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Article

Fiberwise Rossi extension domains for rank-one Matsuki orbits in complex Grassmannians

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Abstract: Let $G_0 = SU(p,q)$ act on the complex Grassmannian $Z = \mathrm{Gr}_k(\mathbb{C}^{p+q})$, and let M be a Matsuki orbit, i.e., the intersection of a G_0-orbit with its Matsuki-dual K-orbit. Such intersections are compact homogeneous real-analytic Cauchy–Riemann (CR) manifolds. This paper studies the holomorphic extension problem for rank-one Matsuki orbits with residual signature $s \geq 2$. The fibers of the Matsuki fibration are isotropic CR spheres in projective spaces of signature $(1, s)$ or $(s, 1)$, whose one-sided Rossi envelope is the corresponding projective ball. We construct the resulting fiberwise Rossi extension domain $\mathcal{D}_{\ell,m,1}$ in a naturally associated projective residual bundle $\mathcal{X}_{\ell,m} \to \mathrm{Gr}_\ell(E^+) \times \mathrm{Gr}_m(E^-)$, where the Matsuki orbit embeds as the boundary. A key clarification is that the evaluation image of $\mathcal{D}_{\ell,m,1}$ in the ambient Grassmannian Z is generally not open, so extended holomorphic functions need not descend. Our main theorem establishes a precise fiberwise Rossi extension: every real-analytic CR function extends holomorphically to $\mathcal{D}_{\ell,m,1}$, and this domain is maximal among those obtained by the fiberwise Rossi-envelope construction. The proof uses horizontal CR vector fields, joint holomorphicity, and fiberwise patching. We also relate the construction to rank-one cycle domains via the evaluation map and formulate the higher-rank problem.

Keywords: CR manifolds; holomorphic extension; projective ball; homogeneous fibration; evaluation map; cycle space; Akhiezer–Gindikin domain

1. Introduction

The study of holomorphic extension of CR functions is a central theme in several complex variables, with deep connections to geometry, representation theory, and partial differential equations. Given a real submanifold M of a complex manifold Z, a fundamental question is whether every CR function defined on M extends holomorphically to some larger domain in Z. This problem has a rich history, beginning with the classical works of Hartogs and continuing through the foundational results of Kohn–Rossi [1], Rossi [2], and Wells [3]; see also the comprehensive surveys of Merker–Porten [4–6].

A particularly important class of CR manifolds arises from the action of real forms on complex flag manifolds. Let G be a connected complex semisimple Lie group, $G_0 \subset G$ a real form, and $Z = G/Q$ a complex flag manifold. The action of G_0 on Z has finitely many orbits. A fundamental theorem due to Matsuki [7] establishes a duality between G_0-orbits and K-orbits, where $K_0 \subset G_0$ is a maximal compact subgroup and $K = K_0^{\mathbb{C}}$. For each G_0-orbit γ, there exists a unique K-orbit κ such that $\gamma \cap \kappa$ is a single K_0-orbit. These intersections are called Matsuki orbits. They are compact homogeneous real-analytic CR manifolds naturally embedded in Z, and their geometric structure has been extensively studied [8,9].

The holomorphic extension problem for Matsuki orbits is both theoretically significant and practically important. On the theoretical side, it bridges CR geometry and complex geometry, revealing how local analytic structures on real submanifolds determine global holomorphic functions. On the practical side, it has applications to the study of cycle spaces and flag domains, which play a central role in representation theory and the geometry of homogeneous spaces. Cycle-space theory, developed by Wolf and further advanced by Fels, Huckleberry, and Wolf [10], provides canonical Stein domains associated with open G_0-orbits. Foundational results on cycle domains were established by Huckleberry [11] and Huckleberry–Wolf [12], while the linear cycle space approach was developed by Wolf–Zierau [13]. In many non-Hermitian cases, the Wolf cycle domain coincides with the Akhiezer–Gindikin universal domain [14, 15]. More recent developments have further elucidated the structure of cycle domains [10,16,17].

Despite this rich theory, explicit computation of envelopes of holomorphy for Matsuki orbits has remained challenging, particularly when the ambient manifold has higher dimension and the orbit structure is complicated. Existing approaches have largely focused on the full abstract Rossi envelope in the ambient manifold, but explicit descriptions have been obtained only in limited cases. The main difficulty lies in the global nature of the extension problem: local fiberwise extensions must be shown to assemble into a global holomorphic function on a maximal domain. Moreover, the natural ambient space for the extension, as we will show, is not always the original manifold itself but rather a larger bundle. This subtle point has been overlooked in previous studies. Recent advances in CR geometry and homogeneous spaces [18–22] have renewed interest in explicit envelope computations. Further contributions in this direction include the works of Merker and Porten [5], which explore related questions in CR geometry and holomorphic extension.

In this paper, we address this problem for a natural class of Matsuki orbits in complex Grassmannians. We consider the action of $G_0 = SU(p, q)$ on $Z = \mathrm{Gr}_k(\mathbb{C}^{p+q})$. The orbit structure admits a concrete description in terms of signature data and intersection dimensions with the positive and negative summands of a fixed Hermitian decomposition $V = E^+ \oplus E^-$. For fixed intersection dimensions (ℓ, m), there is a natural K-equivariant holomorphic fibration over

$$B_{\ell,m} := \mathrm{Gr}_\ell(E^+) \times \mathrm{Gr}_m(E^-).$$

The Matsuki orbit is obtained by imposing an isotropy condition in the residual fibers.

In the rank-one case, where $r = 1$ and $\min\{p - \ell, q - m\} = 1$, the fibers of this fibration are strictly pseudoconvex CR spheres. Writing

$$s = \max\{p - \ell, q - m\},$$

and assuming $s \geq 2$, each fiber is a nondegenerate projective hyperquadric of signature $(1, s)$ or $(s, 1)$. The classical Kohn–Rossi theorem implies that CR functions on each such fiber extend holomorphically to the corresponding projective ball.

A key insight of the present work is that the natural ambient space for the fiberwise extension is not the full Grassmannian Z, but rather a projective residual bundle

$$\mathcal{X}_{\ell,m} := \mathbb{P}(\mathcal{R}_{\ell,m}) \longrightarrow B_{\ell,m},$$

where $\mathcal{R}_{\ell,m}$ is a holomorphic quotient bundle. There is a canonical holomorphic evaluation map $\mathrm{ev} : \mathcal{X}_{\ell,m} \to Z$. This map identifies a transversality locus in $\mathcal{X}_{\ell,m}$ with the original K-orbit $O_{\ell,m}$, but its image on the filled domain is generally not open in Z. This subtle but crucial point—that the fiberwise Rossi extension domain is naturally a domain in $\mathcal{X}_{\ell,m}$, not in Z—is a central contribution of this paper.

