009, the National Natural Science Foundation of China (NSFC) supported a total of 225 teams, with 22 groups receiving (the most) three rounds of funding. We were one of them.
- In 2009 we are also one of the teams selected for international evaluation.
- Professor Chen Yi-Yu, the director of the National Natural Science Foundation of China, wrote in an article published in Science Times on September 2, 2010, that: Probability Theory Research Group at Beijing Normal University led by Professor Chen Mufa (All four groups) are outstanding among the supported groups.
- Our group won the Capital May Day Labor Award and the National May Day Labor Award in 2006.

Thus, our original dream has been realized in 23 years.

3. Interaction with economy: The first precise theory of economic optimization

As an interaction of stochastic mathematics and economics, we have published in Chinese two books:

- Mathematical Theory of Large Scale Optimization in Planned Economy (New Edition), written by Hua Luo-Geng, edited with additional corrections and remarks by Chen Mu-Fa and Shi Hao-Kun, published by Beijing Normal University Press in 2025.
- Hua Luo-Geng's New Theory and Empirical Study on Economic Optimization, by Chen Mu-Fa, Xie Ying-Chao, Chen Bin, Zhou Qin, Yang Ting, Beijing Normal University Press, 2025.

More materials, including theoretical reviews written by Chen Bin, Xie Ying-Chao, Yang Ting, and Zhou Qin on "The Hua–Chen New Theory of Economic Optimization" (in Chinese and English) can be found on the author's homepage.

The precise theory of economic optimization

Of course, this is the first theory in precision science. We have completed a new economic optimization model based on Prof. Hua's work, with necessary revisions and updates. Furthermore, we transform the research on optimization theory of economic models into the study of stochastic mathematics; regarding a transformed transition probability matrix and related others as fundamental invariants in the study. Four major advances have been made: firstly, a ranking and classification method for the economic system of products has been provided; secondly, it provides a relatively complete and efficient algorithm, greatly improving the calculation accuracy; thirdly, a prediction and regulation model was provided for the relationship between production and consumption under different economic growth rates; Fourthly, while ensuring the stability of the economic system, the design and debugging of economic structure optimization have been achieved at the same time. The new theory has been successfully applied to 7 national-level input-output tables (with a period of 30 years), 1 provincial-level input-output table in China, and 2 foreign input-output tables. The above results have been verified to be not only theoretically reliable, but also practically feasible in practice. This precise theory is programmable and has intelligent capability. Wish to apply the new theory soon in practice in our country!

4. Optimization method in deep mathematical research

Interview transcript (Refer to item 21 on author's homepage under the section 'Science Popularization Works'): It seems fair to say that the optimization method always plays a role behind the scenes, no matter what.

The optimization method has influenced my lifelong destiny. Due to the special circumstances of the Cultural Revolution at that time, the graduates of Beijing Normal University's classes of 1969 and 1970 were postponed to 1972; work was only assigned annually. At that time, five of us were assigned to the Affiliated Middle School of Guiyang Normal College (now Guizhou Normal University), basically teaching high school for a lifetime. My luck was that I went to the Beijing Cotton Mill not long before leaving Beijing to listen to Prof. Hua Luo-Geng's promotion on the optimization method. For me, who has been self-learning mathematics for 12 years, I had heard that the optimization method had been successfully applied directly to the production line, I was extremely shocked, so after arriving in Guiyang, I immediately went to the Provincial Science and Technology Information Institute to inquire about any actual information in the area about which department hopes to apply the optimization method. Good luck, I am eager to conduct experiments in the electroplating workshop of Guizhou Automobile Overhaul Factory. So I assist in the factory during my spare time or weekends. After two to three months of experimentation, it was successful. The workers were very happy to write a news report to the provincial radio station. After the news spread to the radio station, more units came to invite me and quickly expanded their influence. Even the provincial Secretary Jia Ting-San, who is in charge of industry, went to the report site to listen to my report. I mentioned that I have been promoting excellence at the Guiyang Lock Factory during these days. The next day, Secretary Jia brought the Provincial Industry Director to Guiyang Lock Factory and said to me,"Teacher Chen, could you please explain in 15 min to Director X about the optimization method". At that time, it was Premier En-Lai Zhou's instruction that Prof. Hua led a small team to promote the Optimization method throughout the country. In 1975, I had the honor of being one of the representatives of Guizhou Province and participating in Hua's team to promote excellence in Shanxi Province. The teachers from the Mathematics Department of Guiyang Normal College were also inspired and strongly requested to transfer me to the Mathematics Department. So, in the autumn of 1974, I joined the Mathematics Department of the College and thus had the basic conditions to do mathematics. When I first arrived, I offered a course on optimization methods. Then take the students to Zunyi to promote the optimization method. During the six years from 1972 to 1978, I visited dozens of factories, and many industries promoted excellence. These six years have been a baptism of the soul for me, and I deeply understand that the workers and peasant masses need science and the country needs science to truly understand what one should do.