Our main result establishes that every real-analytic CR function on a rank-one Matsuki orbit extends holomorphically to the fiberwise filled domain $\mathcal{D}_{\ell,m,1} \subset \mathcal{X}_{\ell,m}$, and that this domain is maximal among domains obtained by the fiberwise Rossi-envelope construction. The proof combines the classical Kohn–Rossi extension theorem with a detailed parameter-dependent extension argument. We explicitly construct horizontal CR vector fields to verify joint holomorphicity in the base and fiber variables, and we use fiberwise uniqueness on balls for patching. We also describe the relation with rank-one cycle domains through the natural evaluation map, clarifying that the projective ball is the evaluation image of the cycle domain, not the cycle domain itself.

Novelty and contribution

The present work differs from previous studies on envelopes of Matsuki orbits in several important ways. First, while earlier approaches have focused on the full abstract Rossi envelope in the ambient manifold, we emphasize the fiberwise nature of the construction and demonstrate that the natural extension domain is not necessarily a domain in the original Grassmannian but rather a domain in a larger residual bundle. Second, we provide explicit constructions of horizontal CR vector fields to prove joint holomorphicity, a technical innovation that avoids the need for global pseudoconvexity assumptions on the total space. Third, we clarify the relationship between fiberwise Rossi domains and cycle domains, showing that the projective ball is the evaluation image of the rank-one cycle domain, not the cycle domain itself. These insights are new to the best of our knowledge and provide a blueprint for the higher-rank case.

Thus, we achieve our aim of providing a complete and precise description of the fiberwise envelope of holomorphy for rank-one Matsuki orbits in complex Grassmannians. A limitation of the present work, which we address in the discussion, is that we do not prove that holomorphic functions on the fiberwise domain descend to the evaluation image in the Grassmannian. We also restrict our analysis to the case $s \geq 2$; the case $s = 1$ corresponds to CR curves, where the extension theory differs substantially. The methods developed here also lay the groundwork for the higher-rank case, which we formulate as a natural open problem.

Organization of the paper

Section 2 presents the materials and methods, including the CR setup and the construction of the residual bundle. A table summarizing the principal symbols and notations is provided at the beginning of Section 2 to improve readability. Section 3 contains the results and discussion, including the main theorem and its proof. Section 4 provides a broader discussion of the implications, limitations, and connections to existing work. Section 5 concludes with a summary and future directions.

2. Materials and methods

This section establishes the mathematical framework and the construction of the fiberwise Rossi extension domain.

For the reader's convenience, we summarize the principal symbols and notations used throughout the paper in **Table 1** below.

Table 1. Principal symbols and notations.

Symbol	Description
Z	Complex Grassmannian $\mathrm{Gr}_k(\mathbb{C}^{p+q})$
G_0	Real form $\mathrm{SU}(p,q)$
K_0	Maximal compact subgroup $S(U(E^+)\times U(E^-))$
K	Complexification $K_0^{\mathbb{C}}$
h	Hermitian form of signature (p,q) on $V=\mathbb{C}^{p+q}$
E^+, E^-	Positive and negative definite summands of V
(ℓ,m,r)	Signature triple of a k-plane
$O_{\ell,m}$	K-orbit defined by intersection dimensions with $E^{\pm}$
$B_{\ell,m}$	Base space $\mathrm{Gr}_\ell(E^+)\times\mathrm{Gr}_m(E^-)$
$\mathcal{R}_{\ell,m}$	Holomorphic residual quotient bundle over $B_{\ell,m}$
$\mathcal{X}_{\ell,m}$	Projective residual bundle $\mathbb{P}(\mathcal{R}_{\ell,m})$
$\mathcal{Q}_\ell^+, \mathcal{Q}_m^-$	Positive and negative residual quotient bundles
ev	Evaluation map $\mathcal{X}_{\ell,m}\to Z$
$\Sigma_{L,M}$	Isotropic CR sphere in the fiber over (L,M)
$\mathcal{B}_{L,M}$	Projective ball (Rossi filling) in the fiber
$\mathcal{M}_{\ell,m,1}$	Lifted Matsuki orbit in $\mathcal{X}_{\ell,m}$
$\mathcal{D}_{\ell,m,1}$	Fiberwise Rossi extension domain in $\mathcal{X}_{\ell,m}$

2.1. CR geometry of Matsuki orbits

We briefly recall the CR structure on Matsuki orbits. Let $M=\gamma\cap\kappa\subset Z$ be a Matsuki orbit. Since M is a K_0-orbit and K_0 is compact, M is compact. It is a real-analytic submanifold of the complex flag manifold Z, and its CR structure is induced by

$$T^{0,1}M := T^{0,1}Z\cap\mathbb{C}TM.$$

At a base point $x_0\in M$, this structure may be described by a CR algebra associated with the pair $(\mathfrak{g}_0,\mathfrak{q})$, where $\mathfrak{q}$ is the parabolic subalgebra attached to Q; see literature [8,9]. This Lie-algebraic description is useful for homogeneity, while the present paper mainly uses the concrete Grassmannian model described below.

In the rank-one situation considered here, the Matsuki fibers are real-analytic hypersurfaces in projective spaces of dimension s. Their real dimension is $2s-1$ and their CR dimension is $s-1$. The assumption $s \geq 2$ guarantees positive CR dimension. Each fiber is a nondegenerate projective hyperquadric of signature $(1, s)$ or $(s, 1)$, hence a strictly pseudoconvex CR sphere after choosing the ball side corresponding to the one-dimensional sign.

The total lifted Matsuki orbit $\mathcal{M}_{\ell,m,1} \subset \mathcal{X}_{\ell,m}$ is a real-analytic hypersurface in the projective residual bundle. Its Levi form is strictly definite in the vertical CR directions and may be degenerate in directions induced from the complex base. For this reason the proof below does not appeal to a global strong-pseudoconvexity theorem on the total space. Instead, it applies Kohn–Rossi/Rossi extension in each strictly pseudoconvex projective fiber and then proves that the resulting extensions are jointly holomorphic in the base and fiber variables.

2.2. The $SU(p, q)$ action and the residual bundle

Let $V = \mathbb{C}^{p+q}$ be equipped with a nondegenerate Hermitian form h of signature (p, q). Fix an h-orthogonal splitting

$$V = E^+ \oplus E^-, \qquad h|_{E^+} > 0, \quad h|_{E^-} < 0, \quad \dim E^+ = p, \quad \dim E^- = q.$$

Let

$$G_0 = SU(V, h) \cong SU(p, q), \quad K_0 = S(U(E^+) \times U(E^-)),$$
$$K = K_0^{\mathbb{C}} \cong S(GL(E^+) \times GL(E^-)).$$

Fix k and set $Z = \mathrm{Gr}_k(V)$. For a k-plane $W \subset V$, the restriction $h|_W$ has a signature triple

$$\mathrm{sig}(W) = (\ell, m, r), \qquad \ell + m + r = k,$$

where ℓ and m are the maximal positive and negative dimensions and $r = \dim \mathrm{Rad}(h|_W)$ is the nullity.

2.3. K-orbits and the holomorphic fibration

Fix integers $\ell, m \geq 0$ and put $r = k - \ell - m$. The basic K-invariants for the action on Z are the intersection dimensions with E^+ and E^-. Define

$$O_{\ell,m} := \{W \in Z : \dim(W \cap E^+) = \ell, \dim(W \cap E^-) = m\}.$$

Let

$$B_{\ell,m} := \mathrm{Gr}_\ell(E^+) \times \mathrm{Gr}_m(E^-).$$

There is a natural map

$$\pi : O_{\ell,m} \longrightarrow B_{\ell,m}, \qquad \pi(W) = (W \cap E^+, W \cap E^-).$$

Proposition 1 (Holomorphic bundle structure)**.** *The map π is K-equivariant and is a holomorphic fiber bundle. For $(L, M) \in B_{\ell,m}$, put*

$$V_{L,M} := (L \oplus M)^{\perp h}.$$

Let $E_L^+ = E^+ \cap L^{\perp h}$ *and* $E_M^- = E^- \cap M^{\perp h}$*. Then*

$$\pi^{-1}(L,M) \cong \{U \in \mathrm{Gr}_r(V_{L,M}) : U \cap E_L^+ = 0,\ U \cap E_M^- = 0\},$$

via $W = L \oplus M \oplus U$*. The residual Hermitian space* $V_{L,M}$ *has signature* $(p-\ell, q-m)$*.*

Proof. The equivariance follows from $K(E^\pm) = E^\pm$. The map is algebraic on the locally closed K-orbit and hence holomorphic. If $W \in \pi^{-1}(L,M)$, then $L \oplus M \subset W$ and $\dim W/(L \oplus M) = r$. The h-orthogonal decomposition $V = (L \oplus M) \oplus V_{L,M}$ identifies this quotient with an r-plane $U \subset V_{L,M}$. To see the equivalence of the intersection conditions, observe that the quotient map $W \mapsto W/(L \oplus M)$ identifies the intersection $W \cap E^+$ with the intersection of the residual plane U with the image of E^+ in the quotient, namely E_L^+. Thus, the condition $\dim(W \cap E^+) = \ell$ is equivalent to $U \cap E_L^+ = 0$; otherwise, the positive dimension would strictly exceed ℓ. Similarly, $\dim(W \cap E^-) = m$ is equivalent to $U \cap E_M^- = 0$. These are open conditions in the Grassmannian. Since L is positive definite and M is negative definite, the residual signature is $(p-\ell, q-m)$. ☐

2.4. The residual quotient bundle and the evaluation map

Before giving the formal construction, we briefly motivate why the residual bundle is needed. Working directly in the Grassmannian Z, a k-plane W is decomposed as $W = L \oplus M \oplus U$, where $L = W \cap E^+$ and $M = W \cap E^-$. The residual part U naturally lives in the quotient space $V/(L \oplus M)$. As the base point (L,M) varies over $B_{\ell,m}$, these quotient spaces assemble into a holomorphic vector bundle $\mathcal{R}_{\ell,m}$. Projectivizing this bundle gives the natural ambient space $\mathcal{X}_{\ell,m}$ for the residual lines. This construction is essential because the evaluation image of the filled domain in Z is generally not open, as we will show in Remark 1.

Let $\mathcal{S}_\ell^+ \to \mathrm{Gr}_\ell(E^+)$ and $\mathcal{S}_m^- \to \mathrm{Gr}_m(E^-)$ denote the tautological bundles, pulled back to $B_{\ell,m}$ without changing notation. Define the holomorphic residual quotient bundle

$$\mathcal{R}_{\ell,m} := (V \times B_{\ell,m})/(\mathcal{S}_\ell^+ \oplus \mathcal{S}_m^-).$$

Its fiber over (L,M) is $V/(L \oplus M)$. Because $L \oplus M$ is nondegenerate for h, the form h induces, through the h-orthogonal representative, a real-analytic nondegenerate Hermitian form $h_{L,M}$ on $\mathcal{R}_{\ell,m}|_{(L,M)}$. The positive and negative residual quotient bundles are

$$\mathcal{Q}_\ell^+ = (E^+ \times B_{\ell,m})/\mathcal{S}_\ell^+, \qquad \mathcal{Q}_m^- = (E^- \times B_{\ell,m})/\mathcal{S}_m^-.$$

They have ranks $p-\ell$ and $q-m$, respectively, and $\mathcal{R}_{\ell,m} \cong \mathcal{Q}_\ell^+ \oplus \mathcal{Q}_m^-$ as a holomorphic vector bundle.

For $r = 1$ put

$$\mathcal{X}_{\ell,m} := \mathbb{P}(\mathcal{R}_{\ell,m}).$$

There is a canonical holomorphic evaluation map

$$\text{ev} : \mathcal{X}_{\ell,m} \longrightarrow Z, \qquad \text{ev}(L, M, [\bar{v}]) = L \oplus M \oplus \mathbb{C}v,$$

where $v \in V$ is any lift of the quotient vector $\bar{v} \in V/(L \oplus M)$. The expression is independent of the lift.

Proposition 2 (Transversality locus)**.** *Let*

$$\mathcal{X}^{\circ}_{\ell,m} := \mathcal{X}_{\ell,m} \setminus (\mathbb{P}(\mathcal{Q}^{+}_{\ell}) \cup \mathbb{P}(\mathcal{Q}^{-}_{m})).$$

Then ev restricts to a biholomorphism

$$\text{ev} : \mathcal{X}^{\circ}_{\ell,m} \xrightarrow{\sim} O_{\ell,m}$$

in the rank-one case.

Proof. If $[\bar{v}] \notin \mathbb{P}(\mathcal{Q}^{+}_{\ell}) \cup \mathbb{P}(\mathcal{Q}^{-}_{m})$, then the line represented by $\bar{v}$ is not contained in the residual positive or residual negative quotient. Equivalently, after using the h-orthogonal representative, the corresponding residual line satisfies the transversality conditions of Proposition 1. Hence $\text{ev}(L, M, [\bar{v}])$ belongs to $O_{\ell,m}$. Conversely, if $W \in O_{\ell,m}$, then $L = W \cap E^{+}$ and $M = W \cap E^{-}$ are uniquely determined, and the quotient line $W/(L \oplus M)$ is transverse to both residual summands. The inverse map is given explicitly by $W \mapsto (W \cap E^{+}, W \cap E^{-}, W/((W \cap E^{+}) \oplus (W \cap E^{-})))$, which is clearly holomorphic on $O_{\ell,m}$. These two constructions are inverse and holomorphic. □

Remark 1. *The preceding proposition applies only on the transversality locus. Once the fiber is filled by the projective ball, points in $\mathbb{P}(\mathcal{Q}^{+}_{\ell})$ or $\mathbb{P}(\mathcal{Q}^{-}_{m})$ may be included, and the evaluation map may fail to be injective. Moreover, the image need not be open in Z. For instance, suppose $p - \ell = 1$ and $q - m = s$. Then*

$$\dim_{\mathbb{C}} \mathcal{X}_{\ell,m} = \ell(p - \ell) + m(q - m) + s = \ell + ms + s,$$

whereas

$$\dim_{\mathbb{C}} Z = k(p + q - k) = (\ell + m + 1)s.$$

The difference is $\ell(s - 1)$. Thus, if $\ell > 0$ and $s \geq 2$, the evaluation image of any open subset of $\mathcal{X}_{\ell,m}$ cannot be open in Z. Similarly, in the case $p - \ell = s$ and $q - m = 1$, the difference is $m(s - 1)$. This dimension count is the reason the present theorem is formulated on the residual projective bundle $\mathcal{X}_{\ell,m}$ rather than as an assertion that the fiberwise filling is a domain in the whole Grassmannian.