Let me now introduce my work on the theory of the optimization method. We know that J. Kiefer's original approach to the optimization method contribution is seeking to apply to any unimodal function under the condition of a fixed number of experiments. The optimal strategy using Fibonacci series: $F_0 = 1, F_1 = 1, F_2 = 2, F_n = F_{n-2} + F_{n-1}, n \geqslant 2$. The conclusion is that if the experiment is fixed at n steps, then score F_n/F_{n+1} should be selected as the first testing point, and then use the symmetrical rule to arrange the second test point. It is called the fractional method. Kieder has also pointed out that if you find it troublesome, then take the limit

$$\omega := \lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \frac{\sqrt{5}-1}{2},$$

which is the well-known golden ratio constant. One can then use it as the first testing point. This method is called the golden ratio method. Prof. Hua directly promoted this optimization method because he proved that the golden ratio method is optimal at infinity, and he regards the fractional method as an approximate special method to the golden ratio method. The theoretical basis is that the fraction sequence $\{F_n/F_{n+1}\}_{n\geqslant 1}$ used in the fractional method is the best approximation sequence for the golden ratio constant ω. This is the Diophantine approximation known in number theory. In summary, both Kiefer's and Hua's optimization methods are based on a predetermined number of testing times, with the former set at a certain finite number of times, and the latter set at infinity. In several years of practice, I feel that it is not feasible to schedule the number of experiments; who knows how many times we need to do it before testing. Satisfactory results can be obtained only through experimentation. So I thought we should take Kiefer's original idea "for any unimodal function" one step further, the optimal method that applies to "any number of experiments simultaneously". According to this viewpoint, I wrote two articles in 1977 and 1995; I wrote another article including multiple testing point scenarios for each batch specifically, if the same even number of testing point projects are arranged for each batch, then at infinity, the optimal strategy in the distance simply does not exist. But our optimal strategy under the condition of an indefinite number of batches is to evenly distribute the testing points for each batch. However, in the special case that each batch of two testing point projects has a unique situation: the first batch of two testing point projects was arranged at 3/7 and 4/7. Starting from the second batch, each one uses the equal division method.

The last two of the four topics mentioned above were carried out in continuation of Prof. Hua's work. The background is: starting from the second year of junior high school (at the age of fifteen), inspired by the story of Prof. Hua's self-taught success and becoming a great mathematician [16], I began to self-study mathematics. When I was at university, I only attended regular courses for seven months and relied on self-study for my whole life. It can be imagined that no one taught me how to self-study back then, but there is no dead end to the road. I remember borrowing a booklet by Prof. Hua from a high school library back then. I was really like. Acquiring the ultimate treasure, one can gain too many lessons from it. In the process of growing up, how to self-study, how to conduct research, everything is based on it. Find the answer in Prof. Hua's writings. I believe I will never forget Hua Loo-Keng's motto: "Intelligence lies in diligence, and genius lies in accumulation".

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