2.5. Matsuki orbits and their lifted fiber description

Let $\gamma_{\ell,m,r} \subset Z$ be the G_0-orbit of k-planes with signature triple (ℓ, m, r). The corresponding Matsuki orbit is

$$M_{\ell,m,r} := \gamma_{\ell,m,r} \cap O_{\ell,m}.$$

Proposition 3 (Fiberwise isotropic Grassmannian)**.** *For each $(L, M) \in B_{\ell,m}$, under the identification of Proposition 1,*

$$M_{\ell,m,r} \cap \pi^{-1}(L, M) = \Sigma_r(V_{L,M}) := \{U \in \mathrm{Gr}_r(V_{L,M}) : h|_U \equiv 0\}.$$

Consequently $\pi|_{M_{\ell,m,r}} : M_{\ell,m,r} \to B_{\ell,m}$ is a smooth K_0-homogeneous fiber bundle whose fibers are compact homogeneous real-analytic CR manifolds.

Proof. Write $W = L \oplus M \oplus U$. Since L is positive definite and M is negative definite, the restriction $h|_W$ has nullity r and exactly ℓ positive and m negative directions if and only if $h|_U \equiv 0$. Indeed, the signature of $h|_W$ is the sum of the signatures of $h|_L$, $h|_M$, and $h|_U$. Since L and M contribute ℓ positive and m negative directions respectively, the residual part U must contribute exactly r null directions, which is equivalent to $h|_U \equiv 0$. A totally isotropic subspace cannot contain a nonzero vector lying entirely in either residual definite summand, so the transversality conditions are automatic on the isotropic locus. The converse is immediate. Homogeneity follows from the transitivity of K_0 on the base and on each isotropic fiber. □

3. Results and discussion

We now present the main results: the rank-one specialization, the parameter-dependent extension lemma, and the theorem with its proof, along with interpretive discussion.

3.1. Rank-one specialization and the ball side

Definition 1 (Rank one)**.** *The Matsuki orbit $M_{\ell,m,r}$ is called rank one if $r = 1$ and*

$$\min\{p - \ell, q - m\} = 1.$$

Set

$$s := \max\{p - \ell, q - m\}.$$

Throughout this section we assume $s \geq 2$.

Define the sign

$$\epsilon_{\ell,m} := \begin{cases} +1, & p - \ell = 1, \\ -1, & q - m = 1. \end{cases}$$

The two alternatives are mutually exclusive under the assumption $s \geq 2$. For $(L, M) \in B_{\ell,m}$ define

$$\Sigma_{L,M} := \{[u] \in \mathbb{P}(\mathcal{R}_{\ell,m}|_{(L,M)}) : h_{L,M}(u, u) = 0\}$$

and

$$\mathcal{B}_{L,M} := \{[u] \in \mathbb{P}(\mathcal{R}_{\ell,m}|_{(L,M)}) : \epsilon_{\ell,m} h_{L,M}(u, u) > 0\}.$$

Then $\Sigma_{L,M} = \partial\mathcal{B}_{L,M}$ and $\mathcal{B}_{L,M}$ is biholomorphic to the unit ball in $\mathbb{C}^s$.

Lemma 1 (Local normal form)**.** *If h has signature $(1, s)$, then in an affine chart $z_0 \neq 0$*

$$\frac{h(z_0 e_0 + w, z_0 e_0 + w)}{|z_0|^2} = 1 - \|w/z_0\|^2.$$

Thus the positive lines form the unit ball and the isotropic lines form its sphere. In signature $(s, 1)$ the same conclusion holds with the sign reversed and the negative lines forming the ball.

Proof. For signature $(1, s)$ choose coordinates with $h(z_0 e_0 + w, z_0 e_0 + w) = |z_0|^2 - \|w\|^2$. The affine coordinates $\zeta = w/z_0$ give $h/|z_0|^2 = 1 - \|\zeta\|^2$. The defining function $\rho(\zeta) = \|\zeta\|^2 - 1$ has positive definite Levi form. The case $(s, 1)$ is obtained by replacing h by $-h$. □

Definition 2 (Fiberwise Rossi domain)**.** *The lifted Matsuki orbit and its fiberwise Rossi filling are*

$$\mathcal{M}_{\ell,m,1} := \{(L, M, [u]) \in \mathcal{X}_{\ell,m} : h_{L,M}(u, u) = 0\}$$

and

$$\mathcal{D}_{\ell,m,1} := \{(L, M, [u]) \in \mathcal{X}_{\ell,m} : \epsilon_{\ell,m} h_{L,M}(u, u) > 0\}.$$

The set $\mathcal{D}_{\ell,m,1}$ is an open connected domain in the complex manifold $\mathcal{X}_{\ell,m}$, and $\partial\mathcal{D}_{\ell,m,1} = \mathcal{M}_{\ell,m,1}$ as a real-analytic hypersurface.

Proof. The induced Hermitian form $h_{L,M}$ on $\mathcal{R}_{\ell,m}$ depends real-analytically on the base and is homogeneous of degree $(1, 1)$ in the fiber variable. Hence the sign condition defining $\mathcal{D}_{\ell,m,1}$ is open and the zero set defining $\mathcal{M}_{\ell,m,1}$ is real-analytic. Fiberwise, the zero set is the nondegenerate projective sphere separating the two sign domains, and the chosen side is the ball. Since the base $B_{\ell,m}$ is connected and the ball fibers are connected, $\mathcal{D}_{\ell,m,1}$ is connected. □

Remark 2. *The evaluation map sends $\mathcal{M}_{\ell,m,1}$ diffeomorphically onto $M_{\ell,m,1} \subset Z$ because isotropic lines automatically satisfy transversality. The map sends $\mathcal{D}_{\ell,m,1}$ into Z, but its image may meet adjacent K-orbits and need not be open. The theorem below asserts extension to $\mathcal{D}_{\ell,m,1}$. It does not assert that the holomorphic extension descends to a single-valued holomorphic function on the possibly singular or non-open evaluation image in Z.*

3.2. A parameter-dependent fiberwise extension lemma

The classical Kohn–Rossi theorem guarantees holomorphic extension of CR functions on a single strictly pseudoconvex hypersurface. However, in our setting, the Matsuki orbit is not a single hypersurface but a fibration over a compact base $B_{\ell,m}$. The extension must be performed simultaneously on each fiber, and we need to ensure that the resulting fiberwise extensions vary holomorphically with the base parameter. This is not a consequence of the classical theorem, which applies fiberwise but does not address the joint holomorphicity in the base variable. The parameter-dependent extension argument below is therefore essential: it provides the analytic tool to prove that the fiberwise extensions assemble into a global holomorphic function on the filled domain $\mathcal{D}_{\ell,m,1}$. This is the key technical step that goes beyond the classical Kohn–Rossi theorem.

The following local lemma is the key analytic tool.

Lemma 2 (Parameter-dependent fiberwise Rossi extension)**.** *Let $U \subset \mathbb{C}^N$ be a polydisc and let*

$$D_U = \{(b, z) \in U \times \mathbb{C}^s : \rho(b, z, \bar{b}, \bar{z}) < 0\}$$

where ρ is real-analytic, each fiber $D_b = \{z : \rho(b, z) < 0\}$ is a bounded strictly pseudoconvex domain with real-analytic boundary Σ_b, and the family varies real-analytically with b. Assume $s \geq 2$. Let

$$\mathcal{M}_U = \{(b, z) : b \in U, z \in \Sigma_b\}.$$

If $f \in \mathrm{CR}^\omega(\mathcal{M}_U)$, then the fiberwise Rossi extensions of $f|_{\Sigma_b}$ assemble to a holomorphic function

$$F \in \mathcal{O}(D_U)$$

whose boundary value on $\mathcal{M}_U$ is f.

Proof. For fixed b, the Kohn–Rossi theorem gives a holomorphic extension of $f_b := f|_{\Sigma_b}$ to D_b. Since ρ and f are real-analytic and the Levi form in the z variables is positive definite on each Σ_b, the extension can be represented locally by a Cauchy–Fantappiè/Henkin integral with a support function depending real-analytically on the parameter; see literature [23, 24]. After shrinking U if necessary, this gives a function $F(b, z)$ that is real-analytic in $(b, \bar{b}, z)$, continuous up to the fiber boundary, and holomorphic in the fiber variable z.

It remains to prove that F is holomorphic in b. Fix a point of $\mathcal{M}_U$. Since $\partial\rho/\partial\bar{z}$ is not identically zero on the hypersurface, after possibly relabelling the fiber coordinates we may assume $\rho_{\bar{z}_s} \neq 0$ in a neighborhood. For each base coordinate b_j define a $(0, 1)$ vector field tangent to $\mathcal{M}_U$ by

$$L_j = \frac{\partial}{\partial \bar{b}_j} - \frac{\rho_{\bar{b}_j}}{\rho_{\bar{z}_s}} \frac{\partial}{\partial \bar{z}_s}.$$

Then $L_j\rho = 0$, so L_j is tangential to $\mathcal{M}_U$. Because f is CR on $\mathcal{M}_U$, one has $L_j f = 0$ on $\mathcal{M}_U$.

The function F is holomorphic in z, hence $\partial F/\partial\bar{z}_s = 0$ in D_b and up to the boundary in the real-analytic sense. Therefore the boundary value of $L_j F$ is precisely the boundary value of $\partial F/\partial\bar{b}_j$. Since $F|_{\mathcal{M}_U} = f$ and $L_j f = 0$, the function

$$G_j(b, z) := \frac{\partial F}{\partial \bar{b}_j}(b, z)$$

vanishes on Σ_b for each fixed b. For fixed b, $G_j(b, \cdot)$ is holomorphic in D_b and continuous on $\overline{D_b}$. By the maximum principle, $G_j(b, \cdot) \equiv 0$ on D_b. Thus $\partial F/\partial\bar{b}_j = 0$ for all j, and F is holomorphic in both base and fiber variables. $\square$

Remark 3. *The proof uses only local data: real-analytic variation of strictly pseudoconvex fibers, the classical one-sided fiberwise Rossi extension, and the CR equations in horizontal directions. It avoids the stronger and generally false assertion that the total boundary is strictly pseudoconvex in every complex tangential direction.*

3.3. The extension theorem

Theorem 1 (Rank-one fiberwise Rossi filling)**.** *Let $M_{\ell,m,1} \subset Z = \mathrm{Gr}_k(\mathbb{C}^{p+q})$ be a rank-one Matsuki orbit with residual signature $s \geq 2$. Let $\mathcal{M}_{\ell,m,1} \subset \mathcal{X}_{\ell,m}$ and $\mathcal{D}_{\ell,m,1} \subset \mathcal{X}_{\ell,m}$ be as in Definition 2. Then every real-analytic CR function on $M_{\ell,m,1}$ extends, after pullback by $\mathrm{ev}|_{\mathcal{M}_{\ell,m,1}}$, to a holomorphic function on $\mathcal{D}_{\ell,m,1}$. More precisely, for every $f \in \mathrm{CR}^{\omega}(M_{\ell,m,1})$ there exists a unique*

$$F \in \mathcal{O}(\mathcal{D}_{\ell,m,1})$$

such that $F = f \circ \mathrm{ev}$ on $\mathcal{M}_{\ell,m,1}$ in the real-analytic boundary-value sense. The domain $\mathcal{D}_{\ell,m,1}$ is maximal among domains obtained by the fiberwise Rossi-envelope construction. We emphasize that this maximality is fiberwise maximality*: it is maximality within the class of domains obtained by applying the Rossi-envelope construction independently in each projective fiber and then globalizing over the base. It is not a claim about the full abstract Rossi envelope of the total CR manifold, which would require additional assumptions.*

Proof. Because ev identifies $\mathcal{M}_{\ell,m,1}$ with $M_{\ell,m,1}$, the pullback $\tilde{f} = f \circ \mathrm{ev}$ is a real-analytic CR function on $\mathcal{M}_{\ell,m,1}$.

Choose a finite cover $\{U_\alpha\}$ of the compact base $B_{\ell,m}$ by holomorphic trivializing neighborhoods for the projective bundle $\mathcal{X}_{\ell,m} \to B_{\ell,m}$. On each U_α the lifted boundary and the filled fibers are represented by a real-analytic family of strictly pseudoconvex hypersurfaces in a fixed projective affine chart. By Lemma 2, the fiberwise Kohn–Rossi/Rossi extensions of $\tilde{f}$ over U_α assemble to a holomorphic function

$$F_\alpha \in \mathcal{O}(\mathcal{D}_{\ell,m,1} \cap p^{-1}(U_\alpha)),$$

where $p : \mathcal{X}_{\ell,m} \to B_{\ell,m}$ is the bundle projection.

If $U_\alpha \cap U_\beta \neq \emptyset$, then F_α and F_β have the same boundary value $\tilde{f}$ on every fiber sphere $\Sigma_{L,M}$ over $(L, M) \in U_\alpha \cap U_\beta$. For fixed (L, M) the difference $F_\alpha - F_\beta$ is holomorphic on the ball $B_{L,M}$ and continuous up to its boundary, and it vanishes on $\partial B_{L,M}$. The maximum principle gives $F_\alpha = F_\beta$ on $B_{L,M}$. Hence the local extensions agree on overlaps and glue to a global holomorphic function $F \in \mathcal{O}(\mathcal{D}_{\ell,m,1})$.

Uniqueness is proved in the same way. If two holomorphic functions on $\mathcal{D}_{\ell,m,1}$ have the same real-analytic boundary value on $\mathcal{M}_{\ell,m,1}$, then their difference vanishes on every fiber boundary sphere. The maximum principle on each ball fiber forces the difference to vanish on every fiber and hence on all of $\mathcal{D}_{\ell,m,1}$.

It remains to justify maximality in the stated sense. For each base point (L, M), the fiber boundary $\Sigma_{L,M}$ is the standard strictly pseudoconvex CR sphere in a projective space of signature $(1, s)$ or $(s, 1)$. Its one-sided Rossi envelope on the chosen side is exactly the projective ball $B_{L,M}$. Therefore any construction that replaces each fiber only by its one-sided Rossi envelope has fiber slice contained in $B_{L,M}$. Taking the union over the base gives a domain contained in $\mathcal{D}_{\ell,m,1}$. Thus $\mathcal{D}_{\ell,m,1}$ is maximal among fiberwise Rossi-envelope constructions. $\square$

Remark 4 (Scope of maximality)**.** *The maximality in Theorem 1 is fiberwise maximality, as emphasized in the theorem statement. The theorem does not claim that $\mathcal{D}_{\ell,m,1}$, or its evaluation image in Z, is the full abstract Rossi envelope of the total CR manifold $M_{\ell,m,1}$. A full abstract envelope statement would require an additional global realization and descent theorem showing that all possible analytic continuations are captured by this fiberwise bundle and, if one works in Z, that the resulting functions are compatible with the identifications of the evaluation map. That additional problem is not addressed here.*

Remark 5 (Non-Steinness)**.** *The domain $\mathcal{D}_{\ell,m,1}$ is generally not Stein. If $p-\ell=1$, then the projectivization $\mathbb{P}(\mathcal{Q}_\ell^+)$ is a section of $\mathcal{X}_{\ell,m} \to B_{\ell,m}$ contained in $\mathcal{D}_{\ell,m,1}$. If $q-m=1$, the analogous section is $\mathbb{P}(\mathcal{Q}_m^-)$. Whenever $B_{\ell,m}$ has positive dimension, this section is a positive-dimensional compact complex subvariety of $\mathcal{D}_{\ell,m,1}$, which cannot occur in a Stein manifold.*

3.4. Discussion of the results

The theorem above provides a complete description of the fiberwise Rossi extension for rank-one Matsuki orbits. Several points merit further discussion.

First, the shift from the Grassmannian Z to the residual bundle $\mathcal{X}_{\ell,m}$ is essential. As shown in Remark 1, the evaluation image of the filled domain is generally not open in Z. This means that the natural object produced by the fiberwise construction is not a domain in the original ambient space but rather a domain in a larger bundle. This observation resolves a potential confusion and clarifies the correct geometric setting for the extension problem.

Second, the parameter-dependent extension lemma (Lemma 2) demonstrates that joint holomorphicity in the base and fiber variables follows from the CR equations in horizontal directions, even though the total boundary is not strictly pseudoconvex. This is a key technical contribution, as it avoids the need for a global pseudoconvexity assumption on the total space.

Third, the maximality statement is deliberately fiberwise. It does not claim that $\mathcal{D}_{\ell,m,1}$ is the full abstract Rossi envelope of the total CR manifold. A full envelope would require a descent theorem showing that all holomorphic functions on $\mathcal{D}_{\ell,m,1}$ that are compatible with the evaluation map descend to Z. This is an interesting open question.

Fourth, we discuss the core limitation of this work: the descent problem. The theorem establishes extension to the fiberwise domain $\mathcal{D}_{\ell,m,1} \subset \mathcal{X}_{\ell,m}$, but it does not prove that holomorphic functions on $\mathcal{D}_{\ell,m,1}$ descend to single-valued holomorphic functions on the evaluation image in Z. A sufficient condition for descent would be the existence of a holomorphic section of the evaluation map over $\mathcal{D}_{\ell,m,1}$, or more generally, the vanishing of the relevant obstruction cocycle. However, such a section generally does not exist due to the non-injectivity of the evaluation map on the filled domain. A counterexample can be constructed by taking two distinct points in $\mathcal{D}_{\ell,m,1}$ that evaluate to the same point in Z via ev; any holomorphic function on the image would have to take the same value at both points, whereas a function on $\mathcal{D}_{\ell,m,1}$ can assign different values. This fundamental obstruction explains why the descent problem is nontrivial and requires additional geometric assumptions, such as the existence of a global holomorphic section or the triviality of the associated bundle.

Fifth, we compare the case $s = 1$ with our framework $s \geq 2$. The case $s = 1$ corresponds to CR curves, where the Matsuki fibers are 1-dimensional CR spheres. The extension theory for CR curves differs substantially from the higher-dimensional case: CR functions on S^1 do not necessarily extend holomorphically to the disk without additional regularity assumptions. Moreover, the Levi form is degenerate in the CR curve case, so the classical Kohn–Rossi theorem does not apply. Our assumption $s \geq 2$ ensures that the fibers have positive CR dimension and strictly definite Levi forms, guaranteeing the applicability of the Kohn–Rossi extension theorem. This distinction is essential: the techniques developed in this paper rely crucially on the strict pseudoconvexity of the fibers, which fails when $s = 1$.

Sixth, the non-Steinness of $\mathcal{D}_{\ell,m,1}$ (Remark 5) is a consequence of the compact base and the presence of compact complex subvarieties. This is consistent with the philosophy that cycle domains are Stein but fiberwise Rossi domains need not be.

Finally, we provide a detailed explicit example to illustrate the geometric features discussed above.

Explicit example: the $(1,1,1)$-orbit in $\mathrm{Gr}_3(\mathbb{C}^8)$. Let $V = \mathbb{C}^8$ carry a Hermitian form h of signature $(2,6)$ and let $Z = \mathrm{Gr}_3(V)$. Fix an h-orthogonal decomposition $V = E^+ \oplus E^-$ with $\dim E^+ = 2$ and $\dim E^- = 6$. Consider the $G_0 = SU(2,6)$-orbit $\gamma_{1,1,1}$ consisting of 3-planes with signature $(1,1,1)$, and the Matsuki orbit

$$M_{1,1,1} = \{W \in \gamma_{1,1,1} : \dim(W \cap E^+) = 1, \dim(W \cap E^-) = 1\}.$$

The base is

$$B_{1,1} = \mathbb{P}(E^+) \times \mathbb{P}(E^-) \cong \mathbb{P}^1 \times \mathbb{P}^5.$$

For base lines $(\ell^+, \ell^-) \in B_{1,1}$, the residual space

$$V_{\ell^+,\ell^-} = (\ell^+ \oplus \ell^-)^{\perp h}$$

has signature $(1,5)$. The Matsuki fiber is the isotropic CR sphere

$$\Sigma_{\ell^+,\ell^-} = \{[v] \in \mathbb{P}(V_{\ell^+,\ell^-}) : h(v,v) = 0\},$$

and its one-sided Rossi filling is the positive projective ball

$$B_{\ell^+,\ell^-} = \{[v] \in \mathbb{P}(V_{\ell^+,\ell^-}) : h(v,v) > 0\}.$$

To see this explicitly, choose coordinates on $V_{\ell+,\ell-}$ with $h(ze_0 + w, ze_0 + w) = |z|^2 - \|w\|^2$, where $z \in \mathbb{C}$ and $w \in \mathbb{C}^5$. In the affine chart $z \neq 0$, with coordinates $\zeta = w/z \in \mathbb{C}^5$, the isotropic condition becomes

$$1 - \|\zeta\|^2 = 0,$$

which is precisely the unit sphere $S^5 \subset \mathbb{C}^5$. The positive ball corresponds to $\|\zeta\| < 1$, which is the standard unit ball $\mathbb{B}^5 \subset \mathbb{C}^5$. Thus, by Lemma 1, $\Sigma_{\ell+,\ell-}$ is strictly pseudoconvex and its one-sided Rossi envelope is exactly $B_{\ell+,\ell-}$.

The fiberwise Rossi domain is therefore

$$\mathcal{D}_{1,1,1} = \bigcup_{(\ell^+,\ell^-)\in\mathbb{P}^1\times\mathbb{P}^5} B_{\ell+,\ell-}$$

which is an open domain in the projective residual bundle $\mathcal{X}_{1,1} \to \mathbb{P}^1 \times \mathbb{P}^5$.

Now we analyze the evaluation image. The residual positive quotient is one-dimensional, and its projectivization gives a center section of the ball bundle. At such a center point, the evaluated plane is

$$W = \ell^+ \oplus \ell^- \oplus E^+_{\ell^+} = E^+ \oplus \ell^-.$$

Hence $\dim(W \cap E^+) = 2$, so the point lies in the adjacent K-orbit $O_{2,1}$ rather than in $O_{1,1}$. This demonstrates that the filled ball $B_{\ell+,\ell-}$ does not remain within the transversality locus; it spills over into adjacent K-orbits.

Moreover, the dimension count confirms that the evaluation image cannot be open in Z. We have

$$\dim_{\mathbb{C}} \mathcal{D}_{1,1,1} = \dim(\mathbb{P}^1 \times \mathbb{P}^5) + 5 = 1 + 5 + 5 = 11,$$

whereas

$$\dim_{\mathbb{C}} \mathrm{Gr}_3(\mathbb{C}^8) = 3 \cdot (8 - 3) = 3 \cdot 5 = 15.$$

Since $\dim_{\mathbb{C}} \mathcal{D}_{1,1,1} = 11 < 15 = \dim_{\mathbb{C}} Z$, the evaluation image of any open subset of $\mathcal{D}_{1,1,1}$ cannot be open in Z. This illustrates precisely why the theorem is formulated on the residual bundle $\mathcal{X}_{1,1}$ rather than as a domain statement in the Grassmannian Z.

Furthermore, the set of added centers (the positive residual lines) forms a compact complex manifold isomorphic to the base $\mathbb{P}^1 \times \mathbb{P}^5$. This demonstrates why the maximal fiberwise Rossi domain contains positive-dimensional compact complex subvarieties and is therefore not a Stein space, as noted in Remark 5.

This example thus concretely illustrates:

1. The strictly pseudoconvex nature of the Matsuki fibers and their Rossi filling as projective balls.
2. The failure of the filled balls to remain within the transversality locus, as they meet adjacent K-orbits.
3. The dimension obstruction preventing the evaluation image from being open in Z.

4. The non-Steinness of the fiberwise Rossi domain due to compact complex subvarieties.

We encourage the reader to view this example as a prototypical illustration of the general theory developed in this paper.

4. Discussion

In this section we place our results in a broader context and discuss their implications, limitations, and connections to existing work.

4.1. Relation to cycle-space theory

Cycle-space theory provides canonical Stein domains associated with open G_0-orbits in flag manifolds. In the rank-one case, the projective ball appearing in our fiberwise construction is the evaluation image of the Wolf cycle domain, not the cycle domain itself. This distinction is important: the cycle domain lives in G/K (dimension $2s$), whereas the projective ball lives in $\mathbb{P}^s$ (dimension s). The evaluation map from G/K to the flag manifold sends the cycle domain onto the projective ball. Thus our fiberwise Rossi domain $\mathcal{D}_{\ell,m,1}$ is a fiberwise version of this evaluation image.

This perspective suggests a deeper connection: the fiberwise Rossi extension construction might be seen as a "localization" of the cycle-domain picture. In higher ranks, one expects the analogous objects to be bounded symmetric domains, which are the evaluation images of higher-rank cycle domains. Our work therefore provides a blueprint for the higher-rank problem.

4.2. Comparison with prior work

Previous results on envelopes of holomorphy for Matsuki orbits and related CR manifolds have primarily focused on the full abstract Rossi envelope within the ambient manifold. The classical approach, as developed in the foundational works of Kohn–Rossi [1], Rossi [2], and Wells [3], establishes the existence of envelopes of holomorphy for real-analytic CR manifolds but does not provide explicit descriptions in homogeneous settings. More recent studies have examined envelopes of holomorphy for specific classes of CR manifolds. For instance, the work of Altomani–Medori–Nacinovich [8] provides a Lie-algebraic classification of CR structures on flag manifolds, while Kim–Zaitsev [18] study CR functions on homogeneous manifolds from a function-theoretic perspective. However, explicit computations of envelopes for Matsuki orbits have remained scarce.

The present work differs from these earlier studies in several significant ways. First, we shift the focus from the abstract envelope in the ambient manifold to the fiberwise construction in a larger bundle. This is not merely a technical choice but a conceptual advancement: as demonstrated in Remark 1, the natural extension domain is not necessarily a domain in the original Grassmannian Z but rather a domain in the residual projective bundle $\mathcal{X}_{\ell,m}$. This observation has been overlooked in previous work.

Second, the use of horizontal CR vector fields to prove joint holomorphicity is a technical innovation. While parameter-dependent integral representations are well-established in the theory of strictly pseudoconvex domains [23, 24], their application to the setting of homogeneous fibrations with non-pseudoconvex total space is novel. Our construction in Lemma 2 explicitly demonstrates how the CR equations in horizontal directions can be leveraged to establish joint holomorphicity without assuming global pseudoconvexity of the total space.

Third, the relationship between fiberwise Rossi domains and cycle domains is clarified. Earlier work, particularly in the cycle-space literature [10,13,16], has focused on the cycle domain itself as a Stein object in G/K. Our results show that the projective ball appearing in the fiberwise construction is the evaluation image of the rank-one cycle domain, not the cycle domain itself. This distinction is subtle but important: it reveals that the fiberwise Rossi construction is a localization of the cycle-domain picture, rather than a direct analog.

In summary, the similarities with prior work include the use of classical CR extension theorems and the connection to cycle-space theory. The key differences are: (1) the identification of the correct ambient space $\mathcal{X}_{\ell,m}$ for the extension; (2) the fiberwise approach with explicit horizontal CR vector fields; and (3) the clarification of the relationship between Rossi domains and cycle domains. These differences constitute the main novelty of the present contribution.

4.3. Limitations and assumptions

Our main result assumes $s \geq 2$. This ensures positive CR dimension and nondegenerate Levi forms on the fibers. The case $s = 1$ would correspond to CR curves, where the extension theory is different and is not covered here. As discussed in Section 3.4, the Levi form degenerates when $s = 1$, and the classical Kohn–Rossi theorem does not apply. The extension theory for CR curves requires different techniques, and the results obtained here do not extend to that case without substantial modification.

We also assume real-analyticity of the CR functions. This is natural for the Rossi envelope theory, but it would be interesting to know whether the result extends to larger classes of CR functions (e.g., continuous or C^∞). Such extensions would require additional regularity theory beyond the scope of this paper.

Another limitation is that we do not address the descent problem: we do not prove that holomorphic functions on $\mathcal{D}_{\ell,m,1}$ descend to single-valued functions on the evaluation image. As discussed in Section 3.4, this is a nontrivial open problem. A sufficient condition for descent would be the existence of a holomorphic section of the evaluation map over $\mathcal{D}_{\ell,m,1}$, but such a section generally does not exist. Counterexamples can be constructed using the non-injectivity of the evaluation map on the filled domain. This remains an open problem that may require additional geometric assumptions or a different analytical approach.

4.4. Broader impact

The methods developed here can be applied to other homogeneous CR manifolds and other real forms. The construction of the residual bundle and the evaluation map is quite general and could be adapted to other flag manifolds. Moreover, the parameter-dependent extension lemma is a versatile tool that could be used in other contexts where fiberwise extensions are needed.

From a broader perspective, this work contributes to the growing literature on the interplay between CR geometry, complex geometry, and representation theory. The fiberwise Rossi extension construction provides a concrete geometric realization of the envelope of holomorphy for a class of homogeneous CR manifolds, and the clarification of the role of the evaluation map offers insights that may be applicable to other homogeneous settings.

The explicit example in Section 3.4 demonstrates the geometric features and serves as a concrete illustration of the theory. We encourage the reader to refer to this example for a concrete visualization of the concepts developed in this paper, particularly the relationship between the Matsuki orbit, the residual bundle, and the evaluation map.

5. Conclusion

In this paper, we have provided a complete and precise description of the fiberwise envelope of holomorphy for rank-one Matsuki orbits in complex Grassmannians $\mathrm{Gr}_k(\mathbb{C}^{p+q})$ under the action of $SU(p,q)$.

Our main contributions are:

(1) **Geometric setup:** We constructed the projective residual bundle $\mathcal{X}_{\ell,m} \to B_{\ell,m}$ and the canonical evaluation map ev : $\mathcal{X}_{\ell,m} \to Z$. We proved that the evaluation image of the filled domain is generally not open in Z, clarifying that the natural fiberwise Rossi extension domain is $\mathcal{D}_{\ell,m,1} \subset \mathcal{X}_{\ell,m}$, not a domain in Z.

(2) **Main theorem:** We proved that every real-analytic CR function on a rank-one Matsuki orbit extends holomorphically to $\mathcal{D}_{\ell,m,1}$, and that this domain is maximal among domains obtained by the fiberwise Rossi-envelope construction. The proof uses a parameter-dependent extension lemma with horizontal CR vector fields and fiberwise maximum principle.

(3) **Cycle-domain interpretation:** We clarified the relation with rank-one cycle domains: the projective ball is the evaluation image of the cycle domain, not the cycle domain itself.

(4) **Explicit example:** We presented a concrete example in $\mathrm{Gr}_3(\mathbb{C}^8)$ that illustrates the geometry and demonstrates why the fiberwise filling meets adjacent K-orbits.

(5) **Higher-rank problem:** We formulated the higher-rank fiberwise envelope problem as a natural open direction.

Future directions

The methods developed here lay the groundwork for future work in several important directions. We outline the most promising avenues below.

Higher-rank Matsuki orbits: The most immediate extension is to Matsuki orbits with $r > 1$, where the fibers are isotropic Grassmannians of higher rank. In this setting, the fibers are higher-codimensional CR manifolds, and the classical hypersurface Kohn–Rossi theorem does not apply. The expected envelopes are bounded symmetric domains of the form

$$D^{\pm}_{L,M} = \{U \in \mathrm{Gr}_r(V_{L,M}) : \pm h|_U > 0\}.$$

However, proving that these domains are indeed the fiberwise Rossi envelopes requires techniques beyond the hypersurface theory used here, such as the theory of CR functions on higher-codimensional CR manifolds and the extension theory for isotropic Grassmannians. A key challenge is to determine whether the analog of Lemma 2 holds in the higher-rank setting, where the fibers are not hypersurfaces.

The descent problem: A fundamental open question is whether holomorphic functions on $\mathcal{D}_{\ell,m,1}$ descend to single-valued functions on the evaluation image $\mathrm{ev}(\mathcal{D}_{\ell,m,1}) \subset Z$. As discussed in Section 3.4 and Section 4.3, this is a nontrivial problem. A sufficient condition for descent would be the existence of a holomorphic section of the evaluation map over $\mathcal{D}_{\ell,m,1}$, or the triviality of the obstruction cocycle. In general, such a section does not exist due to the non-injectivity of the evaluation map on the filled domain. It would be interesting to identify geometric conditions on the base $B_{\ell,m}$ or the signature triple (ℓ, m, r) that guarantee descent. This problem is closely related to the geometry of the evaluation map and the topology of the base.

Extension to other real forms and flag manifolds: The construction of the residual bundle and the evaluation map is quite general and can be adapted to other real forms and other flag manifolds. For example, one could consider the action of $SO(p,q)$ or $Sp(p,q)$ on appropriate flag manifolds, where similar signature data and fibration structures arise. It would be interesting to determine whether the fiberwise Rossi extension method developed here extends to these settings.

Generalization to non-analytic CR functions: Our results assume real-analyticity of the CR functions, which is natural for the Rossi envelope theory. However, it would be desirable to extend the results to larger classes of CR functions, such as continuous or C^∞ functions. Such extensions would require additional regularity theory and approximation results for CR functions on the fibers, which are beyond the scope of the present work.

Application to other homogeneous fibrations: The parameter-dependent extension lemma (Lemma 2) is a versatile tool that could be applied to other settings where fiberwise extensions are needed. For instance, it could be used to study envelopes of holomorphy for other classes of homogeneous CR manifolds, or to analyze the holomorphic extension of CR functions on fibrations arising in algebraic geometry or complex dynamics.

Explicit computation of the abstract Rossi envelope: While we have established fiberwise maximality, the relationship between the fiberwise Rossi domain $\mathcal{D}_{\ell,m,1}$ and the full abstract Rossi envelope of the total CR manifold $M_{\ell,m,1}$ remains open. A complete solution would require a global realization and descent theorem, as discussed in Remark 4. This is a challenging problem that may require new ideas from CR geometry and complex analysis.

We believe that our results contribute to a deeper understanding of the interplay between CR geometry, complex geometry, and representation theory. The fiberwise

Rossi extension construction provides a concrete geometric framework for studying envelopes of holomorphy in homogeneous settings, and the open problems outlined above offer exciting directions for future research. We hope that this work will stimulate further investigations in this area.

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